

## NELSON-SIEGEL MODEL APPROACH TO THE EURO AREA YIELD CURVES

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**Abstract.** In this paper Nelson-Siegel model has been examined. The main purpose of this paper is to fit the best Nelson-Siegel model to the Euro Area yield curves and to compare with Lithuanian yields in order to draw conclusions about readiness of Lithuania to join the Euro Area. In order to succeed in achieving the goal, Euro Area zero coupon bonds have been examined and various static Nelson-Siegel models were developed. Also, the mean of absolute error of the Lithuanian government treasury bills was calculated using the best-fitting Nelson-Siegel model of the Euro Area yields. The results have shown that if the static model is calculated for each year, the yield are described precisely in the model.

**Key words :** spot rate, forward rate, zero coupon yield curve, Euro Area, Nelson-Siegel model.

### 1. Introduction

Due to a very low risk of country bankruptcy, the investment in government bonds and treasury bills are one of the safest ways to get positive interests. One of the most profitable ways to invest in government bonds and treasury bills is investment in zero coupon bonds. These are long term bonds, in which interest is paid just one time during the bonds period.

In the present economy there is a large need to have a quick and simple way to be in charge of a large amount of information and forecast future investments, such as share and bonds ratio, interest rates, changes in unemployment and a lot of other processes. Yield curves, that graphically show us the relationship between yield and maturity, are one of the simplest ways to abstract the financial market and to analyse different risks. The innovators who fulfilled this expectation were Charles R. Nelson and Andrew F. Siegel. They have created a model<sup>1</sup>, that includes few variables, but has good prognoses and characteristics about yields having information on maturity.

For the large part of world economists the yield curves are widely researched. This fact encourages us to examine the Nelson-Siegel model fitted to the Euro Area more in depth. Lithuania became a member of the European Union on May 1, 2004. Since then it took more than 10 years to become a part of the united currency area. One of the main purposes of the current paper is to evaluate whether the Lithuanian yield curve shape has had an impact on joining the Euro Area.

Another purpose of this paper is to find the best Euro Area discount interest rate, corresponding to the fitted Nelson-Siegel model, and to evaluate Lithuanian interest rates, to compare them with the Euro Area, and determine the past preparation to enter the Euro Area in 2007 or 2015.

In the first section of the paper, a theoretical background of the topic concerned is presented. It was used as the basis on which a further research was performed. Available empirical data are presented in the second section. In its subsection, the total static and yearly static Nelson-Siegel models as well as their curves are presented, and the Lithuanian absolute mean error of the best fitting model is given. Finally, in the last part of the paper, conclusions are presented.

### 2. Nelson-Siegel model

#### 2.1. Static model

In 1977, Milton Friedman [2] acknowledged the usefulness of a simple model by modelling an interest rate curve. He has noticed that, using the statistical density function, the analysis process becomes

<sup>1</sup>Nelson-Siegel model was published in 1987 in "The Journal of business" Vol 60, Issue 4

much more effective when all the structural income conditions are described by a few simple parameters. M. Friedman has also observed that it is useful to analyse the density function using a long-term interest rate structure.

While modelling yield curves using yield/maturity data, a novel idea of David Durand [1] offered to approximate the present value function by a piecewise polynomial spline fitted to price data. Gary Shea [5] has noted that the resulting estimate of the function had sudden changes at the beginning and ending of the maturity range. Hence, this model is not valid when attempting to model curves outside the acquired sample data. In 1982, Vasicek and Fong [7] offered an alternative way of modelling interest rates: to use exponential curves instead of polynomial ones as the former need a lesser amount of parameters.

The future stock price can be expressed via current price and yields in the differential equation form. For example, if  $r(m)$  denotes stock selling price at the selling time  $m$  and is a solution of the second order differential equation with real and not equal roots, the result would be as follows:

$$r(m) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) + \beta_2 \exp\left(-\frac{m}{\tau_2}\right), \quad (1)$$

where  $\tau_1$  and  $\tau_2$  are the time constants that describe when the first and second hump appear, respectively, in the monotonic curve, whereas  $\beta_1$  and  $\beta_2$  are defined by primary conditions and have the asymptote  $\beta_0$ . For the last day of payment the yield curve is a selling price mean calculated as follows:

$$R(m) = \frac{1}{m} \int_0^m r(x) dx. \quad (2)$$

It keeps the same shape as it is seen in zero-coupon bond model (1).

While applying this model to treasury bill yields, it has been noticed that quite often, for different values of  $\tau_1$  and  $\tau_2$ , respective fitted values of  $\beta$ 's are approximately the same. So the model could be simplified using one range  $\tau$  instead of two.

While generating an equal-range curves, much simpler model is as follows:

$$r(m) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau}\right) + \beta_2 \frac{m}{\tau} \exp\left(-\frac{m}{\tau}\right). \quad (3)$$

This model can be treated as an approximation of the original model, where the differential order is increased even when the roots are not equal. The last model allows us to summarize the functions of a higher level. While dealing with a yield curve as a function of the selling day for the equal roots, we need to integrate equation (3) from 0 to  $m$  and divide by  $m$  (see (2)):

$$R(m) = \beta_0 + (\beta_1 + \beta_2) \left[1 - \exp\left(-\frac{m}{\tau}\right)\right] \frac{\tau}{m} - \beta_2 \exp\left(-\frac{m}{\tau}\right). \quad (4)$$

If  $m$  approaches the infinity, then the function becomes equal to  $\beta_0$ , however if  $m$  goes to 0, then the function becomes  $\beta_0 + \beta_1$ .

The curve can acquire many shapes: sinusoidal, cubic, piecewise and monotonic. The main hypothesis is that there is a relationship between maturity without any additional variables involved. Another way to make sure that curves are flexible in the second stage models is to divide the curve into three components: the short-term curve  $\psi_1(m) = e^{-m}$ , the middle-term curve  $\psi_2(m) = me^{-m}$ , and the long-term curve  $\psi_3(m) \equiv 1$  (see Fig. 1)<sup>2</sup>.

<sup>2</sup>The picture is taken from [4].

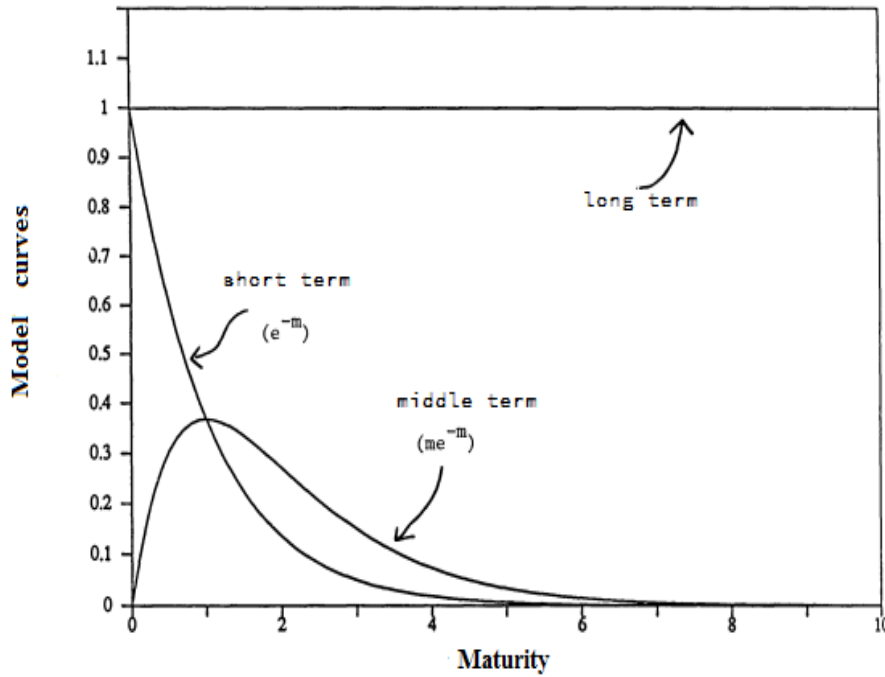


Figure 1: Components of the yield curve

The middle-term curve is the only one which starts at zero and approaches zero, while the short-term curve monotonically approaches zero.

## 2.2. Dynamic model

In 2006, Diebold and Li [6] proposed the dynamic version of the Nelson-Siegel model. First of all, they changed the interpretation of coefficients and introduced new indicators:  $l_t$  indicates a level,  $s_t$  is a slope, and  $c_t$  indicates curvature at the time period  $t$ . They have also noticed that since the model itself is dependent on time, the same factors can define the time. The model is as follows (cf. (4)):

$$R_t(m) = l_t + s_t \frac{1 - e^{-\lambda_t m}}{\lambda_t m} + c_t \left( \frac{1 - e^{-\lambda_t m}}{\lambda_t m} - e^{-\lambda_t m} \right). \quad (5)$$

The parameter  $\lambda = \lambda_t$  gives our model non-linearity and that is why Nelson and Siegel declined this parameter, so that they could evaluate the coefficients in a simpler way, using the ordinary least squares method. However, Diebold and Li [6] offered to evaluate these coefficients in two steps: 1) to calculate the value of parameter  $\lambda$ ; 2) to evaluate parameters  $l_t, s_t, c_t$  by vector autoregression (VAR) model or using another similar form.

## 2.3. Evaluation of the model

In order to evaluate the Nelson-Siegel model first of all prognostic properties of the model are taken into account. It is important for the model to be as specific as possible when describing and depicting the true yield, i.e. the errors of the model should be minimal. When a proper parameter  $\lambda$  is acquired, the model errors are minimal and  $R^2$  is quite high. Prognostic features of the model only slightly depend on correlations between regressors in (5).

## 2.4. Maastricht criteria and the Nelson-Siegel model

**Maastricht criteria** are criteria according to which it is determined when European Union (EU) nations can enter the third European Economic and money union (EMU) stage and change the local currency

to euro. Four main criteria are dictated by the EU contract 121 (1) article. The purpose of these criteria is to keep the prices in Euro zone stable when new members of EU enter this zone and adapt the euro currency.

**Maastricht criteria:**

- inflation cannot exceed more than 1,5 percentile of three EU nations whose inflation rate is the lowest one in relation to the inflation mean;
- the common country budget's deficit cannot exceed 3 % GDP;
- the country's debt cannot exceed 60 % GDP;
- the long-term interest rate cannot be more than that of 3 EU countries in which the inflation rate is lowest, and the last year's long-term interest rate mean more than 2 percentage points;
- the national currency of the country must be stable and at least for two years it cannot cross the required fluctuation boundaries (EU country has to participate in the Currency Exchange mechanism (CEM) II at least for two years, where the allowed currency rate fluctuation boundaries are  $\pm 15\%$  .

Lithuania joined ECM II in 2004 while committing itself to hold the fixed currency rate regimen as well as litas and the euro currency rate without any fluctuations.

Interest rate depicts all of the given criteria because using it investors can estimate the risk that depends on these criteria. Therefore, this paper will not analyse certain criteria such as inflation, county's debt, and budget deficit. As far as Lithuania's readiness to introduce the euro currency is concerned, the interest rate evaluated using the fitted Nelson-Siegel model will be taken as the criterion.

### 3. Euro Area yield curves

The Central bank of Europe is regularly collecting data about the Eurozone interest rate of obligation. Each day, at 12 am in the Central Europe time zone data are transmitted to the data base of the Central bank of Europe. We shall attempt to analyse the obligations, for which the interest is paid when redemption is due to maturity. The data set taken from Eurostat data base covers the period from September 9, 2004 to May 5, 2017. All obligations have maturities from 1 to 30 years.

#### 3.1. Static model

Under the assumption that the curve shape is unaffected by time, the model is static. It is defined by the following formula:

$$R(h) = \beta_0 + \beta_1 \frac{1 - e^{-\lambda h}}{\lambda h} + \beta_2 \left( \frac{1 - e^{-\lambda h}}{\lambda h} - e^{-\lambda h} \right). \quad (6)$$

In the data set taken from Eurostat, the maturity  $h$  and yield of obligation are presented, thus the only parameter  $\lambda$  is missing. Equation (6) is nonlinear in  $\lambda$ , therefore it cannot be evaluated using the ordinary least squares method.

In 1987, Nelson and Siegel [4] offered to fix  $\lambda$  since, in such a case, the equation acquires linear shape and allows the evaluation using a simple ordinary least squares method. In 2006, Diebold and Li [6] noticed that more accurate estimates could be obtained if the equation were analysed in a two steps. Firstly, the best value of  $\lambda$  should be determined and later the coefficients would be evaluated using VAR model or in a similar way.

In this paper, we use the method offered by Nelson and Siegel:

- Values of  $\lambda$  are randomly generated from the interval (0,1), 50 values in total.

- For the each value of  $\lambda$ , two auxiliary variables (functions of the maturity  $h$ ) are created (cf. (6)):

$$a = a_\lambda(h) = \frac{1 - e^{-\lambda h}}{\lambda h},$$

$$b = b_\lambda(h) = \frac{1 - e^{-\lambda h}}{\lambda h} - e^{-\lambda h}.$$

Then the optimal value of  $\lambda$  is chosen using the ordinary least squares method while searching for the least error approximation of the available data by a linear combination of  $a$  and  $b$ .

Thus, for the each of 50  $\lambda$  values, the model

$$R(h) = \beta_0 + \beta_1 a_\lambda(h) + \beta_2 b_\lambda(h) \quad (7)$$

is fitted to the Eurostat data for the redemption durations (maturities)  $h$  ranging from 1 to 30 years. The resulting mean absolute errors are depicted in Fig. 2.

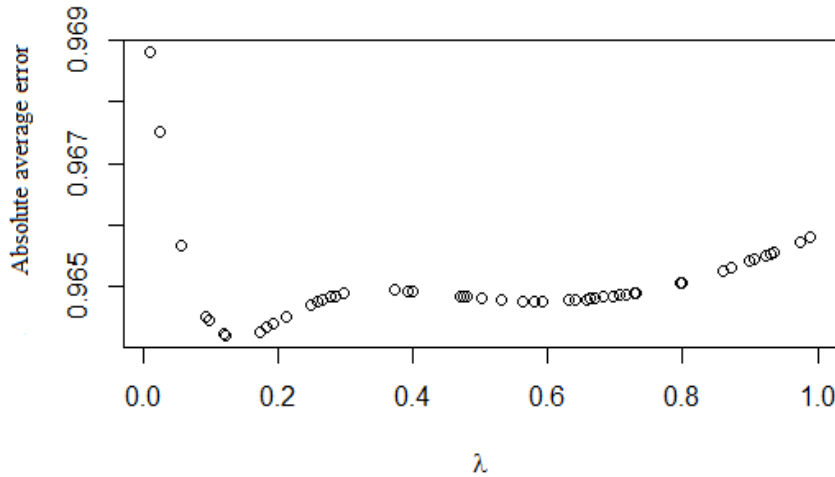


Figure 2: Mean absolute errors of fitted model (7) for different values of  $\lambda$

It is seen that the errors of the model are high reaching nearly up to 1. The least mean absolute error was obtained in the 50th model with  $\lambda = 0.121$ :

$$R(h) = \beta_0 + \beta_1 \frac{1 - e^{-0.121h}}{0.121h} + \beta_2 \left( \frac{1 - e^{-0.121h}}{0.121h} - e^{-0.121h} \right). \quad (8)$$

The least squares estimates of the parameters of the model are given in Table 1, its root-mean-square error  $RMSE = 1.189$ ,  $R^2 = 0.2565$ .

	$\beta_0$	$\beta_1$	$\beta_2$
Coefficient	3.901	2.811	3.236
Standard Error	0.036	0.029	0.096

Table 1: Regression coefficients of the static model

All the parameters are significant. According to  $R^2$ , the fitted model explains only 26% of the response variance. Therefore one can suspect that such a result is due to variability of the model parameters in time and thus the model is not applicable for a long period.

To check whether it is true, three models were created for the data of the September 9, 2004, December 22, 2010, and May 5, 2017, respectively. The parameter estimates and their standard errors of these models are shown in Table 2. The standard errors are given in parentheses.

Day	$\beta_0$	$\beta_1$	$\beta_2$	$R^2$
2004-09-06	5.27 (0.02)	-3.30 (0.04)	3.30 (0.04)	1.00
2010-12-22	5.60 (0.05)	-4.58 (0.09)	4.58 (0.09)	0.9969
2017-05-05	1.82 (0.07)	-2.83 (0.11)	2.83 (0.11)	1.00

Table 2: Parameter estimates and their standard errors (in parentheses) of model (8) for the daily data

A graphical illustration of the results is presented in Fig. 3–5.

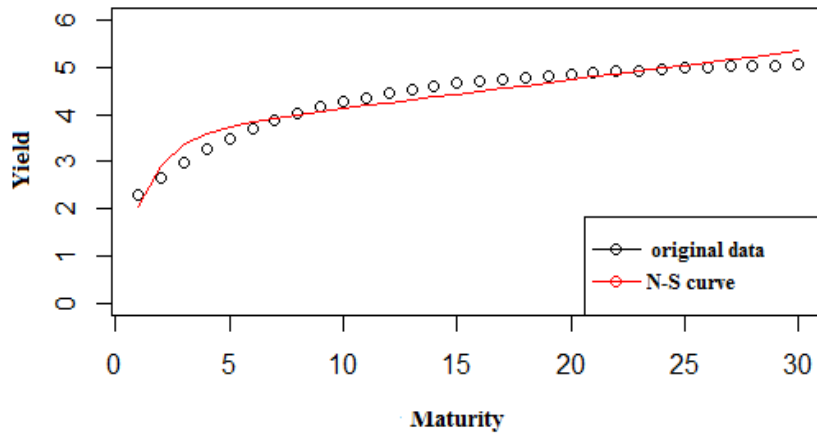


Figure 3: Daily static NS model (2004-09-06)

The results suggest that model (8) can describe the observed yields quite accurately. There are, however, great fluctuations in the model parameters between different periods. It confirms that the static model is not suitable when trying to forecast yield in the long term. Hence, the analysis of the every year static model will be looked further, where the optimal values of  $\lambda$  are searched for each year separately.

### 3.2. Yearly static models

The data covers 14 years. For each year, a separate model have been developed in the same way as for the whole period. The most relevant values of  $\lambda$  and the respective fitted model parameters are given in Table 3.

Notice that, for all fitted models,  $R^2$  is much higher than 50%. Consequently, yield curve predictions based on the yearly-fitted models are much better than yield curve predictions using the model fitted to the whole data. Also note that, for the years 2004, 2015 and 2016, the values of  $\lambda$  are close to zero. This suggests that some of the years should be split more densely.

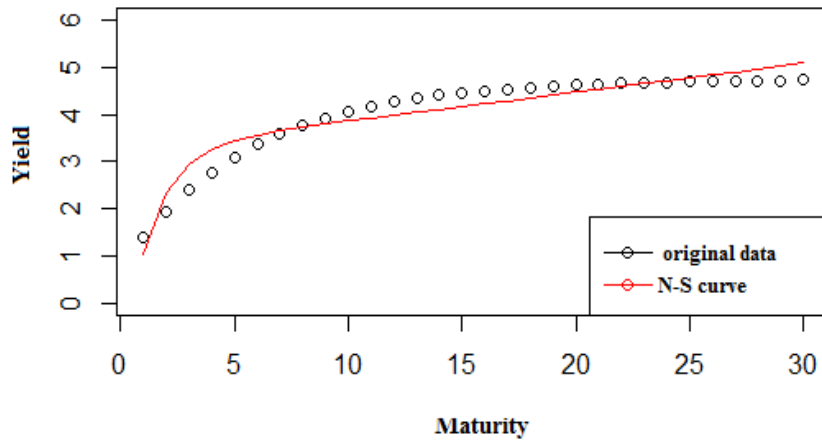


Figure 4: Daily static NS model (2010-12-22)

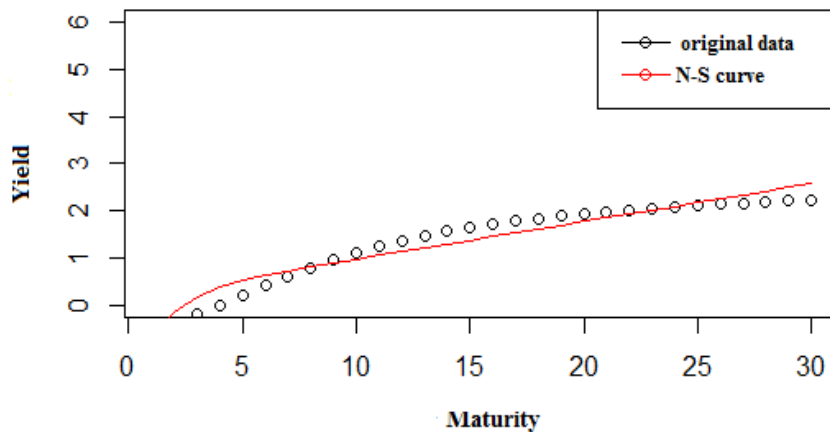


Figure 5: Daily static NS model (2017-05-05)

The smaller values of  $R^2$  in 2006 and 2007 are related to greater changes in the data. In 2006 the slope of the yield curve has changed. The slope is represented by the parameter  $\beta_1$ . In 2007 the curvature of the curve also changed. The curvature is under control of the parameter  $\lambda$ .

Therefore, if we wish to obtain more accurate results, the fluctuation in parameters  $\beta_0$ ,  $\beta_1$ ,  $\beta_2$  should be allowed. One of the ways to do so is to model them as an autoregressive process. However, using fixated parameters, it is possible to get rather good estimates for periods during which there are no significant changes in the curve shapes, e.g. in 2009. The fitted Nelson-Siegel curves of 2006, 2007 and 2009 years are presented in Fig. 6, 7 and 8, respectively.

### 3.3. Comparison with Lithuania's yield curves

After analysing the general Euro Area yield curves, there arose a need to compare how Lithuania's looks in the general Eurozone context. Lithuania has not entered the Eurozone directly after joining

Year	$\lambda$	$\beta_0$	$\beta_1$	$\beta_2$	$R^2$
2004	0.01	24.79 (0.19)	-22.63 (0.20)	22.63 (0.20)	0.93
2005	0.99	2.84 (0.01)	-2.15 (0.05)	2.19 (0.05)	0.81
2006	0.99	3.52( 0.01)	-0.91 (0.04)	0.93 (0.04)	0.53
2007	0.28	4.14 (0.01)	-0.23 (0.02 )	0.23 (0.02 )	0.44
2008	0.28	4.17 (0.01)	-0.87 (0.03)	0.88 (0.03)	0.60
2009	0.18	4.94 ( 0.01)	-4.75 ( 0.02)	4.75 ( 0.02)	0.97
2010	0.99	2.77 ( 0.01)	-6.04 (0.08)	6.11 (0.08)	0.78
2011	0.19	4.61 ( 0.01)	-3.26 (0.02)	3.26 (0.02)	0.92
2012	0.12	4.98 (0.02)	-4.24 (0.03)	4.24 (0.03)	0.89
2013	0.37	2.43 (0.01)	-3.24 (0.02)	3.26 (0.02)	0.85
2014	0.29	1.55 ( 0.02)	-2.32 (0.04)	2.34 (0.04)	0.81
2015	0.01	20.31 (0.17)	-20.74 (0.18)	20.74 (0.18)	0.86
2016	0.01	17.91 (0.15)	-18.61 (0.16)	18.60 (0.16)	0.88
2017	0.01	23.85 (0.11)	-24.65 (0.12 )	24.65 (0.12 )	0.98

Table 3: The parameter estimates of the fitted yearly models

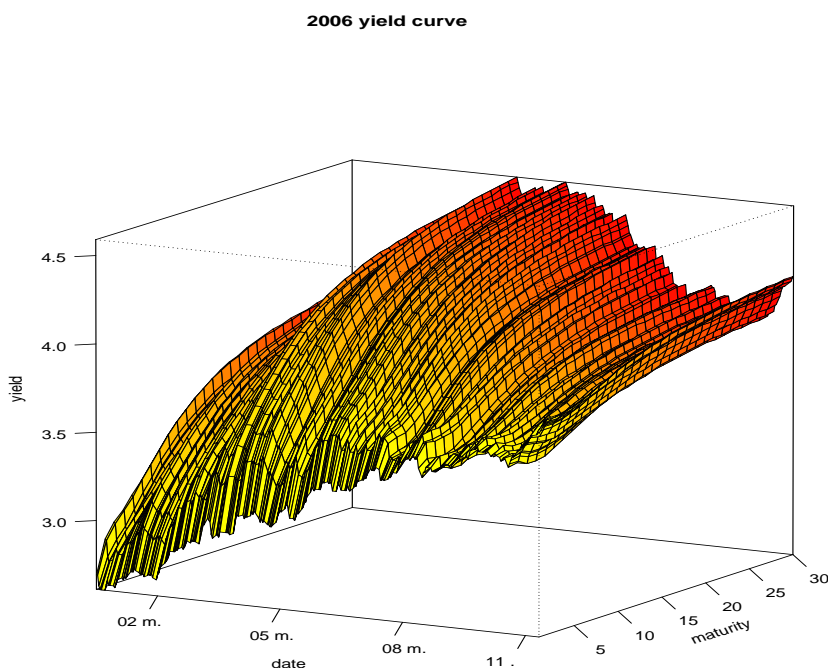


Figure 6: Nelson-Siegel model of 2006

the European Union, thus it is interesting to compare how much Lithuanian yield curve differs from the Euro Area yield curve during the whole period.

Since the duration of the Euro Area yield ranges from one to thirty years, while Lithuania's obligation duration varies a lot (from 10 months to 11 years), the possible yield of the Euro Area is calculated according to Lithuania's obligation durations. Later on, Lithuania's yield deviations are calculated when considering the Eurozone yield. The deviations from the curve are shown in Fig. 9.

The mean absolute error of Lithuania's yield with respect to the Eurozone curve reaches up to



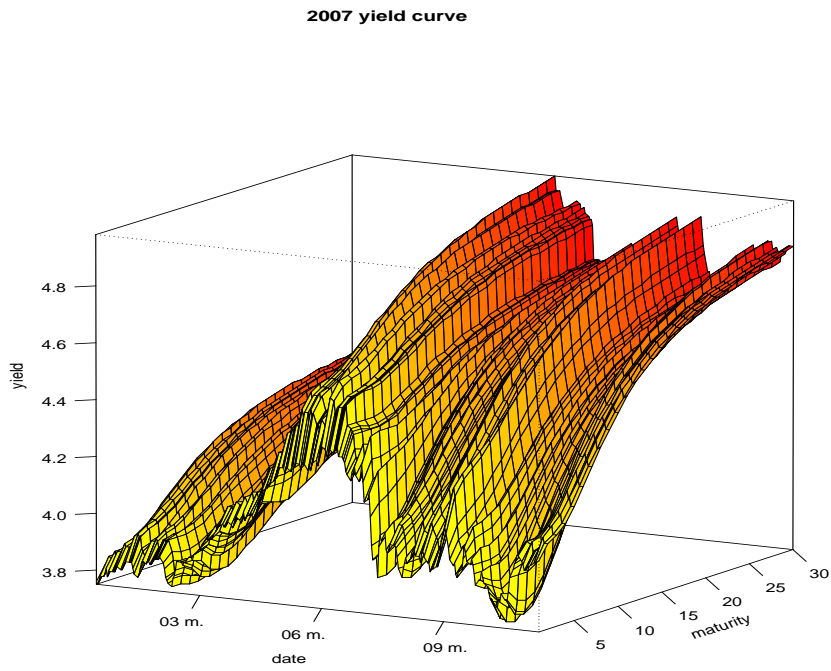


Figure 7: Nelson-Siegel model of 2007

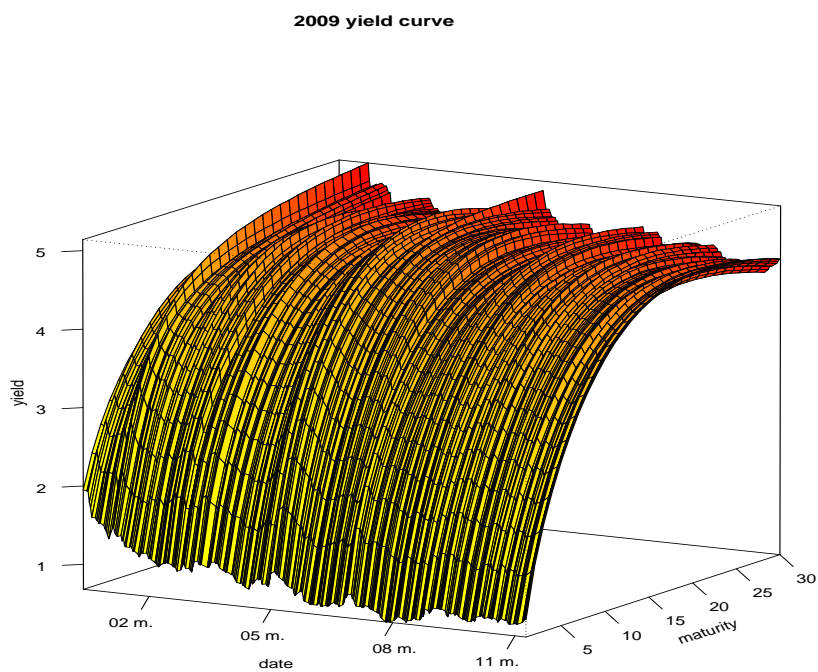


Figure 8: Nelson-Siegel model of 2009

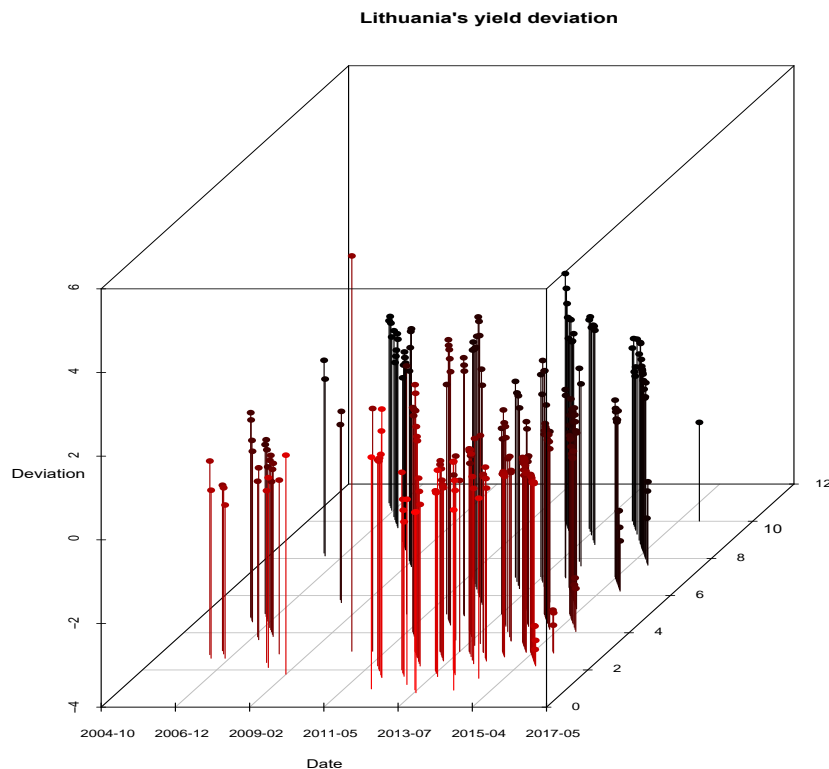


Figure 9: Lithuania's yield curve deviation from the Euro Area curve presented in 3D

0.78% during the whole period. Lithuania's yield was closest to that of the Euro Area around the year of 2006, however, it began to grow after half an year passed. Before 2013 Lithuania's yield went back to the total Eurozone yield curve and until this year it was relatively close.

From the graphical data view it cannot be expected, that in 2007 Lithuania was not ready to introduce the euro currency. However, in the years 2008 - 2013, one can notice the growth of yield above the average of the Eurozone. This result is the one that implies it was still too early for Lithuania to acquire euro in the year 2007. However, in the year 2015 when euro was finally introduced, Lithuania's yield dropped which identifies Lithuania as a reliable borrower for investors. Summing up everything, in the year 2015 Lithuania was ready to introduce the euro currency.

#### 4. Concluding remarks

In the present paper we attempt to analyse zero coupon bond yield curves of the Euro Area and try to identify which Nelson-Siegel model would be best to describe yield curves.

After fitting the Nelson-Siegel model to the Eurozone yield our main purpose was to evaluate Lithuania's yield curve differences from that of the Eurozone and to examine the readiness of Lithuania to introduce the euro in the years 2007 and 2015.

After making the analysis, it has been found that the static Nelson-Siegel model fits quite well in describing yield curves when the model parameters including  $\lambda$  are fitted for each year separately. The same  $\lambda$  could be used for several years in turn, yet sometimes it varies more frequently. It depends on how rapidly the curve curvature is changing.

Lithuania was not ready to introduce euro in 2007 because this year Lithuania's yield peaked rapidly. However, in 2015 Lithuania was already quite ready to enter the Eurozone. This is due to the stability of Lithuania's yield in the period from mid-year of 2014 to 2016. Lithuania's yield was held stable around the Eurozone average which identifies Lithuania as a reliable investment object for the investors.

## References

- [1] Durand, D. 1942: *Basic Yields of Corporate Bonds 1900-1942*, NBER. Retrieved 8 May 2017, from <http://nber.org/books/dura42-1>
- [2] Friedman, M. 1977: Time perspective in demand for money, unpublished paper.
- [3] McCulloch, J. H. 2017: *Measurements the term structure of interest rates*. Retrieved 8 May 2017, from <http://www.econ.ohio-state.edu/jhm/papers/MTTSOIR.pdf>
- [4] Nelson, Ch. R., Siegel, A. F. 1987: Parsimonious Modelling of Yield Curves. *Journal of business*, 60(4), p. 473-489.
- [5] Shea, G. S. 1985: Interest Rate Term Structure Estimation with Exponential Splines: A Note. *Journal of finance*, 40(1), p. 319-325.
- [6] Diebold, F.X., Li, C. 2006: Forecasting the term structure of government bond yields. *Journal of Econometrics*, 130, p. 337-364.
- [7] Vasicek, O. A., Fong, H. G. 1982: Term structure modeling using exponential splines. *Journal of finance*, 37(2), p. 339-348.
- [8] *Official site of Ministry of Finance of the Republic of Lithuania*. Retrieved 8 May 2017, from <http://www.euro.lt/lt/apie-eura/mastrichto-kriterijai>

## NELSON-SIEGEL MODELIO PRITAIKYMAS EURO ZONOS PALŪKANŲ KREIVĖMS

Akvilė Mazanauskaitė

**Santrauka.** Darbe nagrinėjamas Nelson-Siegel palūkanų kreivės modelis. Šio darbo tikslas – parinkti geriausią Nelson-Siegel modelį euro zonai, palyginti Lietuvos palūkanų normas su sumodeliuotomis euro zonos palūkanų normomis ir tuo vadovaujantis, įvertinti Lietuvos pasirengimą įstoti į euro zoną 2007 ir 2015 m. Remiantis euro zonos nulinės atkarpos obligacijų palūkanų normų 2004–2017 m. duomenimis buvo sudaryti statiniai modeliai: viso laikotarpio ir kiekvieniems metams atskirai. Naudojantis parinktu geriausiu euro zonos palūkanų normų modeliu buvo apskaičiuotos vidutinės absoliučios Lietuvos palūkanų normų paklaidos. Gauti rezultatai rodo, kad statinis Nelson-Siegel modelis gana tiksliai aprašo palūkanų normas, jeigu jis sudaromas kiekvieniems metams atskirai.

**Reikšminiai žodžiai:** trumpalaikė palūkanų norma, ilgalaikė palūkanų norma, nulinės atkarpos pajamingumo kreivė, euro zona, Nelson-Siegel modelis.