MATHEMATICAL TRUTH WITHOUT REFERENCE*

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Abstract. *According to a canonical argument for mathematical platonism, if we are to have a uniform semantics which covers both mathematical and non-mathematical language, then we must understand singular terms in mathematics as referring to objects and understand quantifiers as ranging over a domain of such objects, and so treating mathematics as literally true commits us to the existence of (mind-independent, abstract) mathematical objects. In this paper, I argue that insofar as we can provide a uniform semantics for the better part of ordinary, non-mathematical language, we can provide a uniform semantics covering both mathematical and non-mathematical language without thereby committing ourselves to the existence of mathematical objects.*

Keywords: *Mathematics, platonism, anti-realism, non-referential semantics, compositionality, ontological commitment*

Introduction

One canonical argument for mathematical platonism argues for the existence of (mind-independent, abstract) mathematical objects on semantic grounds. It is important for a number of reasons to be able to provide a uniform semantics which covers both mathematical and non-mathematical discourse. In ordinary, non-mathematical language our singular terms refer to objects, and when we quantify, our quantifiers range over such objects. If we want a uniform semantics, we must say that when mathematicians talk about numbers, sets, spaces and so on, they purport to refer to and describe a certain kind of object. Since these are not concrete, physical things, they must be abstract — i.e., non-spatiotemporal and (therefore) causally inert. The position that such objects exist (mind-independently) and are the subject matter of mathematics is platonism about mathematics.

But now we have a problem. If we think that abstract objects are the subject matter of mathematics, then we will face the access problem in various guises. Given that mathematical discourse is about describing causally inert, mind-independent abstract objects, we are at a loss to explain how it is possible for us to have mathematical

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knowledge, given that we have no causal access to such objects. This is just the epistemological horn of Benacerraf's dilemma.[1] But the point generalizes. The way one gains (non-testimonial) mathematical knowledge is by means of proofs, which can be explained without reference to any sort of mathematical object. But if mathematical objects have no clear role to play in the practice of working mathematicians, then this leaves us at a loss to see what motivates claims that such a special class of objects exists and how, in such a practice, we manage to refer to them.[2]

Given these difficulties, we might want to deny the existence of abstract mathematical objects. But if we continue to accept that mathematical singular terms *purport* to refer to such (non-existent) objects and that quantifiers in mathematical language *purport* to range over such objects, then we must either introduce considerable awkwardness into our semantics or revise mathematical language, neither of which is desirable.

In this paper, I will argue that insofar as we can provide a uniform semantics for even large expanses of non-mathematical language, a uniform semantics for mathematical and non-mathematical language does not force an ontology of abstract objects upon us. The gist of my argument is this: even if we ignore mathematical language, there is no univocal sense of "domain" and "object" which makes sense of singular terms and quantification in ordinary, non-mathematical language. If we want to give a plausible, uniform semantics for these parts of non-mathematical language, we should give up the objectual approach to singular terms and quantifiers—at least as a general approach. And if we do this, we can with little difficulty arrive at a semantics covering both mathematical and non-mathematical language that has the appropriate kind of uniformity.

The semantic argument for platonism

The guiding idea behind the semantic argument for platonism is that we should be able to integrate with relative ease the semantics of mathematical language and the semantics of the rest of our language. There is good reason to think that an important constraint on an acceptable position in the philosophy of mathematics is that it should explain how mathematical language fits in with the rest of our language. Or, at the very least, it should not preclude an adequate semantics which covers both mathematical and non-mathematical language.

Being able to apply the same semantic approach to both mathematical and non-mathematical language is a theoretical virtue in that (all things being roughly equal) it yields a simpler, more elegant theory, which uses fewer basic principles to explain a greater variety of cases. But, more importantly, a unified semantics for mathematical and non-mathematical language is essential

[1] See Benacerraf (1973). Benacerraf's dilemma is roughly this: if we have an account of mathematical truth which is at least parallel with the semantics of non-mathematical language, then this implies the existence of abstract objects. This is inconsistent, however, with an adequate epistemological account of mathematics (which Benacerraf thinks must be consistent with a causal theory of knowledge). Platonism achieves semantic adequacy at the expense of epistemic adequacy. If we reject platonism, we obtain epistemic adequacy at the expense of semantic adequacy. The aim of this paper is to contest the latter claim.

[2] See Hodes (1984) for a good exposition of the problem of reference.

if we are to understand the application of mathematics, central to which are "mixed sentences," sentences which contain both mathematical and non-mathematical vocabulary. These include (for example) mundane statements like "The number of apples on the table is odd" as well as sentences of physical theories couched in the language of mathematics. Without a unified semantics, we are at a loss to explain these sentences and their crucial role in the application of mathematics.

The platonist has a straightforward way to give a unified semantics for mathematical and non-mathematical language. The platonist holds that there are abstract, mind-independent mathematical objects, which pure mathematics describes. So the platonist can say that an atomic sentence of the form "*a* is *F*" in the mathematical case is true if and only if the object to which the singular term *a* refers has the property expressed by the predicate *F*.[3] And this is just what we would get from a standard semantic theory for non-mathematical language. Likewise, the platonist can say that, in both the mathematical case and the non-mathematical case, an existentially quantified sentence of the form "There is an *F*" is true if and only if some object in the domain over which the quantifier ranges has the property expressed by *F*. This and other similar accounts I will call referential accounts, as they cash out the truth conditions of such expressions in terms of objects to which singular terms refer.

Now, if we accept this semantic account in the mathematical case but reject platonism, we run into a problem. We must either revise the truth values we ascribe to mathematical propositions in problematic ways,[4] or not take mathematical language at face value.[5] But neither option is very satisfying. The former is almost untenable; if anything is true, "2 + 2 = 4" is. And the latter would at the very least introduce considerable awkwardness into our semantics. On the other hand, if we reject this semantic account in the mathematical case, then we seem to face difficulty in providing a unified semantics for mathematical and non-mathematical language. In the rest of this paper, I will argue that we should reject referential semantics as a blanket approach to the semantics of non-mathematical language. If this argument is successful, then the referential approach to mathematical language loses a great deal of its appeal. I will then argue that we can provide a unified semantics for mathematical and non-mathematical language that has the right kind of uniformity even if we reject the referential approach.

[3] In the rest of this paper, I will sometimes speak as if the only atomics were composed of a singular term and a one-place predicate, but this is just for convenience in exposition. When this happens, what I say will also apply (with standard modifications) to sentences built from an n-tuple of singular terms and an n-place predicate.

[4] For an example of this, see Field (1980). According to Field, assuming that mathematical objects do not exist, existential claims like "there are at least three prime numbers" are false and universal claims like "all numbers are greater than or equal to seventeen" are vacuously true, since there are no numbers. But to call this unintuitive would be an understatement. After all, 2 is prime, 3 is prime and 5 is prime, and sixteen is strictly less than seventeen.

[5] For instance, we may take the paraphrases proposed in Field (1989) as giving us the logical form of sentences of mathematics. "Two is prime" would then mean "According to arithmetic, two is prime." Or, following Hellman, we may provide modal paraphrases: "Two is prime" would then mean "Necessarily, that two is prime holds in any structure satisfying the Peano axioms, and it is possible for such a structure to exist." (Hellman 1996)

The referential approach to non-mathematical language

Essential to the semantic argument for platonism is the thesis that, in ordinary, non-mathematical language, singular terms always purport to refer to objects, and predicates purport to ascribe properties to such objects. An atomic sentence "*a* is *F*" is true if and only if the object to which *a* refers has the property expressed by *F*. Likewise with quantifiers outside of mathematics: "there is an *F*" is true if and only if there exists an object in the domain of quantification which has the property expressed by *F*. If we are to take mathematical sentences of the same form at face value, then they too involve reference to objects and their properties.

The question now is whether this is how singular terms and quantifiers in ordinary, non-mathematical language work. And the right answer seems to me to be a resounding *yes and no*. Certainly, we often use sentences of the form "*a* is *F*" to say of an object that it has a certain property. For instance, when I say, "Whiskers is black," I pick out an object (a certain cat named Whiskers) and ascribe to it the property of being black. Likewise with existentially quantified sentences. When I say, "There is a pen on the table," I mean in some sense that there exists an object which has the properties of being a pen and being on the table.

But this does not seem to generalize to all uses of such language. Here, in very broad strokes, is why. Say the relevant sense of object is the sense in which both Whiskers and a pen are objects. Then "Pride is a sin," which is also of the form "*a* is *F*," appears to resist treatment in terms of objects. Likewise with numerous other standard uses of singular terms and quantifiers. Consider, for example:

(1) The French Revolution preceded the dissolution of the Holy Roman Empire.
(2) 1963 was an eventful year.
(3) There is a solution to this problem.
(4) There are words in the English language that polite people do not use.

We can easily multiply examples of the same kind. So now the problem for the referential approach is this: can we have univocal senses of "object" and "domain" such that everything that is the semantic value of a singular term in cases like (1)-(4) is an object and every domain of quantification is a domain of objects? It looks like the only way to do this and cover all such cases is to identify objects with semantic values of singular terms.[6]

We can achieve a uniform semantics in this way, but only to the detriment of informativeness. For now if we say that all singular terms refer to objects, this just tells us that all singular terms have semantic values. And all that this means is that our semantic theory assigns them values as a part of a theoretical apparatus meant to account for the compositionality of our language — to explain the role of the semantic properties of singular terms in determining the semantic

[6] For a similar argument concerning the semantics of first-order logic, see Ben-Yami (unpublished manuscript). Ben-Yami uses similar considerations to argue against a model-theoretic and for a substitutional interpretation of quantifiers in first-order logic. The model-theoretic approach treats all atomics as if they were made true in the same way, but there is no univocal sense of its key terms (like object and domain) which supports this. An argument is logically valid on the basis of its form alone, and how atomics are made true is irrelevant to our logical understanding of quantification.

properties of complex expressions in which they occur.

Now the question is whether assigning semantic values to singular terms commits us to the existence of objects in the sense relevant to the semantic argument for platonism. As Rayo astutely observes, semantic values are just pieces of theoretical machinery, and philosophers typically do not take sentences containing predicates, logical connectives, and so on to engender a commitment to the existence of the semantic values of these sorts of expressions — e.g., extensions, functions, etc. The cases of singular terms and quantifiers are special in this respect. (Rayo 2009)

What explains the special status of singular terms and quantifiers? It would seem to be that the structure of an atomic sentence is supposed to correspond in a substantial way to the structure of the world. The world is divided up into objects with properties, and the structure of atomic sentences mirrors the relation between object and property.

But it looks much less plausible that the world is divided up into objects when we understand objects in the very broad sense required for a uniform referential semantics. And much of what we do with assertions is not about limning the structure of the world, so it is implausible that common linguistic structures should be meant to reflect a common structure of the world. In any case, if the supporter of the referential approach insists on this point, then she seems at the very least to be imposing a preconception of how language works on the opponent of the referential approach, one that the opponent of the referential approach need not accept.

And so the semantic argument for platonism loses much of its bite. An opponent of platonism can say that singular terms have semantic values and figure into a compositional semantics for mathematics in the same way as non-mathematical singular terms do. But the fact that such terms have semantic values which figure in the same way into our semantic theory as those of non-mathematical singular terms means only that the same compositional principles will apply to both — not that both commit us to the existence of certain objects in the world.

Toward a unified non-referential semantics

At this point, we seem to have two options if we want both to reject platonism and to provide a unified semantics for mathematical and non-mathematical language. It appears open to us to provide a standard referential semantics with the deflated sense of object that allows us to cover all non-mathematical cases. But we might also provide a semantics that allows atomics to be made true in a number of different ways (i.e. not just by an object's having a property) and thus provide a non-referential semantics.

The former option appears to be advocated by Rayo and perhaps also by Azzouni. Rayo argues that we can preserve a standard semantics for mathematical language but reject the claim that mathematical language commits us to the existence of abstract objects. We continue to ascribe semantic values to singular terms in mathematics, and, like ordinary predicates, we treat mathematical predicates as functions from the semantic values of singular terms to truth values. But we distinguish between the semantic values of singular terms and the objects that must exist for mathematical sentences to be true (Rayo 2009).

Similarly, Azzouni distinguishes between ontological commitment and quantifier commitment. A theory engenders a quantifier commitment if and only if it entails existentially quantified sentences, while it engenders an ontological commitment to a (kind of) object if and only if that (kind of) object must exist for the sentence to be true (Azzouni 2004). Mathematical objects are "ultrathin posits" in that sentences about them may be true though these objects do not exist (Azzouni 1994, 2004). On a natural reading of Azzouni, mathematical objects *qua* ultrathin posits serve as semantic values of mathematical singular terms but need not exist for the truth conditions of a mathematical sentence to be satisfied.

But here is the problem. Reference *per se* and the objects that serve as semantic values of singular terms play a role in securing compositionality, but do not figure into the truth conditions of mathematical sentences. But, we might think, for the objects to which the singular terms of a sentence putatively refer to drop out of the picture when it comes time to give the truth conditions of that sentence just is for the singular terms to be non-referential. And if this is the case, a referential semantics covering these terms will be both misleading and artificial. Of course, we can assign to mathematical singular terms whatever semantic values we like, and we can find functions to serve as the semantic values of mathematical predicates which allow us to assign the right truth values to mathematical atomic sentences. But this surely will not be particularly illuminating, and it will gloss over the real difference between how the truth values of such sentences are determined and how those of sentences like "Whiskers is black" are.

That reference ultimately drops out of the picture on such a referential approach would seem to favour the second, non-referential approach. Here is the core idea. We need uniformity in our semantics in order to be able to give a unified semantics for mathematical and non-mathematical language which allows us to make sense of mixed sentences. And for this, we do need uniformity, but not in how atomics are made true. We need uniformity in how the semantic properties of complex expressions depend on those of their parts in non-atomic cases.

Note that this does not mean that the composite structure of atomics is idle; it only means that we can get our semantics for mathematics to play nicely with the rest of our semantics without treating mathematical atomics and non-mathematical atomics in the same way. So long as we can provide a more or less systematic way to deal with atomics in terms of their parts in each case, compositionality remains secure.

We can clearly achieve this sort of uniformity in the case of logical connectives. In order to understand semantically the sentence "Three is odd and Barack Obama is the 44^{th} president of the United States," all we need is a systematic way to get from the semantic properties of each of the conjuncts to the semantic properties of the whole, and this will be just the same when one or both conjuncts is a mathematical sentence as when each conjunct is a non-mathematical sentence. The conjunction will be true if and only if each conjunct is true, regardless of whether they are true in virtue of their referring to objects with certain properties or their being true amounts to something else — say, their being provable in arithmetic.

We can likewise understand quantification in terms of atomics. While quantifiers are ordinarily understood along the lines of model theory — in terms of objects and domains — we may understand them substitutionally. “Some *F* is *G*” is true just in case, for some singular term *a*, “*a* is *F*” and “*a* is *G*” are both true. Likewise, “All *F*s are *G*s” is true just in case, for every singular term *a* such that “*a* is *F*” is true, so is “*a* is *G*.” Now, the substitutional interpretation is not uncontroversial, especially as an account of the *meaning* of the quantifiers.[7] But what is important is not whether substitutional quantification is the only game in town or whether it always provides the best understanding of the meaning of a given quantifier. Rather, it is important that it gives us a tool with which we can understand quantified sentences on the basis of their parts in both the mathematical and non-mathematical cases.

We must also deal with “logically atomic” mixed sentences like “The number of apples on the table is odd.” But this too can be accommodated, and most of the difficulty is just in giving the semantics of expressions like “the number of,” a difficulty that the platonist also faces. In both cases, however, all we need is a way to get from “the number of apples on the table” to a number term (or the semantic value of a number term), and then we can treat this sentence as we would any other mathematical atomic sentence. We could thus say, for example, that “The number of apples on the table is odd” is true if and only if “*n* is odd” is true, where *n* is the number of apples.

Of course, this is only a sketch, and a fuller account would have to be considerably more nuanced. But, in broad strokes, this is the shape that a unified, non-referential account might take. This strikes me as a very plausible way to understand how mathematical and non-mathematical language fit together, and in general I think that a non-referential semantics is best suited to capture the way mathematical language is used in both pure and applied mathematics. But whatever one thinks about these matters, if we *can* have such an account, then there is no *semantic* barrier to rejecting platonism. Our language does not force upon us an ontology of abstract objects.

7 See, for example, van Inwagen (1981).

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MATEMATINĖ TIESA BE REFERENCIJOS

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Santrauka. Pagal kanoninį argumentą, remiantį matematinį platonizmą, vieninga semantika, apimanti matematinę ir nematematinę kalbą, įmanoma tik jei matematikos singuliarinius terminus laikysime nurodančiais objektus, o kvantorius – apimančiais tokių objektų sritį, todėl jei matematikos teiginius laikome teisingais tiesiogine prasme, tai įpareigoja mus pripažinti (nuo mąstymo nepriklausomų, abstrakčių) matematinių objektų egzistavimą. Šiame straipsnyje siekiama įrodyti, kad jei mes galime sukurti vieningą semantiką reikšmingai daliai kasdienės nematematinės kalbos, tai galime sukurti vieningą semantiką apimančią matematinę ir nematematinę kalbą, neįsipareigodami matematinių objektų egzistavimui.

Pagrindiniai žodžiai: matematika, platonizmas, antirealizmas, nereferentinė semantika, kompozicionalumas, ontologinis įsipareigojimas.

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