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BAYESIAN ESTIMATION OF THE PARAMETER OF THE *p*-DIMENSIONAL SIZE-BIASED RAYLEIGH DISTRIBUTION

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Abstract. In this paper, we have derived the probability density function of the size-biased *p*-dimensional Rayleigh distribution and studied its properties. Its suitability as a survival model has been discussed by obtaining its survival and hazard functions. We also discussed Bayesian estimation of the parameter of the size-biased *p*-dimensional Rayleigh distribution. Bayes estimators have been obtained by taking quasi-prior. The loss functions used are squared error and precautionary.

Key words: Size-biased distribution, Bayes theorem, squared error loss function, precautionary loss function, quasi-prior.

1. Introduction

The Rayleigh distribution is frequently employed by engineers, physicists and other scientists as a model for the analysis of data resulting from investigations involving wave propagation, radiation and related inquiries. It was first derived by Lord Rayleigh [3] in connection with a study of acoustic problems. The p.d.f. of the *p*-dimensional Rayleigh distribution is given as

$$h(x;\, p,\, \theta) = \frac{2^{-\frac{(p-2)}{2}}}{\theta\Gamma\left(\frac{p}{2}\right)}\left(\frac{x}{\theta}\right)^{p-1} exp^{\left\{-\frac{1}{2}\left(\frac{x}{\theta}\right)^2\right\}};\ x > 0,\ \theta > 0. \tag{1}$$

The rth moment about the origin is

$$\mu_r' = \frac{2^{\frac{r}{2}}\theta^r \Gamma\frac{r+p}{2}}{\Gamma\left(\frac{p}{2}\right)}\ .$$

Therefore

$$E(X) = \frac{\theta\sqrt{2}\Gamma\left(\frac{1+p}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \quad \text{and} \quad V(X) = 2\theta^2\left[\frac{\Gamma\left(\frac{2+p}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} - \left(\frac{\Gamma\left(\frac{1+p}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\right)^2\right].$$

Although the general form of the Rayleigh distribution with p >3 might have limited applications. The Rayleigh distribution with p.d.f. with p =1 is sometimes called the folded Gaussian, the folded normal, or the half-normal distribution.

In the Bayesian approach it is assumed that the parameter θ is itself a random variable. In this paper, we consider the Bayesian analysis (estimation) problem of the scale parameter of a p-dimensional size-biased Rayleigh distribution using the squared error and precautionary loss functions under quasi-prior.

When observation is selected with probability proportional to its size, the resulting distribution is called size-biased. Statistical analysis based on size-biased samples has been studies in detail since the early 70's. The concept of size-biased sampling was mainly developed by Rao and Zelen & Feinleib [6]. The size-biased distribution occurs naturally for some sampling plans in biometry, wildlife studies and survival analysis, among other.

Consider the *p*-dimensional Rayleigh distribution [1] whose p.d.f. is given by equation (1). Now using the relationship

$$f(x) = \frac{xh(x,\theta)}{E(X)},$$

we get the p.d.f. of size-biased *p*-dimensional Rayleigh distribution as

$$f(x;p,\theta) = \frac{2^{\frac{(1-p)}{2}}}{\theta\Gamma\left(\frac{1+p}{2}\right)}\left(\frac{x}{\theta}\right)^p exp^{\left\{-\frac{1}{2}\left(\frac{x}{\theta}\right)^2\right\}}, \qquad x > 0,\ \theta > 0. \tag{2}$$

Moments

$$E(X^r) = \int_0^\infty x^r f(x)dx\ , \qquad \mu_r' = \frac{2^{\frac{r}{2}}\,\theta^r\Gamma(\frac{p+r+1}{2})}{\Gamma(\frac{p+1}{2})}\ .$$

Now

$$\mu_1' = \frac{2^{\frac{1}{2}}\,\theta\Gamma(\frac{p+2}{2})}{\Gamma(\frac{p+1}{2})}, \qquad \mu_2' = \frac{2\,\theta^2\Gamma(\frac{p+3}{2})}{\Gamma(\frac{p+1}{2})}, \qquad \mu_3' = \frac{2^{\frac{3}{2}}\,\theta^3\Gamma(\frac{p+4}{2})}{\Gamma(\frac{p+1}{2})}, \qquad \mu_4' = \frac{4\,\theta^4\Gamma(\frac{p+5}{2})}{\Gamma(\frac{p+1}{2})}\ .$$

Therefore

$$E(X) = \frac{\theta\sqrt{2}\Gamma\left(\frac{p+2}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)}, \qquad V(X) = \mu_2' - {\mu_1'}^2\ ,$$

and

$$V(X) = 2\theta^2 \left[\frac{\Gamma\left(\frac{p+3}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)} - \left(\frac{\Gamma\left(\frac{p+2}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)}\right)^2\right].$$

Cumulative distribution function

The cumulative distribution function of a size-biased *p*-dimensional Rayleigh distribution is

$$F(x) = \int_0^x f(x)dx = \frac{2^{\frac{(1-p)}{2}}}{\theta\Gamma\left(\frac{1+p}{2}\right)}\int_0^x \left(\frac{x}{\theta}\right)^p exp^{\left\{-\frac{1}{2}\left(\frac{x}{\theta}\right)^2\right\}}\,dx\ .$$

Survival function

The survival function of a size-biased *p*-dimensional Rayleigh distribution is

$$S(x) = 1 - F(x) = 1 - \frac{2^{\frac{(1-p)}{2}}}{\theta\Gamma\left(\frac{1+p}{2}\right)}\int_0^x \left(\frac{x}{\theta}\right)^p exp^{\left\{-\frac{1}{2}\left(\frac{x}{\theta}\right)^2\right\}}\,dx\ .$$

Hazard function

The hazard function of a size-biased *p*-dimensional Rayleigh distribution is

$$H(x) = \frac{f(x)}{S(x)} = \frac{\frac{2^{\frac{(1-p)}{2}}}{\theta\Gamma\left(\frac{1+p}{2}\right)}\left(\frac{x}{\theta}\right)^p exp^{\left\{-\frac{1}{2}\left(\frac{x}{\theta}\right)^2\right\}}}{1 - \frac{2^{\frac{(1-p)}{2}}}{\theta\Gamma\left(\frac{1+p}{2}\right)}\int_0^x\left(\frac{x}{\theta}\right)^p exp^{\left\{-\frac{1}{2}\left(\frac{x}{\theta}\right)^2\right\}}dx}.$$

2. Estimation of the Parameter

(a) Maximum Likelihood Estimation (MLE)

The estimation of the parameter of the size-biased *p*-dimensional Rayleigh distribution is obtained by the method of MLE using equation (2). The likelihood function of the size-biased *p*-dimensional Rayleigh distribution is as follows:

$$L(\theta) = \prod_{i=1}^n f\,(x_i\ ;\ \theta) = \left(\frac{2^{\frac{(1-p)}{2}}}{\Gamma\left(\frac{1+p}{2}\right)}\right)^n \left(\prod_{i=1}^n x_i^p\right)\left(\frac{1}{\theta}\right)^{np+n}.e^{-\left(\sum_{i=1}^n x_i^2\right)\left(\frac{1}{2\theta^2}\right)}.$$

Now the log likelihood function is given by

$$\log L = \log L(\theta) = n\left[\log\left(\frac{2^{\frac{(1-p)}{2}}}{\Gamma\left(\frac{1+p}{2}\right)}\right)\right] + \log\left(\prod_{i=1}^n x_i^p\right) + (np+n)\left[\log\left(\frac{1}{\theta}\right)\right] - \left(\frac{1}{2\theta^2}\right)\left(\sum_{i=1}^n x_i^2\right)\ . \qquad (3)$$

Differentiating equation (3) w.r.t. θ and setting the results equal to zero, we have

$$\frac{d\log L}{d\theta} = \frac{-(np+n)}{\theta} + \left(\sum_{i=1}^n x_i^2\right)\frac{1}{\theta^3}\ .$$

Now, $\frac{d\log L}{d\theta} = 0$, leads to

$\hat{\theta} = \sqrt{\frac{(\sum_{i=1}^{n} x_i^2)}{(np+n)}}$ or $\sqrt{\frac{S}{(np+n)}}$.

(b) Bayesian Estimation

Under Bayesian analysis the fundamental problems are those of the choice of a prior distribution $g(\theta)$ and a loss function *l*. Let us suppose that very little information is available about the parameter (the suitable prior for this case is given in [4]. Assuming independence among the parameters, consider a quasi prior

$$g(\theta) = \frac{1}{\theta^d}; \quad \theta > 0,\ d > 0. \tag{4}$$

Loss function

Let θ be an unknown parameter of some distribution $f(x \mid \theta)$ and suppose we estimate θ by some statistic $\hat{\theta}$. Let $l(\hat{\theta}, \theta)$ represent the loss incurred when the true value of the parameter is θ and we are estimating θ by the statistic $\hat{\theta}$.

Squared error loss function (SELF)

Squared error loss function $l(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$. The Bayes estimator under the above loss function is the posterior mean, i.e.

$$\hat{\theta}_B = E_\pi(\theta), \tag{5}$$

where E_π stands for the posterior expectation.

Precautionary Loss Function

Norstrom [2] introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss functions with quadratic loss function as a special case [5]. A very useful and simple asymmetric precautionary loss function is given as

$$l(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\hat{\theta}}.$$

The Bayes estimator under this precautionary loss function is denoted by $\hat{\theta}_P$, and is obtained by solving the following equation:

$$\hat{\theta}_P = E_\pi(\theta)^2.$$

The joint density (i.e., likelihood) function of the size-biased *p*-dimensional Rayleigh distribution is given by

$$f(\underline{x}|\theta) = \left(\frac{2^{\frac{(1-p)}{2}}}{\Gamma\left(\frac{1+p}{2}\right)}\right)^n \left(\prod_{i=1}^{n} x_i^p\right) \left(\frac{1}{\theta}\right)^{np+n} . e^{-\left(\sum_{i=1}^{n} x_i^2\right)\left(\frac{1}{2\theta^2}\right)}. \tag{6}$$

Now using Bayes' theorem, the joint density function (6) along with the prior (4), we obtain the following joint posterior density function of the size-biased *p*-dimensional Rayleigh distribution

$$f(\theta \mid \underline{x}) = \frac{f(x|\theta) g(\theta)}{\int_0^\infty f(x|\theta) g(\theta) d\theta},$$

which on substituting the value of $g(\theta)$ and $f(\underline{x}|\theta)$ gives

$$f(\theta \mid \underline{x}) = \frac{\left(\frac{1}{\theta}\right)^{np+n+d} e^{\left(-\left(\frac{S}{2\theta^2}\right)\right)} S^{\frac{np+n+d-1}{2}}}{2^{\frac{np+n+d-3}{2}} \Gamma\left(\frac{np+n+d-1}{2}\right)}. \tag{7}$$

The Bayes estimator under squared error loss function is the posterior mean given by

$$\hat{\theta}_S = \int_0^\infty \theta\, f(\theta|\underline{x})\, d\theta. \tag{8}$$

Substituting the value of $f(\theta \mid \underline{x})$ from equation (7) in equation (8) and solving it, we get

$$\hat{\theta}_S = \int_0^\infty \theta\, \frac{\left(\frac{1}{\theta}\right)^{np+n+d} e^{\left(-\left(\frac{S}{2\theta^2}\right)\right)} S^{\frac{np+n+d-1}{2}}}{2^{\frac{np+n+d-3}{2}} \Gamma\left(\frac{np+n+d-1}{2}\right)} d\theta. \tag{9}$$

Solving equation (9), we obtain

$$\hat{\theta}_S = \frac{S^{\frac{1}{2}}\ \Gamma(\frac{np+n+d-2}{2})}{2^{\frac{1}{2}}\Gamma(\frac{np+n+d-1}{2})}\ . \tag{10}$$

Using (5), the Bayes estimator under the precautionary loss function comes out to be

$$\hat{\theta}_P = [E_\pi(\theta^2)]^{\frac{1}{2}} = \left[\int_0^\infty \theta^2\ f(\theta \mid \underline{x})\ \mathrm{d}\theta\right]^{\frac{1}{2}}$$

which on simplification leads to

$$\hat{\theta}_P = \left(\frac{S^{\frac{1}{2}}\left(\frac{np+n+d-1}{2}\right)}{2}\right)^{\frac{1}{2}}. \tag{11}$$

3. Conclusion

In this paper we have obtained the p.d.f. of the size-biased p-dimensional Rayleigh distribution and studied its properties. We also obtained its survival function and hazard function. The maximum likelihood estimator of the size-biased *p*-dimensional Rayleigh distribution is,

$$\hat{\theta} = \sqrt{\frac{(\sum_{i=1}^{n} x_i^2)}{(np+n)}}.$$

Also, we have obtained the Bayes estimator of the parameter θ under the squared error loss function and precautionary loss function given in equations (10) and (11).

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p-MAČIO RAYLEIGHO SKIRSTINIO PARAMETRO PROPORCINGAJAME DIDUMUI ĖMIME BAYESO ĮVERTINIMAS

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Santrauka. Šiame straipsnyje išvedama p-mačio Rayleigho skirstinio tankio proporcingajame didumui ėmime formulė ir tiriamos jo savybės. Jo pritaikomumą išgyvenamumo analizėje nusako išlikimo ir rizikos funkcijos. Taip pat aptariamas Rayleigho skirstinio parametro proporcingajame didumui ėmime Bayeso įvertinimas. Bayeso įvertiniai gaunami pasitelkiant kvaziapriorinį skirstinį. Naudojama kvadratinė ir įspėjamoji nuostolių funkcija.

Reikšminiai žodžiai: proporcingojo didumui ėmimo skirstinys, Bayeso teorema, kvadratinė nuostolių funkcija, įspėjamoji nuostolių funkcija, kvaziapriorinis skirstinys.