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LEE–CARTER MORTALITY FORECASTING

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Abstract. In this paper, we focus on Lee–Carter mortality forecasting. Model residuals and future mortality trends are explored. Predictions of the force of mortality for France, Belarus and Lithuania are provided and compared. Several modifications of the model are applied to Lithuanian mortality data in order to obtain the most precise forecast.

Keywords: mortality forecasting, Lee–Carter model.

1. Introduction

Mortality is not constant over time; moreover, it changes differently in different age groups. Therefore, it is important to identify the mortality trend and be able to predict mortality rates accurately. From an economic perspective, inaccurate forecast can make a negative influence on life insurance companies and the pension system (both the state social insurance system and private pension funds).

There are several ways of forecasting mortality. In recent years, the Lee–Carter model has been the most widely used one. This model was originally applied in 1992 to US data and later to mortality data of many other countries such as Canada (Nault, 1993), Austria (Carter and Prkawetz, 2001), Italy (Giacometti, Bertocchi, Rachev, Fabozzi, 2012). Moreover, the majority of new attempts to forecast mortality refer to the Lee–Carter model; therefore, it has many modifications (see Renshaw and Haberman, 2003; Debon, Montes and Puig, 2007).

To our knowledge, Lithuanian mortality has not been predicted; therefore, the objective of this paper is to apply the Lee–Carter model to Lithuanian mortality data and identify the best fitting modification. The paper also compares the applicability of the model to populations with different mortality profiles. For this purpose, besides Lithuania, France and Belarus were chosen.

2. Lee–Carter model

Suppose that $m_{x,t}$ is the death rate for age x in year t, i.e. the ratio between the total number of deaths in the population of age x in year t and the total population of age x in year t $(N_{x,t})$:

$$m_{x,t} = \frac{D_{x,t}}{N_{x,t}},$$

and $\mu_{x,t} = \ln(m_{x,t})$ – empirical force of mortality. Lee and Carter [5] suggested a linear form for the force of mortality $\mu_{x,t}$:

$$\mu_{x,t} = \ln(m_{x,t}) = \alpha_x + \beta_x k_t + \varepsilon_{x,t},\ x = 1,\dots,A; t = 1,\dots,T, \tag{2.1}$$

where α_x, β_x are age-specific parameters, k_t – time-specific parameter, and $\varepsilon_{x,t}$ – independent identically distributed Gaussian errors $N(0,\sigma^2)$. Parameters α_x show the general rate of mortality for a certain age, and parameters k_t – the general rate of mortality for a certain time. It can be easily proved that the expression (2.1) of the force of mortality $\mu_{x,t}$ is invariant with respect to the transformations:

$$(\beta_x, k_t) \to (c\beta_x, k_t/c);\ (\alpha_x, k_t) \to (\alpha_x - c\beta_x, k_t + c) \text{ for some } c \in \mathbb{R}\backslash\{0\}.$$

So we can require the parameters β_x, k_t to satisfy these conditions:

$$\sum_{x=1}^{A} \beta_x = 1; \ \sum_{t=1}^{T} k_t = 0. \tag{2.2}$$

These conditions ensure the unambiguousness of parameters β_x and k_t.

Given the restriction of $\sum_{t=1}^{T} k_t = 0$, parameters α_x, $x = 1,...,A$, are estimated by averages of the force of mortality over a time period, i.e.

$$\tilde{\alpha}_x = \frac{1}{T}\sum_{t=1}^{T} \mu_{x,t}, \text{ for all } x = 1,...,A. \tag{2.3}$$

Then random variables $\mu_{x,t} - \tilde{\alpha}_x = \beta_x k_t + \varepsilon_{x,t}$, $x = 1,...,A$; $t = 1,...,T$, are Gaussian $N(\beta_x k_t, \sigma^2)$. According to [1] and [5], the optimal method to find the estimators of parameters β_x and k_t is SVD (singular value decomposition) of the matrix of variables $z_{x,t} = \mu_{x,t} - \tilde{\alpha}_x$, $x = 1,...,A$; $t = 1,...,T$.

Given the matrix $\mathbf{Z} = (z_{x,t})_{x=1,...,A;\, t=1,...,T}$, we can compute normalized eigenvectors $\mathbf{u}_1 = (u_{1,1},...,u_{1,T})^T$ and $\mathbf{v}_1 = (v_{1,1},...,v_{1,A})^T$ of the matrices $\mathbf{Z}^T\mathbf{Z}$ and $\mathbf{Z}\mathbf{Z}^T$, which correspond to the largest eigenvalue λ_1. Then the estimators of β_x, $x = 1,...,A$, which satisfy the conditions (2.2) and estimators of k_t, $t = 1,...,T$, are:

$$\hat{\boldsymbol{\beta}} = \left(\hat{\beta}_1,...,\hat{\beta}_A\right)^T = \frac{\mathbf{v}_1}{\sum_{j=1}^{A} v_{1,j}}, \quad \tilde{\mathbf{k}} = \left(\tilde{k}_1,...,\tilde{k}_T\right)^T = \lambda_1 \left(\sum_{j=1}^{A} v_{1,j}\right) \mathbf{u}_1. \tag{2.4}$$

For the two values - the number of deaths estimated by the model and the real number of deaths - to be equal, we need another clarification of parameters k_t, $t = 1,...,T$. Estimators $\bar{\bar{k}}_t$ can be found from equations:

$$D_t = \sum_{x=1}^{A} N_{x,t} \exp\left(\tilde{\alpha}_x + \hat{\beta}_x \bar{\bar{k}}_t\right), \ t = 1,...,T, \tag{2.5}$$

where D_t is the total number of deaths in year t and $N_{x,t}$ has been defined earlier.

We need one more transformation for estimators of α_x and k_t to satisfy conditions (2.2). We define

$$\hat{k}_t = \bar{\bar{k}}_t - \frac{1}{T}\sum_{t=1}^{T} \bar{\bar{k}}_t, \text{ for all } t = 1,...,T, \quad \hat{\alpha}_x = \tilde{\alpha}_x + \hat{\beta}_x \frac{1}{T}\sum_{t=1}^{T} \bar{\bar{k}}_t, \text{ for all } x = 1,...,A. \tag{2.6}$$

In order to forecast the force of mortality, Lee and Carter [5] assume that parameters α_x and β_x are constant over time and k_t is a stochastic process. In the same article, they suggest using random walk with a drift for k_t, i.e.

$$\hat{k}_t = \hat{k}_{t-1} + \theta + \xi_t, \ t \geq 2, \tag{2.7}$$

where ξ_t are independent zero-mean Gaussian with variance σ^2_{rw}. The maximum likelihood estimates of θ and σ^2_{rw} are

$$\hat{\theta} = \frac{\hat{k}_T - \hat{k}_1}{T-1}; \ \hat{\sigma}^2_{rw} = \frac{1}{T-1}\sum_{t=1}^{T-1} (\hat{k}_{t+1} - \hat{k}_t - \hat{\theta})^2. \tag{2.8}$$

We get that $\hat{k}_{T+\Delta t} = \hat{k}_T + (\Delta t)\hat{\theta} + \sqrt{\Delta t}\,\tilde{\xi}$, where $\tilde{\xi}$ are Gaussian $N(0, \sigma^2_{rw})$. The forecast of force of mortality in year $T + \Delta t$ can be approximated as follows:

$$\hat{\mu}_{x,T+\Delta t} = \hat{\alpha}_x + \hat{\beta}_x \left(\hat{k}_T + (\Delta t)\hat{\theta}\right) = \hat{\alpha}_x + \hat{\beta}_x \left(\hat{k}_T + \Delta t \frac{\hat{k}_T - \hat{k}_1}{T-1}\right). \tag{2.9}$$

3. Empirical data analysis

Mortality data, population size and the number of deaths in Lithuania, France and Belarus are taken from the Berkeley Human Mortality Database, University of California (www.mortality.org). Lithuanian data is available for the period from 1959 to 2010, for ages 0 to 110 years, for men and women separately and together.

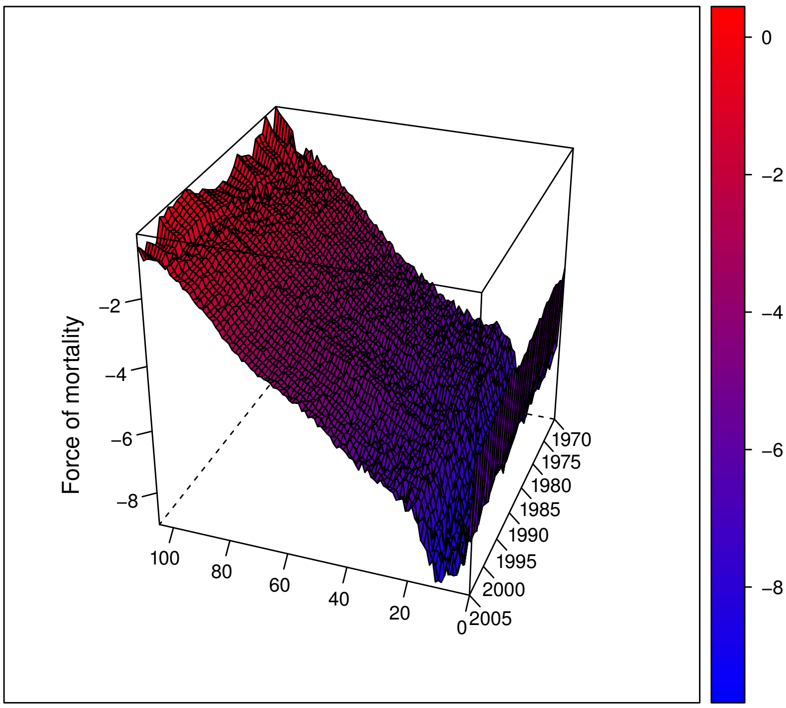

Figure 3.1: Empirical force of mortality of the Lithuanian population

We use the data from 1970 for our calculations because the data until 1970 might be unreliable. From Figure 3.1, where the empirical force of mortality of the Lithuanian population is shown, we can see that the optimal age interval for modelling is 20 to 90 years. Analogous surfaces are similar for France and Belarus, therefore, for these countries, we use the same time and age intervals. Later, the two data subsets will be taken for men and women separately:

- *Subset 1*: 1970 to 2005, 20 to 90 years,
- *Subset 2*: 2006 to 2010 (2009 for France), 20 to 90 years.

From the first subset, the parameters of the model are estimated. From the second subset, we can compare the models and the estimates of the force of mortality with the empirical data.

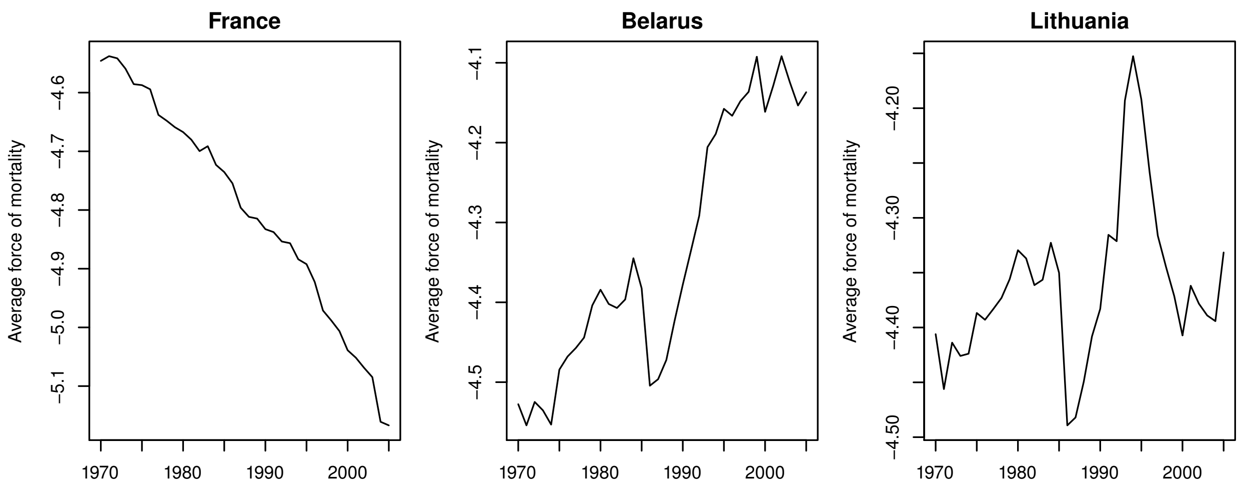

Figure 3.2: Average empirical force of mortality

Figure 3.2 plots the average empirical force of mortality $\bar{\mu}_t, t = 1970, \dots, 2005$, $\left(\bar{\mu}_t = \frac{1}{71}\sum_{x=20}^{90}\mu_{x,t}\right)$ for France, Belarus and Lithuania for the period 1970 to 2005. The mortality of the French population is characterised by a significant dawnward trend; the average empirical force of mortality of Belarus increases over time, but with some fluctuations. Meanwhile, the average empirical force of mortality of Lithuania fluctuates without any visible trend.

4. $\mu_{x,t}$ forecast for France

From equations (2.2)–(2.8), we calculate $\hat{\alpha}_x$, $\hat{\beta}_x$, $x = 20,\dots,90$, and $\hat{k}_t, t = 1970,\dots,2005$, for the French data. The results are shown in Figure 4.1. We assume that $\hat{k}_t$, $t = 1971,\dots,2005$, modelled for the French data, is a first-order autoregressive process, i.e.

$$\hat{k}_t = \theta\hat{k}_{t-1} + \theta_0 + \xi_t, t = 1971,\dots,2005,$$

with independent Gaussian errors $\xi_t \sim N(0,\sigma^2_{rw})$.

Using the augmented Dickey–Fuller (ADF) test, we test the hypothesis:

H_0: process has a unit root,	or	$H_0 : \theta = 1$,
H_1: process does not have a unit root.		$H_1 : \theta \neq 1$.

We get that the p-value is 0.0951 for men and 0.0148 for women. Therefore, we cannot reject the null hypothesis that there is a unit root in men's $\hat{k}_t$, when the significance level is less than 0.0951, and in women's $\hat{k}_t$, when the significance level is less than 0.0148. Though for women we should reject H_0, when the significance level is more than 0.0148, from Figure 4.1 we can see that the plots of $\hat{k}_t$ are very similar for men and women, and we do not reject H_0 for men's $\hat{k}_t$. Therefore, for both men and women, we model $\hat{k}_t$ as a random walk with a drift. The maximum likelihood estimate of the drift $\hat{\theta} = -1.3615$ for women and $\hat{\theta} = -1.130$ for men.

We notice that errors ξ_t are substantially Gaussian for both men and women (the p-values of the Kolmogorov–Smirnov test are 0.6654 for men and 0.1193 for women, the p-values of the χ^2 test are, respectively, 0.7685 for men and 0.1046 for women). The means of errors ξ_t are 0, $\hat{\sigma}_{rw} = 1.1872$ for men and $\hat{\sigma}_{rw} = 1.7452$ for women. Having the values of $\hat{\alpha}_x$, $\hat{\beta}_x$ and $\hat{k}_t$, using the expression (2.9), we calculate $\hat{\mu}_{x,t}, t = 2006,\dots,2009$. The results are shown in figures 4.2 and 4.3.

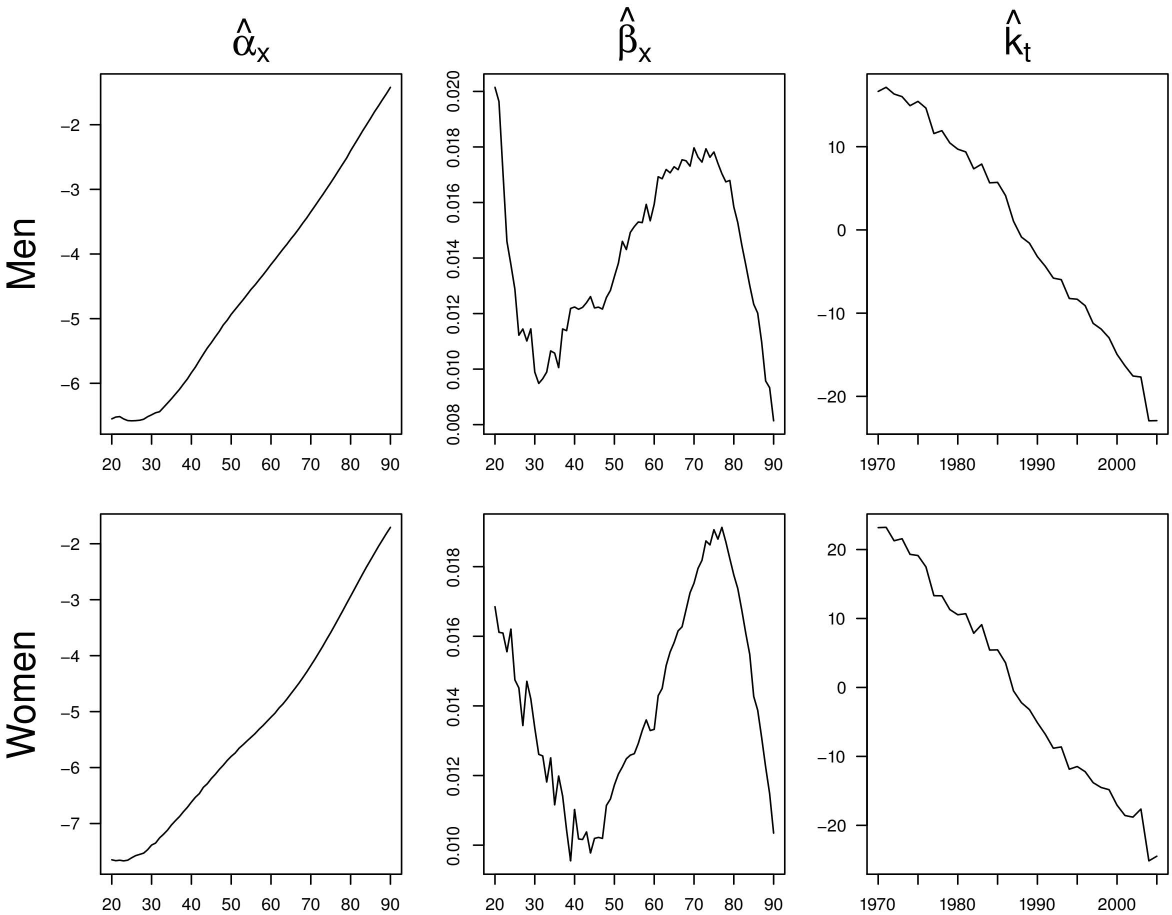

Figure 4.1: $\hat{\alpha}_x, \hat{\beta}_x, \hat{k}_t$ for men (top) and women (bottom)

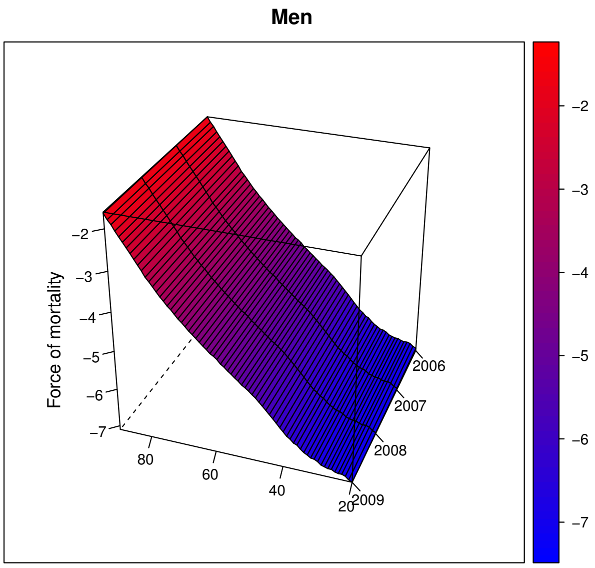

Figure 4.2: $\hat{\mu}_{x,t}$

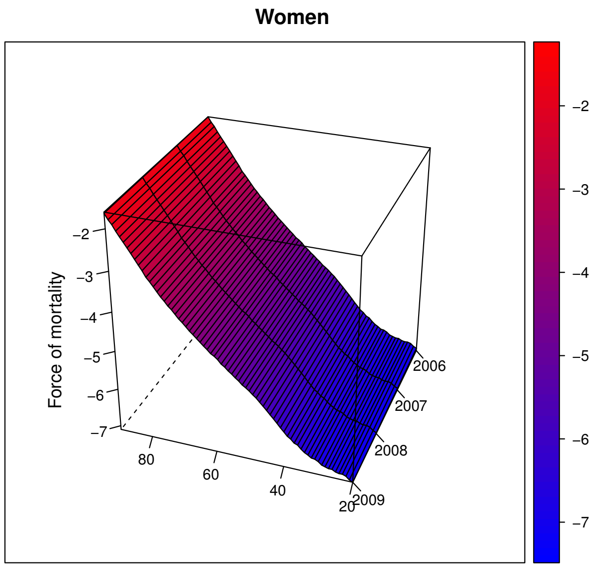

Figure 4.3: $\hat{\mu}_{x,t}$

5. $\mu_{x,t}$ forecast for Belarus

In the same way as for France, we calculate $\hat{\alpha}_x$, $\hat{\beta}_x$, $x = 20,...,90$, and $\hat{k}_t$, $t = 1970,...,2005$, for Belarus. The results are shown in Figure 5.3. The p-values of the ADF test are 0.5794 for men and 0.297 for women.

Since we do not reject the null hypothesis that there is a unit root, we model $\hat{k}_t$ as a random walk with a drift. The maximum likelihood estimate of the drift is $\hat{\theta} = 0.411$ for women and $\hat{\theta} = 0.8802$ for men. The errors ξ_t are substantially Gaussian for both men and women (the p-values of the Kolmogorov–Smirnov test are 0.4144 for men and 0.1984 for women, the p-values of the χ^2 test are, respectively, 0.4361 for men and 0.24232 for women). The means of errors ξ_t are 0, $\hat{\sigma}_{rw} = 2.6276$ for men and $\hat{\sigma}_{rw} = 2.4816$ for women. Having $\hat{\alpha}_x$, $\hat{\beta}_x$, $\hat{k}_t$, using (2.9), we calculate $\hat{\mu}_{x,t}, t = 2006,...,2010$. The results are shown in figures 5.1 and 5.2.

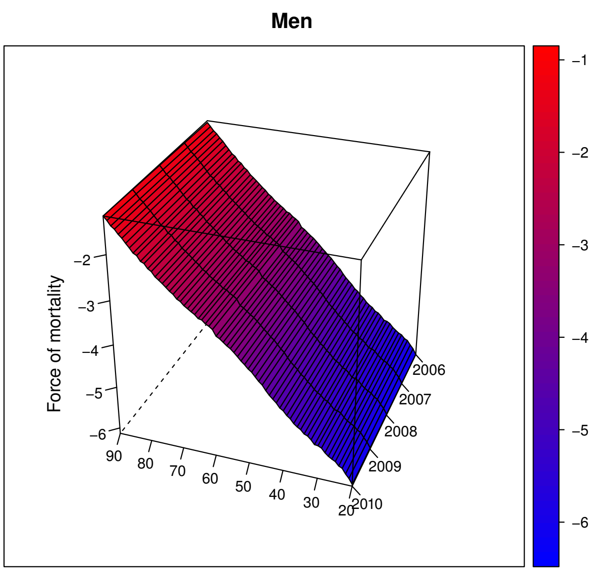

Figure 5.1: $\hat{\mu}_{x,t}$

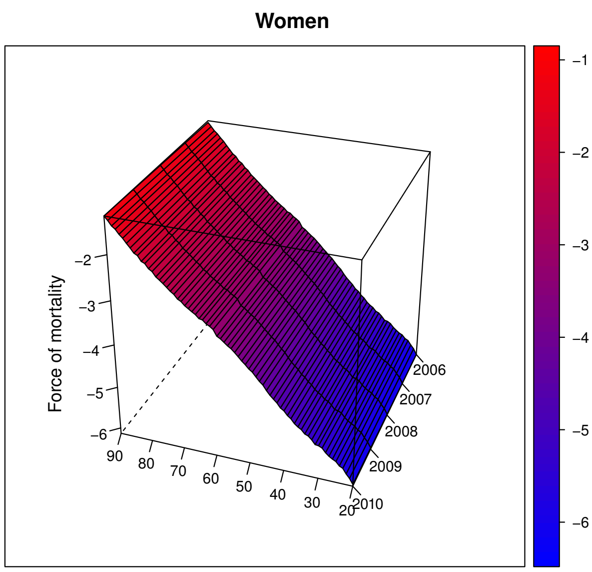

Figure 5.2: $\hat{\mu}_{x,t}$

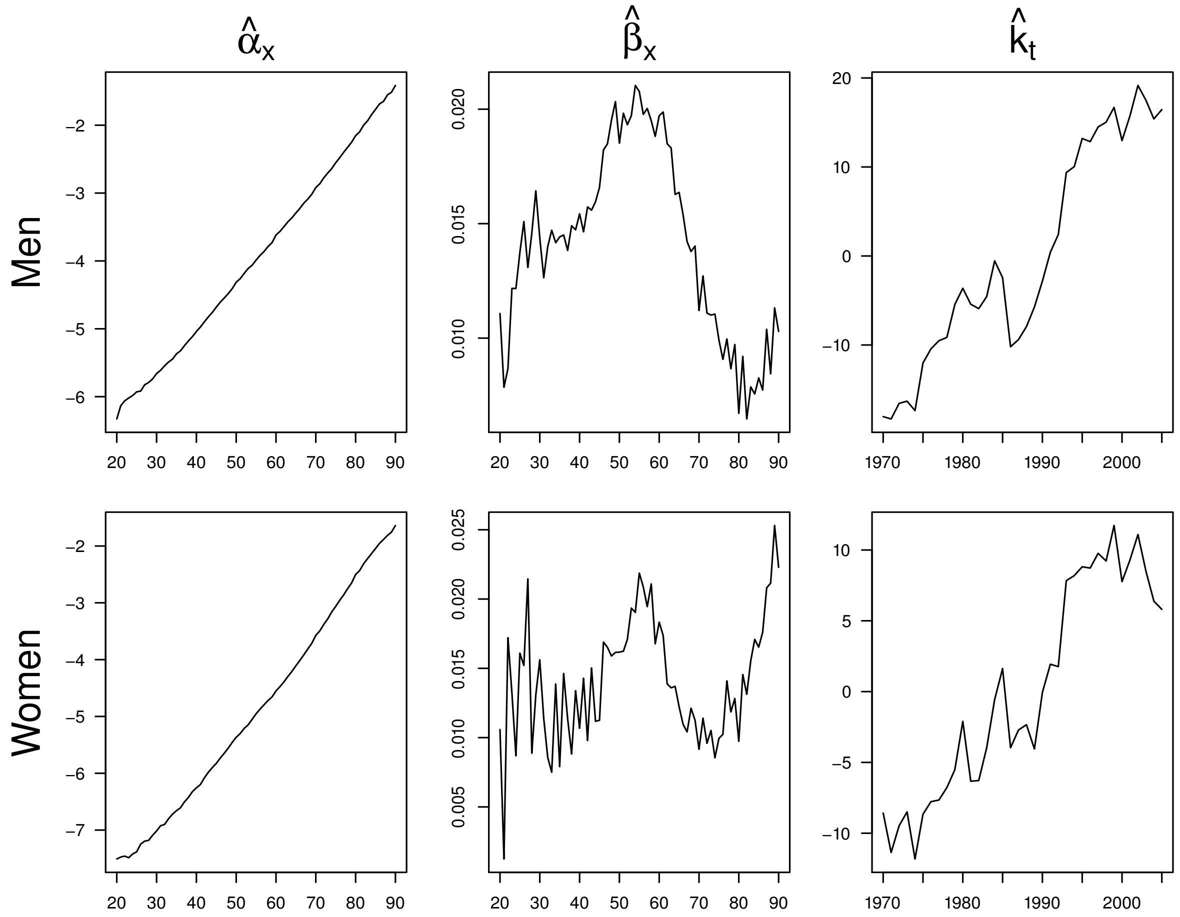

Figure 5.3: $\hat{\alpha}_x, \hat{\beta}_x, \hat{k}_t$ for men (top) and women (bottom)

6. $\mu_{x,t}$ forecast for Lithuania

6.1. Classical model

We calculate $\hat{\alpha}_x$, $\hat{\beta}_x$, $x = 20,...,90$, and $\hat{k}_t$, $t = 1970,...,2005$, for Lithuanian data. The estimates are shown in Figure 6.1.1. The p-values of the ADF test are 0.4711 for men and 0.4377 for women.

Hence we model $\hat{k}_t$ as a random walk with a drift. The maximum likelihood estimate of the drift is $\hat{\theta} = -0.3792$ for women and $\hat{\theta} = 0.3889$ for men. The errors $\hat{k}_t$ are substantially Gaussian for both men and women (means are 0, $\hat{\sigma}_{rw} = 3.6219$ for women and $\hat{\sigma}_{rw} = 3.1181$ for men, the p-values of normality tests are 0.5959 for women and 0.4333 for men of the Kolmogorov–Smirnov test and 0.9614 for women and 0.1480 for men of the χ^2 test).

6.2. $\hat{\mathbf{k}}_\mathbf{t}$ with one and two lags

Let us consider that $\hat{k}_t$ is a random process with one lag, i.e. a second-order autoregressive process with a drift:

$$\hat{k}_t = \theta\hat{k}_{t-1} + \theta_0 + \theta_1\left(\hat{k}_{t-1} - \hat{k}_{t-2}\right) + \xi_t, t = 1971,...,2005,$$

with independent Gaussian errors $\xi_t \sim N(0, \sigma_{rw}^2)$. Having applied the ADF test, we do not reject the null hypothesis that there is a unit root (p-values are 0.2426 for men and 0.4328 for women); therefore, we assume that $\theta = 1$ while modelling $\hat{k}_t$. Least squares estimates are $\theta_0 = -0.2113$, $\theta_1 = -0.0827$ for women and $\theta_0 = 0.4743$, $\theta_1 = 0.1317$ for men. The errors ξ_t are substantially Gaussian for both men and women (means are 0, $\hat{\sigma}_{rw} = 3.4463$ for women and $\hat{\sigma}_{rw} = 3.052$ for men, the p-values of normality tests are 0.5959 for women and 0.8608 for men of the Kolmogorov–Smirnov test and 0.9614 for women and 0.1062 for men of the χ^2 test).

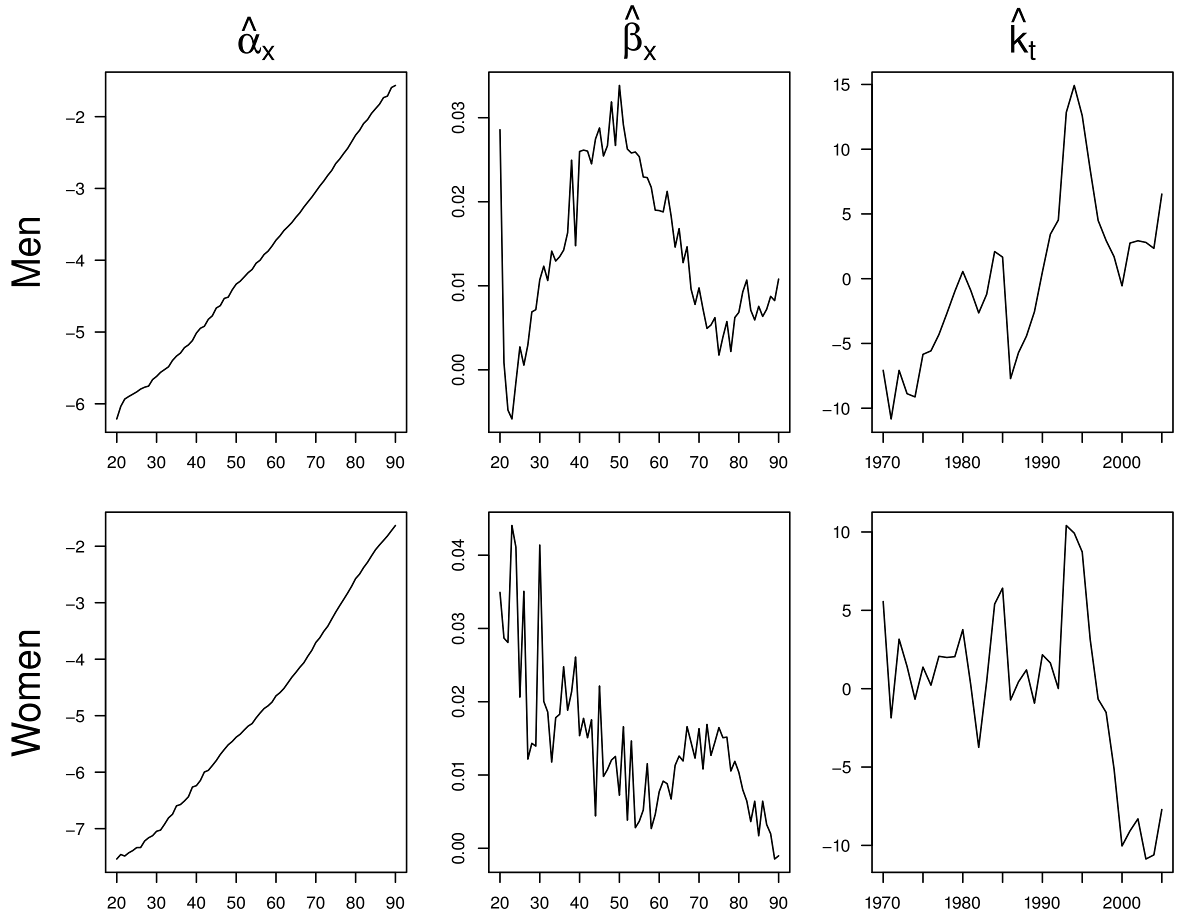

Figure 6.1.1: $\hat{\alpha}_x$, $\hat{\beta}_x$ ir $\hat{k}_t$ for men (top) and women (bottom)

Now let us consider that $\hat{k}_t$ is a random process with two lags, i.e. a third-order autoregressive process with a drift:

$$\hat{k}_t = \theta\hat{k}_{t-1} + \theta_0 + \theta_1\left(\hat{k}_{t-1} - \hat{k}_{t-2}\right) + \theta_2\left(\hat{k}_{t-2} - \hat{k}_{t-3}\right) + \xi_t,\, t = 1971, ..., 2005,$$

with independent Gaussian errors $\xi_t \sim N(0, \sigma^2_{rw})$ and $\theta = 1$ (p-values of the ADF test are 0.2774 for men and 0.6371 for women). Least squares estimates are $\theta_0 = 0.3602$, $\theta_1 = -0.0165$, $\theta_2 = -0.1976$ for women and $\theta_0 = 0.4743$, $\theta_1 = 0.1973$, $\theta_2 = -0.0882$ for men. The errors ξ_t are substantially Gaussian for both men and women (means are 0, $\hat{\sigma}_{rw} = 3.3087$ for women and $\hat{\sigma}_{rw} = 3.0078$ for men, the p-values of normality tests are 0.9083 for women and 0.8335 for men of the Kolmogorov–Smirnov test and 0.9022 for women and 0.7739 for men of the χ^2 test).

6.3. Adjusted Lee–Carter model

According to [4], we modify Lee–Carter model:

$$\mu_{x,t} = \alpha_x + \beta1_x k1_t + \beta2_x k2_t + \varepsilon_{x,t},\, x = 20, ..., 90;\, t = 1970, ..., 2005,$$

where $\varepsilon_{x,t} \sim N(0, \sigma^2)$ are Gaussian errors. For the estimation of parameters $\beta1_x, k1_t, \beta2_x, k2_t$, we also use the SVD method, but in the decomposition of the matrix $z_{x,t} = \mu_{x,t} - \tilde{\alpha}_x$ we take two singular values and get:

$$\hat{\boldsymbol{\beta}}1 = \left(\hat{\beta}1_1, ..., \hat{\beta}1_A\right)^T = \frac{\mathbf{v_1}}{\sum\limits_{j=1}^{A} v_{1,j}}, \quad \tilde{\mathbf{k}}1 = \left(\tilde{k}1_1, ..., \tilde{k}1_T\right)^T = \lambda_1\left(\sum_{j=1}^{A} v_{1,j}\right)\mathbf{u}_1,$$

$$\hat{\boldsymbol{\beta}}2 = \left(\hat{\beta}2_1, ..., \hat{\beta}2_A\right)^T = \frac{\mathbf{v_2}}{\sum\limits_{j=1}^{A} v_{2,j}}, \quad \bar{\bar{\mathbf{k}}}\mathbf{2} = \left(\bar{\bar{\mathbf{k}}}\mathbf{2_1}, ..., \bar{\bar{\mathbf{k}}}\mathbf{2_T}\right)^{\mathbf{T}} = \lambda_2\left(\sum_{\mathbf{j=1}}^{\mathbf{A}} \mathbf{v_{2,j}}\right)\mathbf{u_2}.$$

As in the classical model, the estimated and empirical number of deaths must be equal; therefore, we find the estimates $\bar{\bar{k}}1_t$ from the equations:

$$D_t = \sum_{x=1}^{A} N_{x,t} \exp\left(\tilde{\alpha}_x + \hat{\beta}1_x \bar{\bar{k}}1_t + \hat{\beta}2_x \bar{\bar{k}}2_t\right),\ t = 1, \dots, T.$$

Since the parameters $k1_t$ and $k2_t$ must satisfy the constraints

$$\sum_{t=1}^{T} k1_t = 0, \quad \sum_{t=1}^{T} k2_t = 0,$$

an additional transformation is made:

$$\hat{k}1_t = \bar{\bar{k}}1_t - \frac{1}{T}\sum_{t=1}^{T} \bar{\bar{k}}1_t, \quad \hat{k}2_t = \bar{\bar{k}}2_t - \frac{1}{T}\sum_{t=1}^{T} \bar{\bar{k}}2_t, \quad \hat{\alpha}_x = \tilde{\alpha}_x + \hat{\beta}1_x \frac{1}{T}\sum_{t=1}^{T} \bar{\bar{k}}1_t + \hat{\beta}2_x \frac{1}{T}\sum_{t=1}^{T} \bar{\bar{k}}2_t.$$

The estimates $\hat{k}1_t$, $\hat{k}2_t$ and $\hat{\beta}1_t$, $\hat{\beta}2_t$ are shown in Figure 6.3.3. We do not reject the unit root hypothesis for both $\hat{k}1_t$ and $\hat{k}2_t$. The p-values of the ADF test for $\hat{k}1_t$ are 0.5333 for men and 0.3898 for women; for $\hat{k}2_t$: 0.6132 for men and 0.3087 for women. Hence we model $\hat{k}1_t$ and $\hat{k}2_t$ as random walks with a drift. For men, $\hat{\theta} = 0.3747$ for the process $\hat{k}1_t$ and $\hat{\theta} = -0.0398$ for the process $\hat{k}2_t$. For women, $\hat{\theta} = -0.422$ for the process $\hat{k}1_t$ and $\hat{\theta} = 0.0648$ for the process $\hat{k}2_t$. The errors of the process $\hat{k}1_t$ are substantially Gaussian for both men and women, while for the process $\hat{k}2_t$ we should reject the normality hypothesis when the significance level is more than 0.0415 for men and more than 0.0351 for women. The means of errors are 0, $\hat{\sigma}_{rw}$ and the p-values of the Kolmogorov–Smirnov and the χ^2 tests are given in Table 6.3.1. $\hat{\mu}_{x,t}, t = 2006, \dots, 2010$ are shown in figures 6.3.1 and 6.3.2.

Table 6.3.1: The characteristics of the errors.

		$\hat{\sigma}_{rw}$	p-value of K–S test	p-value of χ^2 test
Men	$\hat{k}1_t$	3.0952	0.3743	0.3851
	$\hat{k}2_t$	0.6068	0.02	0.0415
Women	$\hat{k}1_t$	3.1739	0.1655	0.5438
	$\hat{k}2_t$	0.7497	0.0074	0.0351

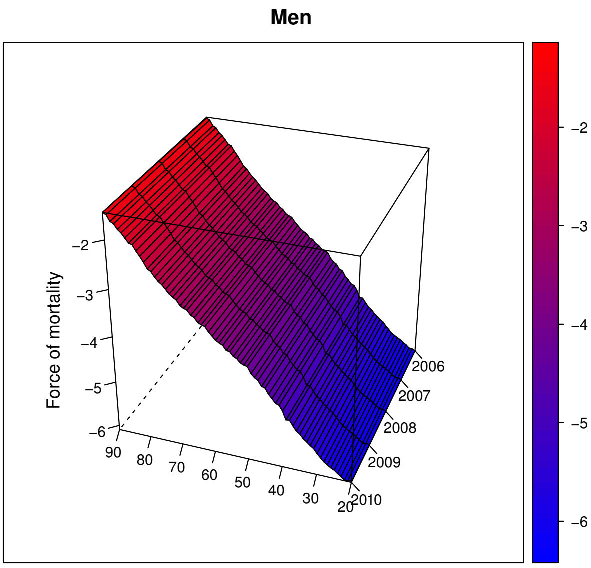

Figure 6.3.1: $\hat{\mu}_{x,t}$

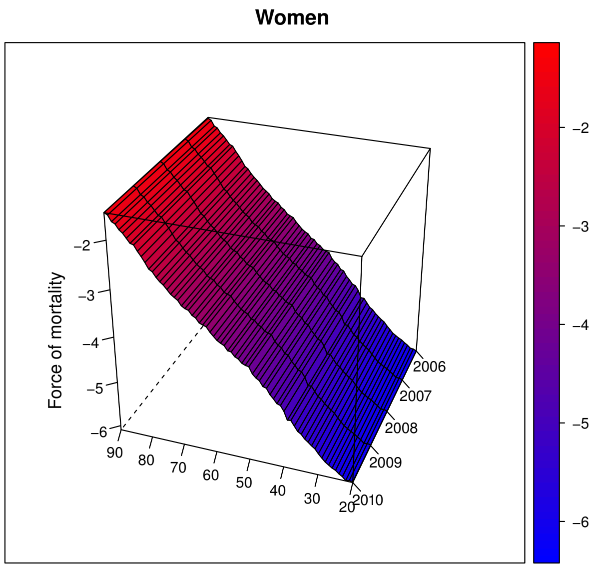

Figure 6.3.2: $\hat{\mu}_{x,t}$

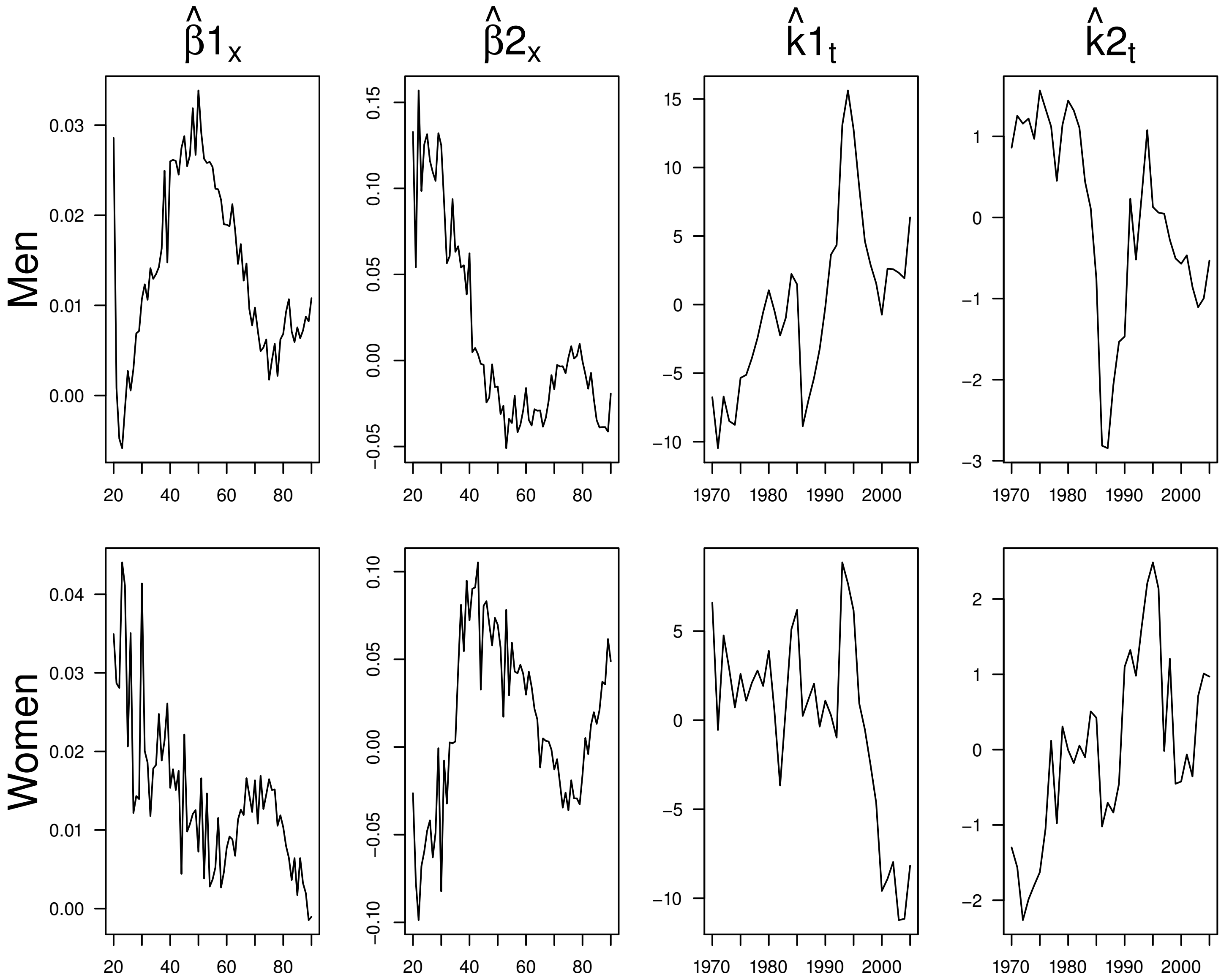

Figure 6.3.3: $\hat{\beta}1_x$, $\hat{\beta}2_x$, $\hat{k}1_t$ and $\hat{k}2_t$ for men (top) and women (bottom)

6.4. Comparison of the models

We will use a square root of a mean square error to compare which model is the most suitable for Lithuanian data and most accurately predicts $\mu_{x,t}$:

$$\sqrt{MSE} = \sqrt{\frac{1}{71}\sum_{x=20}^{90}(\hat{\mu}_{x,t} - \mu_{x,t})^2},$$

where $t = 1970, ..., 2005$ for model errors and $t = 2006, ..., 2010$ for forecast errors.

The average $\sqrt{MSE}$ of a classical model is 0.1052 for men and 0.1578 for women, the average $\sqrt{MSE}$ of adjusted model is 0.105 for men and 0.1566 for women. $\sqrt{MSE}$ of forecast is reported in Table 6.4.1. It is evident that we do not reduce or even increase $\sqrt{MSE}$ by adding lags to the prediction of $\hat{k}_t$ and therefore do not make the prediction more accurate. However, the adjustment of the model reduces the $\sqrt{MSE}$ of both model and the forecast and this means that the adjusted model is the most suitable for Lithuanian data and most accurately predicts $\mu_{x,t}$.

Table 6.4.1 shows that the accuracy of a longer-term (4–5 years) forecast for women is only slightly smaller than that of a short-term (1–2 years). The accuracy of a short-term forecast for men is much better than for women; however, with an increase in the period, the accuracy decreases much faster, and longer-term forecasts for men are less accurate than for women.

Table 6.4.1: $\sqrt{MSE}$ of a forecast

	Men $\sqrt{MSE}$					Women $\sqrt{MSE}$				
Model/year	2006	2007	2008	2009	2010	2006	2007	2008	2009	2010
Classical	0.123	0.1154	0.1515	0.2595	0.3055	0.1793	0.1759	0.1775	0.1512	0.1811
1 lag	0.1236	0.1134	0.1598	0.2738	0.3206	0.1798	0.1753	0.1765	0.1543	0.1879
2 lags	0.124	0.1136	0.1561	0.2667	0.3124	0.1803	0.1802	0.1805	0.1483	0.1761
Adjusted	0.1047	0.1056	0.1305	0.2325	0.2762	0.1755	0.1572	0.1618	0.1518	0.1795

7. Comparison of the countries

We calculate $\sqrt{MSE}$ (from 1970 to 2005) of the Belarusian and French models and compare them with the $\sqrt{MSE}$ of the Lithuanian adjusted model. On average, the French $\sqrt{MSE}$ are the smallest and the Lithuanian – the biggest. We also calculate the $\sqrt{MSE}$ of the forecast (from 2006 to 2010 for Belarus and to 2009 – for France). Comparing them with analogous $\sqrt{MSE}$ of the Lithuanian adjusted model forecast, Figure 7.1 shows that a short-term (1–2 years) Lithuanian forecast for men is more accurate than the corresponding Belarusian and French forecasts. However, the Lithuanian forecast for men becomes less accurate with an increase in the period. Meanwhile, the Lithuanian forecast for women is less accurate than forecasts for Belarus and France (except for a 5-year forecast, which is more accurate than Belarusian). Hence the Lee–Carter model most accurately describes and predicts mortality in France. This is what might have been expected because from Figure 3.2 it is obvious that French mortality varies over time the least.

Figure 7.2 shows the forecast of the force of mortality for the year 2011. The forecast is obtained by taking data from 1970 to 2010 (to 2009 for France), because if we take data only until the year 2005, we get a less accurate forecast. Female mortality remains lower than the male one in younger age, while in older age female and male mortality becomes similar.

In Figure 7.3, forecasts for the force of mortality for the years 2011 and 2060 are compared. It is seen that the forecasted mortality of French women and men will significantly decrease over 50 years. The forecasted mortality of Belarusian women will slightly, that of Belarusian men – quite significantly increase. Meanwhile, the mortality of Lithuanian women, especially of younger ones, will decrease. The mortality of younger Lithuanian men will decrease, while that of older men – increase.

Table 7.1: $\sqrt{MSE}$ of model

Country	France		Belarus		Lithuania	
Sex	Men	Women	Men	Women	Men	Women
Average	**0.0664**	**0.0623**	**0.0762**	**0.1005**	**0.105**	**0.1566**

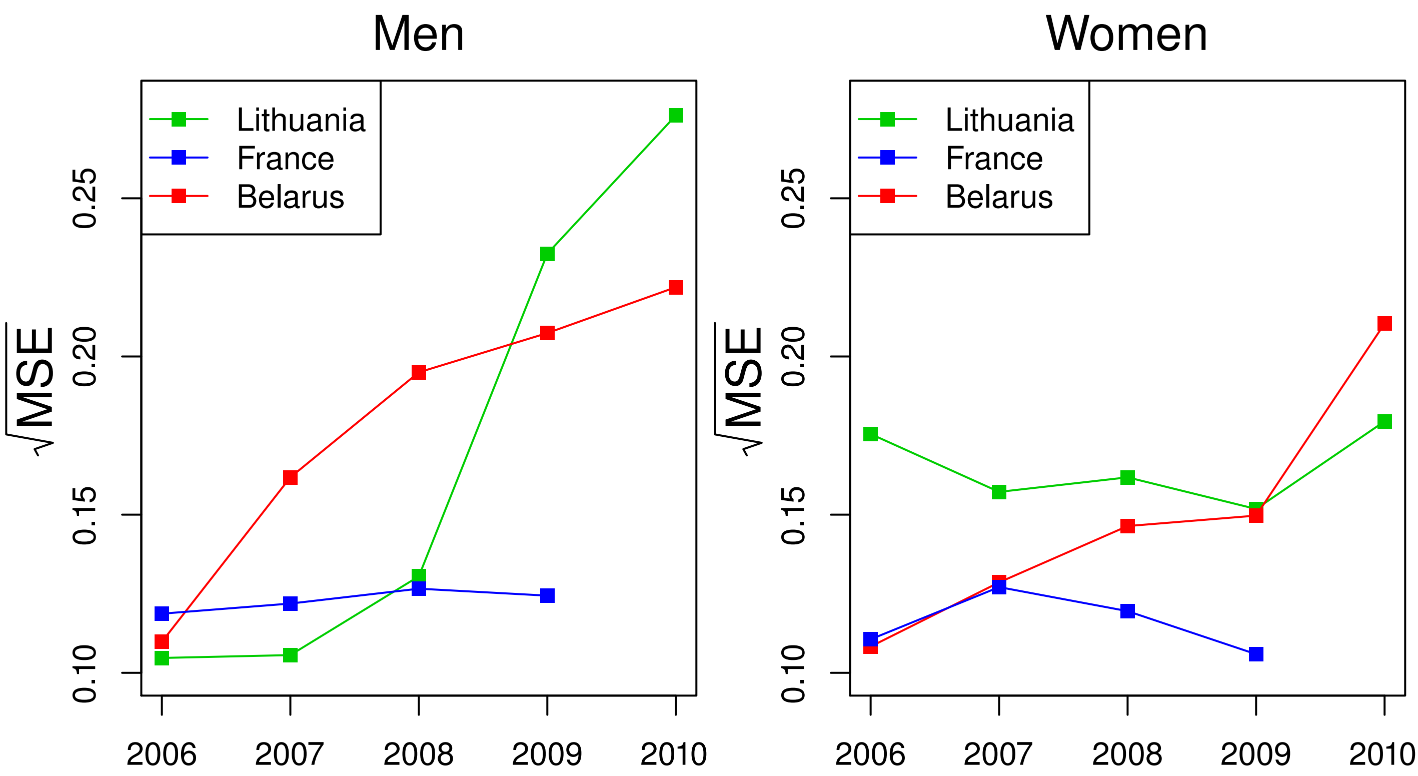

Figure 7.1: $\sqrt{MSE}$ of forecast for men (on the left) and for women (on the right)

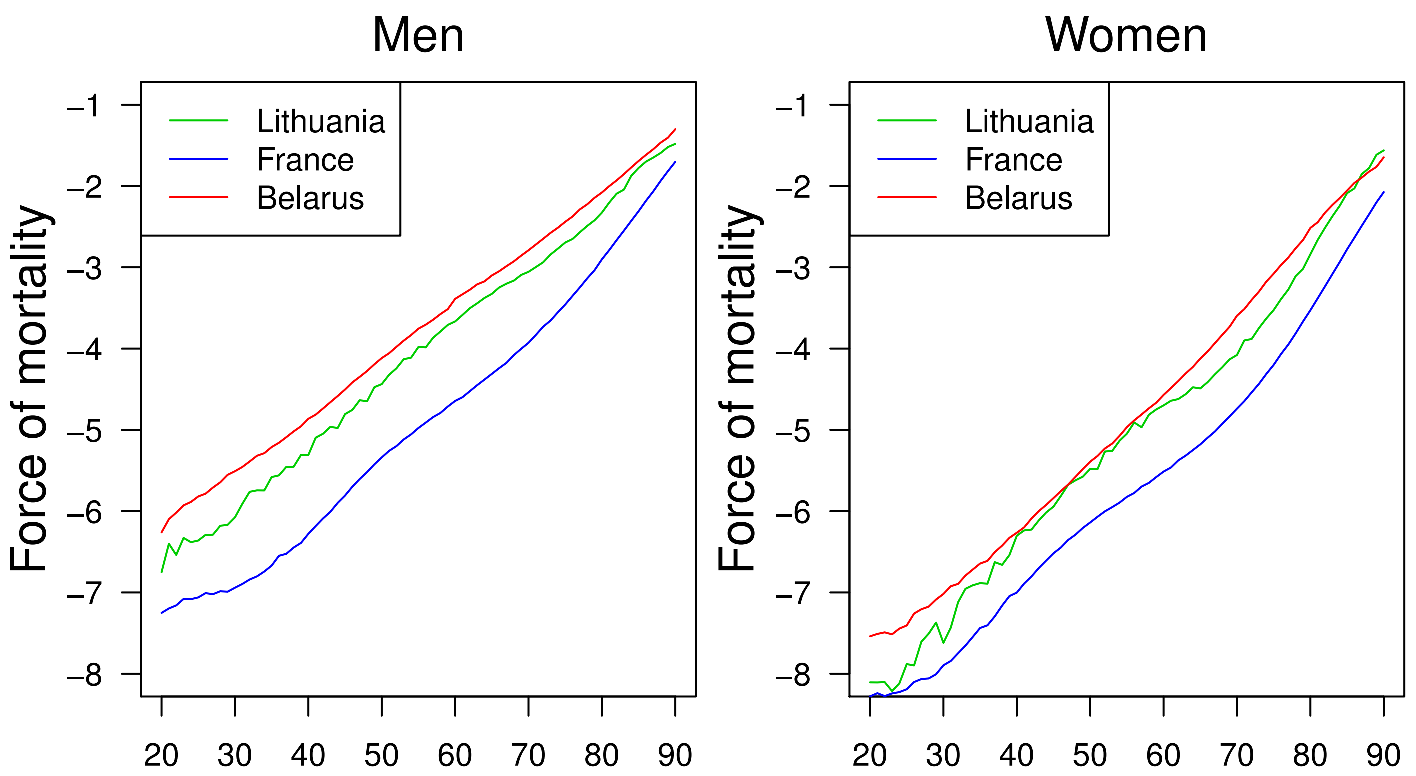

Figure 7.2: $\hat{\mu}_{x,2011}$ for men (on the left) and for women (on the right)

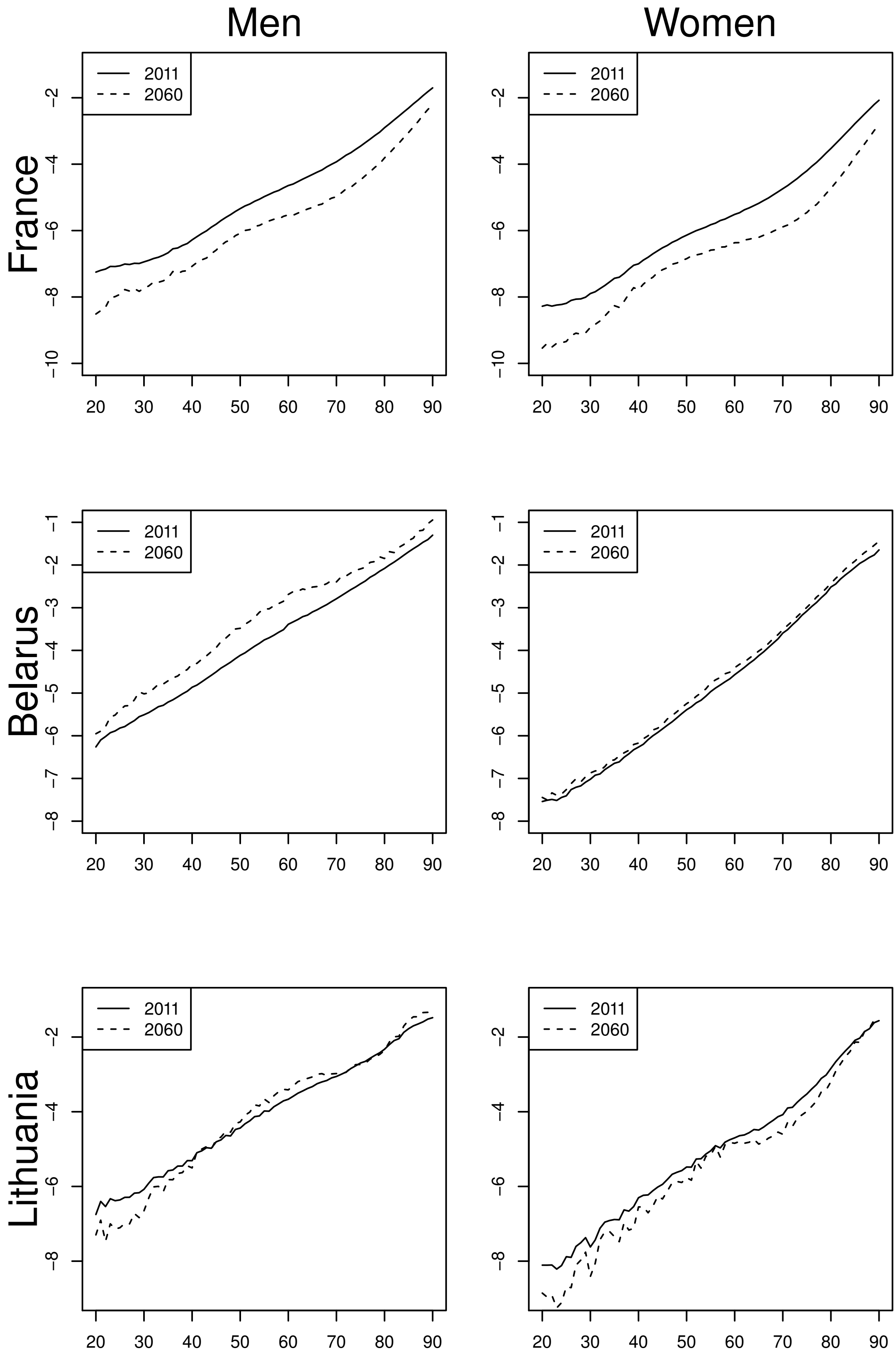

Figure 7.3: $\hat{\mu}_{x,2011}$ and $\hat{\mu}_{x,2060}$ for men (on the left) and for women (on the right)

8. The average number of years of life remaining

Suppose ${}_kp_{x,t}$ is the probability that an individual who is x years old in year t will survive at least k years, $q_{x,t}$ – the probability that the individual will die during the next year. We estimate the probability $q_{x,t}$ with the mortality rate $m_{x,t} = e^{\hat{\mu}_{x,t}}$. It is easy to see that

$$ {}_kp_{x,t} = {}_1p_{x,t} \cdot {}_1p_{x+1,t+1} \cdot \ldots \cdot {}_1p_{x+k-1,t+k-1}, \text{ for all } k,x,t. $$

Moreover, ${}_1p_{x,t} = 1 - q_{x,t}$, for all x and t.

For the calculation of the average number of years of life remaining for an individual who is x years old (e_x), we will use the formula:

$$ e_x = \sum_{k=1}^{\infty} {}_kp_{x,t}. $$

The derivation of this formula is shown in [2].

We forecast the force of mortality for ages $x = 20,\ldots,90$; therefore, we assume that the probability for an individual to survive more than 91 years is very small. Hence e_x, $x = 20,\ldots,90$, are estimated as follows:

$$ \hat{e}_x = \sum_{k=1}^{91-x} {}_kp_{x,2012}, \qquad \text{or} \qquad \hat{e}_x = \sum_{k=1}^{91-x} \prod_{l=0}^{k-1} \left(1 - e^{\hat{\mu}_{x+l,2012+l}}\right). $$

In Table 8.1, e_x, $x = 20,\ldots,80$, for Belarusian, French and Lithuanian men and women are presented.

Table 8.1: Average number of years of life remaining for the year 2012

	France		Belarus		Lithuania			France		Belarus		Lithuania	
Age	Men	Women	Men	Women	Men	Women	Age	Men	Women	Men	Women	Men	Women
20	63.6	67.6	39.5	54.9	46.7	60.3	51	31.2	35.5	17.7	26.5	20.9	29.7
21	62.5	66.6	38.7	54	45.8	59.2	52	30.2	34.5	17.1	25.6	20.2	28.8
22	61.5	65.5	37.9	53	44.9	58.2	53	29.3	33.5	16.5	24.8	19.5	27.9
23	60.4	64.5	37.1	52.1	43.9	57.2	54	28.3	32.5	16	24	18.9	27
24	59.3	63.4	36.4	51.1	43	56.2	55	27.4	31.5	15.4	23.1	18.2	26.1
25	58.3	62.4	35.6	50.2	42.1	55.1	56	26.5	30.6	14.9	22.3	17.6	25.3
26	57.2	61.3	34.9	49.2	41.2	54.1	57	25.6	29.6	14.3	21.5	17	24.4
27	56.1	60.3	34.1	48.3	40.3	53.1	58	24.6	28.6	13.8	20.7	16.3	23.5
28	55.1	59.3	33.4	47.3	39.4	52.1	59	23.7	27.6	13.3	19.9	15.8	22.7
29	54	58.2	32.6	46.4	38.6	51.1	60	22.8	26.6	12.8	19.1	15.2	21.9
30	52.9	57.2	31.9	45.5	37.7	50	61	21.9	25.6	12.3	18.3	14.6	21
31	51.9	56.1	31.1	44.5	36.8	49	62	21	24.7	11.8	17.5	14	20.2
32	50.8	55.1	30.4	43.6	35.9	48	63	20.2	23.7	11.4	16.8	13.5	19.4
33	49.7	54	29.7	42.7	35.1	47	64	19.3	22.7	10.9	16	12.9	18.5
34	48.7	53	29	41.7	34.2	46	65	18.4	21.8	10.5	15.3	12.4	17.7
35	47.6	51.9	28.2	40.8	33.3	45	66	17.6	20.8	10	14.5	11.9	16.9
36	46.5	50.9	27.5	39.9	32.5	44	67	16.7	19.9	9.6	13.8	11.4	16
37	45.5	49.8	26.8	39	31.7	43	68	15.9	18.9	9.1	13.1	10.9	15.2
38	44.4	48.8	26.1	38	30.8	42	69	15.1	18	8.7	12.4	10.4	14.4
39	43.4	47.8	25.4	37.1	30	41	70	14.3	17.1	8.3	11.7	9.9	13.6
40	42.3	46.7	24.7	36.2	29.2	40.1	71	13.5	16.1	7.8	11.1	9.4	12.8
41	41.3	45.7	24	35.3	28.4	39.1	72	12.7	15.2	7.4	10.4	8.8	12.1
42	40.2	44.7	23.3	34.4	27.6	38.1	73	11.9	14.3	7	9.8	8.3	11.3
43	39.2	43.6	22.7	33.5	26.8	37.2	74	11.2	13.4	6.7	9.2	7.9	10.5
44	38.2	42.6	22	32.6	26	36.2	75	10.4	12.5	6.3	8.6	7.4	9.8
45	37.2	41.6	21.4	31.7	25.2	35.3	76	9.7	11.7	5.9	8	6.9	9.1
46	36.1	40.6	20.7	30.8	24.5	34.3	77	9	10.8	5.5	7.5	6.5	8.4
47	35.1	39.6	20.1	29.9	23.7	33.4	78	8.3	10	5.2	6.9	6	7.7
48	34.1	38.5	19.5	29.1	23	32.4	79	7.6	9.2	4.8	6.4	5.6	7
49	33.2	37.5	18.9	28.2	22.3	31.5	80	7	8.3	4.5	5.9	5.1	6.4
50	32.2	36.5	18.3	27.3	21.6	30.6							

9. Conclusions

While comparing the suitability of the Lee–Carter model for different countries, we have obtained that the model most accurately describes and predicts mortality for France. Only short-term (1–2 years) forecasts for the Belarusian and Lithuanian mortality are accurate, while the accuracy of longer-term (4–5 years) forecasts is lower, especially for men. The accuracy of forecast for France does not decrease with an increase in the period, while for women it even increases.

From these results, we can conclude that the Lee–Carter method is most suitable for populations with a clear upward or downward mortality trend over time. It is incorrect to assume that the force of mortality has a linear form if mortality varies considerably over time and linearity is the key assumption of the model.

We have not obtained an accurate forecast using the classical Lee–Carter model because Lithuanian mortality has varied significantly in recent decades. Modified models can be applied when the assumption of linearity is not satisfied. One of such modifications is the adjusted model. We have obtained more accurate results by applying it to Lithuanian data.

The Lee–Carter model is based on the assumption that the model errors are independent identically distributed Gaussian random variables. This assumption is not only difficult to verify but also unrealistic in many cases. This problem may be solved by modelling the dependence of residuals or choosing other random variables to model the errors and modifying the model in this way.

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MIRTINGUMO PROGNOZAVIMAS LEE-CARTER'IO METODU

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Santrauka. Darbe nagrinėjamas Lee–Carter'io metodas mirtingumui prognozuoti. Analizuojamos modelio liekanos bei mirtingumo tendencijos. Pateikiamos ir palyginamos mirtingumo galios prognozės Prancūzijai, Baltarusijai ir Lietuvai. Siekiant gauti tiksliausią prognozę Lietuvos mirtingumo duomenims taikomos kelios metodo modifikacijos.

Reikšminiai žodžiai: mirtingumo prognozavimas, Lee–Carter'io metodas.