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Fixed points for rational-type contractions in soft b-metric spaces with applications to integral and fractional problems

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Abstract. We study the topology and fixed point theory of soft b-metric spaces. In contrast to the soft metric case, we show by an explicit example that, when $s > 1$, a soft open ball need not be open in the induced topology τ_d, which indicates a genuine divergence between the ball structure and τ_d. We also discuss compactness-related consequences of this mismatch between ball covers and τ_d-open covers. On the fixed point side, we establish existence and uniqueness results for Reich–Ćirić-type contractions and for a rational-type contractive condition in complete soft b-metric spaces together with an α–ψ extension. As applications, we obtain solvability of a class of soft integral equations and of a Caputo fractional boundary value problem via an associated integral operator. Finally, we provide an explicit iterative scheme for rational contractions and a brief computational illustration in comparison with a standard linear contraction.

Keywords: soft b-metric space, soft topology, rational contraction, fixed point, computational illustration.

1 Introduction

Soft set theory, introduced by Molodtsov [17], offers a convenient language for treating uncertainties that are not easily handled by classical methods. A quantitative direction was initiated by Das and Samanta [10] through soft metric spaces. Further developments of soft set theory have shown that the framework is flexible enough to admit refined algebraic and operational structures; see, for example, the generalized relaxed operations studied in [5]. Relaxations of the triangle inequality appear earlier in the literature on quasi-metric spaces; see, for example, Vulpe, Ostrajkh, and Khojman [27].

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The notion of a b-metric space in the sense of Bakhtin [6] and Czerwik [8], where the triangle inequality is relaxed by a coefficient $s \geqslant 1$, provides a useful setting for fixed point methods. For a historical survey of early developments in b-metric fixed point theory and related aspects, we refer to Berinde and Păcurar [7]. Soft b-metric spaces arise naturally from combining these two lines of research. They have been studied in various forms in the recent literature; see, for instance, Wadkar et al. [28] and Abbas et al. [2]. On the fixed point theory side, important foundational results for soft metric spaces were obtained by Abbas, Murtaza, and Romaguera [1], which motivated further developments in generalized soft metric frameworks. For background on cone metric spaces and fixed point methods, we refer to Radenović [18] and to the survey by Janković, Kadelburg, and Radenović [14].

Fixed point theory has also been developed in several metric-type structures involving fuzziness, graph constraints, neutrosophic uncertainty, and proximity-type phenomena. For instance, graphical fuzzy metric spaces and their applications to fractional differential equations were investigated in [22], while multivalued neutrosophic fractals and Hutchinson–Barnsley-type operators were studied in [21]. Optimal fuzzy best proximity coincidence points in non-Archimedean fuzzy metric spaces were considered in [15], and interpolative Ćirić–Reich–Rus-type best proximity point results with applications were obtained in [23]. These works show that fixed point methods remain effective even when the underlying geometry is no longer governed by the classical metric axioms.

In spite of the rapid growth of fixed point results in soft b-metric spaces, a comparably explicit account of the induced topological structure is still relatively scarce. In many works, one proceeds as if soft open balls were automatically soft open sets, in the same way as in classical soft metric spaces. However, when $s > 1$, this intuition is not reliable since the same difficulty is already present in ordinary b-metric spaces: a ball defined by the distance does not have to be open in the topology naturally induced by that distance. This observation is not merely terminological; it affects arguments where ball covers are treated as open covers or where ball-based local reasoning is used without passing to the induced topology.

Related contractive conditions of rational-type have been investigated extensively in generalized metric frameworks; see, for instance, [3]. In this direction, (α, ψ)-rational-type contractions in Jleli–Samet generalized metric spaces and rational-type contractions in b-metric spaces have been studied in [11, 13, 25]. Additional remarks and refinements on rational contractions can be found in [12]. More recent applications of rational-type fixed point principles to models arising in economics and equilibrium problems were obtained in [4]. Integral and fractional equations provide another important motivation for such methods. Unique solvability of integral equations in generalized fuzzy metric-type settings was studied in [24], while fractional integral equations and fixed point approaches were considered in [29]. Geraghty-type fixed point equations with applications to integral equations were investigated in [26]. In addition, iterative and theoretical treatments of (k, z)-Riemann–Liouville fractional integral equations were developed in [9]. These contributions are close in spirit to the application-oriented part of the present paper, but our approach differs in that we combine rational contractive estimates with the topology of soft b-metric spaces and explicitly separate ball-based arguments from τ_d-topological arguments.

Our first objective is to clarify and formalize this topological distinction in the setting of soft b-metric spaces. We define the topology τ_d associated with a soft b-metric and present a concrete counterexample showing that a soft open ball may fail to be τ_d-open when $s > 1$. At the same time, the induced topology retains familiar separation and countability properties; in particular, it is Hausdorff and first countable, so sequential arguments formulated in τ_d behave in the usual manner. We also show that the discrepancy between ball-based structures and the induced topology has concrete consequences for compactness-type arguments. In particular, Section 5 demonstrates that ball-compactness and τ_d-compactness are no longer equivalent when $s > 1$, which prevents a direct transfer of compactness-based fixed point principles from classical metric and soft metric spaces to the soft b-metric setting.

The second aim of the paper is to develop fixed point principles under contractive hypotheses tailored to the soft b-metric setting. The systematic comparison of contractive conditions in metric spaces goes back to the work of Rhoades [19, 20], and it continues to motivate many generalizations in metric-type and b-metric settings. We prove an existence and uniqueness theorem for mappings satisfying a Reich–Ćirić-type inequality and then establish a rational-type contraction principle together with a natural extension involving α-admissibility and a comparison function. Finally, we apply the rational contraction theorem to derive existence and uniqueness results for a class of soft integral equations and for a fractional boundary value problem through an associated integral operator.

Computational aspects of fixed point results in soft settings have received comparatively less attention. For this reason, we include an explicit Picard-type iteration associated with the rational contractive condition and provide a computational illustration on a concrete example. We also compare the resulting iterates with those generated by a standard linear contraction.

The paper is organized as follows. Section 2 contains the basic preliminaries on soft sets and soft b-metric spaces. Section 3 studies the induced topology and clarifies the possible failure of ball openness when $s > 1$. Section 4 presents the fixed point theorems, including Reich–Ćirić-type and rational contractive conditions together with an α–ψ extension. Section 5 discusses compactness issues arising from the mismatch between ball covers and τ_d-open covers. Section 6 introduces the iterative algorithm and gives a computational comparison. Finally, Section 7 presents applications to integral and fractional differential equations.

2 Preliminaries

Throughout the paper, X denotes a nonempty set and S a nonempty set of parameters. We summarize below the notions from soft set theory and soft b-metric spaces that will be needed in what follows.

A soft set over X is a mapping $F : S \to P(X)$, where $P(X)$ denotes the power set of X. This concept was introduced by Molodtsov [17] and later developed in various directions. A soft point is a particular soft set x_e defined by $x_e(e) = \{x\}$ for some $x \in X$

and $e \in S$, while $x_e(\lambda) = \emptyset$ for all $\lambda \in S$ with $\lambda \neq e$. The family of all soft points on X will be denoted by $SP(X)$ (cf. [10, 28]).

Definition 1. Let $X \neq \emptyset$, let S be a nonempty parameter set, and fix $s \geqslant 1$. A function $d : SP(X) \times SP(X) \to [0, \infty)$ is said to be a *soft b-metric* provided that, for all $x_e, y_u, z_v \in SP(X)$,

(i) $d(x_e, y_u) = 0$ if and only if $x_e = y_u$;
(ii) $d(x_e, y_u) = d(y_u, x_e)$;
(iii) $d(x_e, z_v) \leqslant s(d(x_e, y_u) + d(y_u, z_v))$.

Then (X, d, S, s) is called a *soft b-metric space*.

Definition 1 extends the soft metric framework of Das and Samanta [10] by allowing a relaxed triangle inequality, namely the b-type inequality with coefficient $s \geqslant 1$ in the sense of Bakhtin [6] and Czerwik [8]. Variants of soft b-metric spaces and related fixed point results have appeared in the literature; see, e.g., [2, 28].

Let $\{x^{(n)}\} \subseteq SP(X)$. We say that $x^{(n)}$ *converges* to $x^* \in SP(X)$ if

$$d\big(x^{(n)}, x^*\big) \underset{n\to\infty}{\longrightarrow} 0.$$

The sequence is called *Cauchy* if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x^{(n)}, x^{(m)}) < \varepsilon$ whenever $m, n \geqslant N$; equivalently,

$$d\big(x^{(n)}, x^{(m)}\big) \underset{n,m\to\infty}{\longrightarrow} 0.$$

A soft b-metric space (X, d, S, s) is *complete* if each Cauchy sequence in $SP(X)$ converges (with respect to d) to a point of $SP(X)$.

Remark 1. Unlike the metric case, a b-metric is not required to be (jointly) continuous in its two arguments; see, for instance, [7, Ex. 3.1]. For this reason, in the proofs of the fixed point results, we do not justify limit transitions by continuity of d. Instead, we use only the b-triangle inequality combined with $\limsup$-type estimates.

3 Topology of soft *b*-metric spaces

We next analyze the topological framework arising from a soft b-metric. The point that requires particular care is that, when the coefficient s is greater than one, the distance balls do not necessarily behave as open sets in the topology determined by the corresponding ball-neighbourhood structure. This phenomenon is already familiar in classical b-metric spaces; see, for example, [7]. In the soft setting, however, it has additional importance because several arguments in the literature implicitly pass from ball-based statements to topological statements. Such a passage is harmless in soft metric spaces, but it is not automatic in soft b-metric spaces.

3.1 Failure of ball openness and interior points

We first fix the topology that will be used throughout the paper.

Definition 2. Let (X, d, S, s) be a soft b-metric space. We denote by τ_d the family of all subsets $U \subseteq SP(X)$ with the following property: for every $x_e \in U$, there exists $r > 0$ such that

$$B(x_e, r) := \{y_u \in SP(X)\colon\ d(x_e, y_u) < r\} \subseteq U.$$

Then τ_d is a topology on $SP(X)$, called the topology induced by the soft b-metric d.

The family in Definition 2 is closed under arbitrary unions and finite intersections, and hence it is indeed a topology. The wording of the definition is intentional. When $s = 1$, the balls form the usual base for the induced topology. When $s > 1$, however, a ball need not be a τ_d-open set. Thus τ_d is determined by the ball-neighbourhood condition, but the family of all balls should not be treated as a topological base without additional assumptions.

We now give a concrete example showing that the obstruction is real and not merely a matter of terminology.

Proposition 1. *There exists a soft b-metric space (X, d, S, s) with $s > 1$ such that a soft open ball is not open in the topology τ_d.*

Proof. Let

$$X = \{a, b\} \cup \{c_n,\ n \in \mathbb{N}\},$$

and let $S = \{e\}$. Define $d : SP(X) \times SP(X) \to [0, \infty)$ by

$$d(a_e, b_e) = 1, \qquad d\big(a_e, (c_n)_e\big) = 2, \qquad d\big(b_e, (c_n)_e\big) = \frac{1}{n},$$

and, for $n \neq m$,

$$d\big((c_n)_e, (c_m)_e\big) = \frac{1}{n} + \frac{1}{m}.$$

Also, put $d(p, p) = 0$ for every $p \in SP(X)$ and extend the definition by symmetry. The first two axioms of a soft b-metric are immediate. For the relaxed triangle inequality with coefficient 2, the only nontrivial cases occur, up to symmetry, when the left-hand side is one of the distances 1, 2, or $1/n + 1/m$. These cases are controlled by

$$2 \leqslant 2\left(1 + \frac{1}{n}\right), \qquad 1 \leqslant 2\left(2 + \frac{1}{n}\right),$$

and by the identity

$$d\big((c_n)_e, (c_m)_e\big) = d\big((c_n)_e, b_e\big) + d\big(b_e, (c_m)_e\big).$$

The remaining cases follow directly from symmetry and positivity of the distances. Thus $(X, d, S, 2)$ is a soft b-metric space.

Consider the ball $B(a_e, 3/2)$. Since $d(a_e, b_e) = 1 < 3/2$, we have $b_e \in B(a_e, 3/2)$. On the other hand, $d(a_e, (c_n)_e) = 2 > 3/2$ for every $n \in \mathbb{N}$, so none of the points $(c_n)_e$ belongs to this ball.

We claim that b_e is not a τ_d-interior point of $B(a_e, 3/2)$. Let $\delta > 0$ be arbitrary. Choose $n \in \mathbb{N}$ such that $1/n < \delta$. Then

$$d\big(b_e, (c_n)_e\big) = \frac{1}{n} < \delta,$$

and hence $(c_n)_e \in B(b_e, \delta)$. But $(c_n)_e \notin B(a_e, 3/2)$. Therefore no ball centred at b_e is contained in $B(a_e, 3/2)$. Consequently, $B(a_e, 3/2)$ is not τ_d-open. □

The preceding example also clarifies a subtle point concerning interior points of balls. The centre of a ball is always an interior point in the sense of the topology τ_d, but this need not be true for every point of the same ball.

Lemma 1. *Let (X, d, S, s) be a soft b-metric space, and let $A \subseteq SP(X)$. A point $p \in A$ belongs to $\operatorname{Int}_{\tau_d} A$ if and only if there exists $r > 0$ such that $B(p, r) \subseteq A$. Consequently, for every $x_e \in SP(X)$ and every $r > 0$, the centre x_e is a τ_d-interior point of the ball $B(x_e, r)$. However, when $s > 1$, an arbitrary point of $B(x_e, r)$ need not be a τ_d-interior point.*

Proof. If $p \in \operatorname{Int}_{\tau_d} A$, then there exists a τ_d-open set U such that $p \in U \subseteq A$. By Definition 2, there is $r > 0$ with $B(p, r) \subseteq U \subseteq A$.

Conversely, assume that $B(p, r) \subseteq A$ for some $r > 0$. Let

$$V = \big\{q \in SP(X)\colon\ B(q, \rho) \subseteq A \text{ for some } \rho > 0\big\}.$$

Then $p \in V$ and $V \subseteq A$. We show that V is τ_d-open. Take $q \in V$ and choose $\rho > 0$ such that $B(q, \rho) \subseteq A$. If $z \in B(q, \rho/(2s))$, then $\rho - sd(q, z) > 0$. For any $w \in SP(X)$ satisfying

$$d(z, w) < \frac{\rho - sd(q, z)}{s},$$

the soft b-triangle inequality gives

$$d(q, w) \leqslant s\big(d(q, z) + d(z, w)\big) < \rho.$$

Thus $B(z, (\rho - sd(q, z))/s) \subseteq A$, and hence $z \in V$. Therefore $B(q, \rho/(2s)) \subseteq V$, so V is τ_d-open. It follows that $p \in \operatorname{Int}_{\tau_d} A$.

Taking $A = B(x_e, r)$ gives the assertion about the centre since $B(x_e, r) \subseteq B(x_e, r)$. The final assertion follows from Proposition 1, where $b_e \in B(a_e, 3/2)$ but b_e is not a τ_d-interior point of that ball. □

Although individual balls may fail to be open, the induced topology still has the expected separation and countability properties.

Theorem 1. *Assume that (X, d, S, s) is a soft b-metric space. Then the topological space $(SP(X), \tau_d)$ is Hausdorff and first countable.*

Proof. Let $x_e, y_u \in SP(X)$ with $x_e \neq y_u$ and put $r = d(x_e, y_u)/(4s)$. By Lemma 1, x_e is a τ_d-interior point of $B(x_e, r)$ and y_u is a τ_d-interior point of $B(y_u, r)$. Therefore

$$B^\circ(x_e, r) := \mathrm{Int}_{\tau_d} B(x_e, r), \qquad B^\circ(y_u, r) := \mathrm{Int}_{\tau_d} B(y_u, r)$$

are τ_d-open neighbourhoods of x_e and y_u, respectively. If z_v belonged to their intersection, then $z_v \in B(x_e, r) \cap B(y_u, r)$, and so

$$d(x_e, y_u) \leqslant s\big(d(x_e, z_v) + d(z_v, y_u)\big) < 2sr = \frac{1}{2} d(x_e, y_u),$$

which is impossible. Hence the two neighbourhoods are disjoint, and $(SP(X), \tau_d)$ is Hausdorff.

To prove first countability, fix $x_e \in SP(X)$. We show that

$$\left\{ \mathrm{Int}_{\tau_d} B\left(x_e, \frac{1}{n}\right),\ n \in \mathbb{N} \right\}$$

is a countable local base at x_e. Let $U \in \tau_d$ be a neighbourhood of x_e. By Definition 2, there exists $r > 0$ such that $B(x_e, r) \subseteq U$. Choose $n \in \mathbb{N}$ with $1/n < r$. Then

$$x_e \in \mathrm{Int}_{\tau_d} B\left(x_e, \frac{1}{n}\right) \subseteq B\left(x_e, \frac{1}{n}\right) \subseteq B(x_e, r) \subseteq U.$$

Thus $(SP(X), \tau_d)$ is first countable. □

Remark 2. Theorem 1 shows that the induced topology is well behaved even though the balls themselves may fail to be open. The difficulty is therefore not a failure of the topology, but a failure of the unrestricted ball structure to coincide with the open-set structure. In particular, identifying ball covers with open covers, or assuming that balls form a base, is not justified in general when $s > 1$.

The next result records the metric case as a limiting situation in which the usual relation between balls and topology is recovered.

Proposition 2. *Let (X, d, S, s) be a soft b-metric space with $s = 1$. Then every soft open ball belongs to τ_d.*

Proof. Let $x_e \in SP(X)$, and let $r > 0$. Take $y_u \in B(x_e, r)$ and put $\varepsilon = r - d(x_e, y_u)$. If $d(y_u, z_v) < \varepsilon$, then the triangle inequality gives

$$d(x_e, z_v) \leqslant d(x_e, y_u) + d(y_u, z_v) < r.$$

Thus $B(y_u, \varepsilon) \subseteq B(x_e, r)$. Hence every point of $B(x_e, r)$ is a τ_d-interior point, and consequently, $B(x_e, r)$ is τ_d-open. □

Remark 3. The obstruction exhibited in Proposition 1 is caused by the coefficient $s > 1$. When $s = 1$, the soft b-metric reduces to a soft metric in the sense of [10], and the usual ball-topology correspondence is recovered.

3.2 Ball covers and compactness

Since soft open balls need not be open in τ_d, it is important to distinguish between τ_d-open covers and covers formed by soft open balls.

Definition 3. A set $K \subseteq SP(X)$ is called ball-compact if every cover of K by soft open balls admits a finite subcover.

This notion is useful for describing ball-based compactness arguments, but it should not be confused with compactness in the induced topology.

Remark 4. Ball-compactness is not a purely topological notion when $s > 1$ because a cover by balls need not be a τ_d-open cover. Consequently, compactness in $(SP(X), \tau_d)$ and compactness defined through ball covers should not be interchanged without additional hypotheses.

Proposition 3. *Let (X, d, S, s) be a soft b-metric space. If every soft open ball $B(x_e, r)$ is τ_d-open, then every τ_d-compact subset $K \subseteq SP(X)$ is ball-compact.*

Proof. Assume that all balls are τ_d-open, and let K be τ_d-compact. Let $\{B(x_i, r_i)\}_{i\in I}$ be a cover of K by soft open balls. By the assumption, each $B(x_i, r_i)$ is a τ_d-open set. Thus this family is a τ_d-open cover of K. The τ_d-compactness of K gives a finite subcover, and hence K is ball-compact. □

Remark 5. Proposition 3 explains why the implication "τ_d-compact $\Rightarrow$ ball-compact" is automatic in the metric case and in any soft b-metric space whose balls are τ_d-open. When $s > 1$, this mechanism may fail because ball covers are not necessarily open covers in τ_d. This is the source of the compactness mismatch discussed later in Section 5.

4 Fixed point results

In this section, we establish fixed point principles in complete soft b-metric spaces by imposing two different contractive frameworks. For each framework, we derive both existence and uniqueness of a soft fixed point via a Picard-type iterative construction. The proofs are based on Picard iteration and use the coefficient s only through the b-triangle inequality. This metric approach avoids the topological ambiguities discussed in Section 3 when $s > 1$.

4.1 Reich–Ćirić-type contractions

We begin the fixed point analysis with a Reich–Ćirić-type inequality, which incorporates both the interpoint distance and the iterative defects. This setting is convenient in soft b-metric spaces, where the relaxed triangle inequality is the only structural tool needed in the proofs.

Theorem 2. *Let (X, d, S, s) be a complete soft b-metric space with $s \geqslant 1$, and let $T : SP(X) \to SP(X)$ be a mapping such that*

$$d(Tx_e, Ty_u) \leqslant \alpha d(x_e, y_u) + \beta d(x_e, Tx_e) + \gamma d(y_u, Ty_u) \tag{1}$$

for all $x_e, y_u \in SP(X)$, where $\alpha, \beta, \gamma \geqslant 0$ satisfy $s(\alpha + \beta + \gamma) < 1$. Then T admits a unique soft fixed point.

Proof. Choose $x^{(0)} \in SP(X)$ and define $x^{(n+1)} = Tx^{(n)}$ for $n \in \mathbb{N}$.

Applying (1) with $x_e = x^{(n-1)}$ and $y_u = x^{(n)}$ yields

$$d\big(x^{(n)}, x^{(n+1)}\big) \leqslant (\alpha + \beta) d\big(x^{(n-1)}, x^{(n)}\big) + \gamma d\big(x^{(n)}, x^{(n+1)}\big).$$

Hence

$$(1 - \gamma) d\big(x^{(n)}, x^{(n+1)}\big) \leqslant (\alpha + \beta) d\big(x^{(n-1)}, x^{(n)}\big).$$

Set $k = (\alpha + \beta)/(1 - \gamma)$. The condition $s(\alpha + \beta + \gamma) < 1$ implies $sk < 1$, and therefore

$$d\big(x^{(n)}, x^{(n+1)}\big) \leqslant k^n d\big(x^{(0)}, x^{(1)}\big), \quad n \in \mathbb{N}.$$

Let $m > n$. Repeated use of the b-triangle inequality gives

$$\begin{aligned} d\big(x^{(n)}, x^{(m)}\big) \leqslant{}& s d\big(x^{(n)}, x^{(n+1)}\big) + s^2 d\big(x^{(n+1)}, x^{(n+2)}\big) \\ &+ \cdots + s^{m-n} d\big(x^{(m-1)}, x^{(m)}\big), \end{aligned}$$

and hence

$$d\big(x^{(n)}, x^{(m)}\big) \leqslant sk^n d\big(x^{(0)}, x^{(1)}\big) \sum_{i=0}^{m-n-1} (sk)^i.$$

Since $sk < 1$, it follows that $\{x^{(n)}\}$ is Cauchy. Completeness yields $x^* \in SP(X)$ with $x^{(n)} \to x^*$.

To show that x^* is a fixed point, apply (1) with $x_e = x^{(n)}$ and $y_u = x^*$:

$$d\big(x^{(n+1)}, Tx^*\big) \leqslant \alpha d\big(x^{(n)}, x^*\big) + \beta d\big(x^{(n)}, x^{(n+1)}\big) + \gamma d(x^*, Tx^*).$$

Taking $\limsup_{n\to\infty}$ in the above inequality yields

$$\limsup_{n\to\infty} d\big(x^{(n+1)}, Tx^*\big) \leqslant \gamma\, d(x^*, Tx^*).$$

On the other hand, the b-triangle inequality gives

$$d(x^*, Tx^*) \leqslant s\big(d\big(x^*, x^{(n+1)}\big) + d\big(x^{(n+1)}, Tx^*\big)\big).$$

Letting $n \to \infty$ and using $x^{(n)} \to x^*$, we obtain

$$d(x^*, Tx^*) \leqslant s \limsup_{n\to\infty} d\big(x^{(n+1)}, Tx^*\big) \leqslant s\gamma d(x^*, Tx^*).$$

Since $s\gamma < 1$, it follows that $d(x^*, Tx^*) = 0$, hence $Tx^* = x^*$.

For uniqueness, if y^* is another fixed point, then (1) gives

$$d(x^*, y^*) \leqslant \alpha d(x^*, y^*).$$

Since $\alpha \leqslant \alpha + \beta + \gamma < 1/s \leqslant 1$, we have $\alpha < 1$, hence $x^* = y^*$. □

4.2 Rational-type contractions

Rational-type contractive inequalities constitute a flexible alternative to linear contractions since they involve both the interpoint distance and the iteration defects $d(x_e, Tx_e)$ and $d(y_u, Ty_u)$. In the soft b-metric setting, such conditions allow one to obtain a Banach-type conclusion by purely metric arguments without any continuity requirement on d.

Theorem 3. *Let (X, d, S, s) be a complete soft b-metric space with $s \geqslant 1$. Let $T : SP(X) \to SP(X)$ satisfy*

$$d(Tx_e, Ty_u) \leqslant \lambda \frac{d(x_e, Tx_e)d(y_u, Ty_u)}{1 + d(x_e, y_u)} + \mu d(x_e, y_u) \tag{2}$$

for all $x_e, y_u \in SP(X)$, where $\lambda, \mu \geqslant 0$ and $\lambda + s\mu < 1$. Then T has a unique soft fixed point.

Proof. Choose $x^{(0)} \in SP(X)$ and define $x^{(n+1)} = Tx^{(n)}$ for $n \in \mathbb{N}$. If $x^{(n)} = x^{(n+1)}$ for some n, then $x^{(n)}$ is a fixed point.

Assume $x^{(n)} \neq x^{(n+1)}$ for all n and set

$$d_n = d\big(x^{(n)}, x^{(n+1)}\big), \quad n \in \mathbb{N}.$$

Applying (2) with $x_e = x^{(n-1)}$ and $y_u = x^{(n)}$ gives

$$\begin{aligned} d_n = d\big(Tx^{(n-1)}, Tx^{(n)}\big) &\leqslant \lambda \frac{d_{n-1}d_n}{1 + d\big(x^{(n-1)}, x^{(n)}\big)} + \mu d\big(x^{(n-1)}, x^{(n)}\big) \\ &= \lambda \frac{d_{n-1}d_n}{1 + d_{n-1}} + \mu d_{n-1}. \end{aligned}$$

Hence

$$(1 - \lambda)d_n \leqslant \mu d_{n-1}.$$

Put $k = \mu/(1 - \lambda)$. Since $\lambda + s\mu < 1$, we have $sk < 1$, and therefore

$$d_n \leqslant k^n d_0, \quad n \in \mathbb{N}.$$

Let $m > n$. Using the b-triangle inequality repeatedly,

$$d\big(x^{(n)}, x^{(m)}\big) \leqslant sd_n + s^2 d_{n+1} + \cdots + s^{m-n} d_{m-1} \leqslant sk^n d_0 \sum_{i=0}^{m-n-1} (sk)^i.$$

Since $sk < 1$, the above estimate shows that $\{x^{(n)}\}$ is a Cauchy sequence in $SP(X)$. By completeness of (X, d, S, s), there exists $x^* \in SP(X)$ such that $x^{(n)} \to x^*$.

Now apply (2) with $x_e = x^{(n)}$ and $y_u = x^*$ to obtain

$$\begin{aligned} d\big(x^{(n+1)}, Tx^*\big) &= d\big(Tx^{(n)}, Tx^*\big) \\ &\leqslant \lambda \frac{d(x^{(n)}, x^{(n+1)})d(x^*, Tx^*)}{1 + d(x^{(n)}, x^*)} + \mu d\big(x^{(n)}, x^*\big). \end{aligned}$$

Taking $\limsup_{n\to\infty}$ and using $d_n \to 0$ and $d(x^{(n)}, x^*) \to 0$, we get

$$\limsup_{n\to\infty} d\big(x^{(n+1)}, Tx^*\big) = 0. \tag{3}$$

On the other hand, by the b-triangle inequality, we have

$$d(x^*, Tx^*) \leqslant s\big(d\big(x^*, x^{(n+1)}\big) + d\big(x^{(n+1)}, Tx^*\big)\big).$$

Taking $\limsup_{n\to\infty}$ on both sides and using $d(x^*, x^{(n+1)}) \to 0$ together with (3), we obtain

$$d(x^*, Tx^*) \leqslant s \cdot 0 = 0.$$

Hence $d(x^*, Tx^*) = 0$, and therefore $Tx^* = x^*$.

For uniqueness, let y^* be another fixed point of T. Then, by (2),

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leqslant \lambda \frac{d(x^*, Tx^*)d(y^*, Ty^*)}{1 + d(x^*, y^*)} + \mu d(x^*, y^*).$$

Since $Tx^* = x^*$ and $Ty^* = y^*$, the first term vanishes, and we get

$$d(x^*, y^*) \leqslant \mu d(x^*, y^*).$$

Moreover, $\lambda + s\mu < 1$ implies $\mu < 1$. Consequently, $(1-\mu)d(x^*, y^*) \leqslant 0$, so $d(x^*, y^*) = 0$, which yields $x^* = y^*$. □

Corollary 1. *If $\lambda = 0$, then* (2) *reduces to a Banach-type contraction with constant μ in the soft b-metric setting.*

Corollary 2. *If $s = 1$, then Theorem* 3 *reduces to the corresponding fixed point result in soft metric spaces.*

Example 1. Let $X = \mathbb{R}$ and let $S = \{e\}$. Define $d(x_e, y_e) = |x - y|^2$, which is a soft b-metric with coefficient $s = 2$. Let $T(x_e) = (\alpha x)_e$ with $|\alpha| < 1$. Then $d(Tx_e, Ty_e) = |\alpha|^2 d(x_e, y_e)$, so (2) holds with $\lambda = 0$ and $\mu = |\alpha|^2$. If $|\alpha| < 1/\sqrt{2}$, then $\lambda + s\mu = 2|\alpha|^2 < 1$, and T has a unique soft fixed point.

4.3 An $\alpha-\psi$ extension

In many applications the contractive condition is required only along admissible pairs or along the Picard orbit. To cover such situations, we extend Theorem 3 by combining α-admissibility with a comparison function ψ.

Definition 4. A function $\psi : [0, \infty) \to [0, \infty)$ is called a comparison function if $\psi(0) = 0$ and $\psi(t) < t$ for all $t > 0$.

Definition 5. Let $\alpha : SP(X) \times SP(X) \to [0, \infty)$. A mapping $T : SP(X) \to SP(X)$ is called α-admissible if, for all $x_e, y_u \in SP(X)$, $\alpha(x_e, y_u) \geqslant 1$ implies $\alpha(Tx_e, Ty_u) \geqslant 1$.

Definition 6. A complete soft b-metric space (X, d, S, s) is said to be α-*regular* if the following condition holds: whenever a sequence $\{x^{(n)}\} \subseteq SP(X)$ converges to some $x^* \in SP(X)$ and

$$\alpha\big(x^{(n)}, x^{(n+1)}\big) \geqslant 1 \quad \text{for all } n \in \mathbb{N},$$

then

$$\alpha\big(x^{(n)}, x^*\big) \geqslant 1 \quad \text{for all } n \in \mathbb{N}.$$

Theorem 4. *Let (X, d, S, s) be a complete soft b-metric space. Assume that $T : SP(X) \to SP(X)$ is α-admissible and that (X, d, S, s) is α-regular. Assume that there exist a comparison function ψ satisfying $\psi(t) \leqslant t$ for all $t \geqslant 0$ and constants $\lambda, \mu \geqslant 0$ with $\lambda + s\mu < 1$ such that, for all $x_e, y_u \in SP(X)$,*

$$\alpha(x_e, y_u)\, d(Tx_e, Ty_u) \leqslant \psi\left(\lambda \frac{d(x_e, Tx_e)d(y_u, Ty_u)}{1 + d(x_e, y_u)} + \mu d(x_e, y_u)\right). \tag{4}$$

Assume also that there exists $x^{(0)} \in SP(X)$ such that $\alpha(x^{(0)}, Tx^{(0)}) \geqslant 1$. Then T has a fixed point. If, in addition, $\alpha(x^*, y^*) \geqslant 1$ holds for any two fixed points x^*, y^* of T, then the fixed point is unique.

Proof. Choose $x^{(0)} \in SP(X)$ with $\alpha(x^{(0)}, Tx^{(0)}) \geqslant 1$ and define $x^{(n+1)} = Tx^{(n)}$. Since T is α-admissible, it follows that $\alpha(x^{(n)}, x^{(n+1)}) \geqslant 1$ for all n.

Apply (4) with $x_e = x^{(n-1)}$ and $y_u = x^{(n)}$. Using $\psi(t) \leqslant t$ and $\alpha(x^{(n-1)}, x^{(n)}) \geqslant 1$, we obtain

$$d\big(x^{(n)}, x^{(n+1)}\big) \leqslant \lambda \frac{d(x^{(n-1)}, x^{(n)})d(x^{(n)}, x^{(n+1)})}{1 + d(x^{(n-1)}, x^{(n)})} + \mu d(x^{(n-1)}, x^{(n)}).$$

Let $d_n = d(x^{(n)}, x^{(n+1)})$. Then $(1 - \lambda)d_n \leqslant \mu d_{n-1}$. Put $k = \mu/(1 - \lambda)$. The condition $\lambda + s\mu < 1$ implies $sk < 1$. Hence $d_n \leqslant k^n d_0$ for all n.

Using the b-triangle inequality as in Theorem 3, one obtains

$$d\big(x^{(n)}, x^{(m)}\big) \leqslant sk^n d_0 \sum_{i=0}^{m-n-1} (sk)^i, \quad m > n,$$

so $\{x^{(n)}\}$ is Cauchy. Completeness yields $x^{(n)} \to x^*$ for some $x^* \in SP(X)$.

By α-regularity, $\alpha(x^{(n)}, x^*) \geqslant 1$ for all n. Apply (4) with $x_e = x^{(n)}$ and $y_u = x^*$. Using $\alpha(x^{(n)}, x^*) \geqslant 1$ and $\psi(t) \leqslant t$, we get

$$d\big(x^{(n+1)}, Tx^*\big) \leqslant \lambda \frac{d(x^{(n)}, x^{(n+1)})d(x^*, Tx^*)}{1 + d(x^{(n)}, x^*)} + \mu d(x^{(n)}, x^*).$$

Taking $\limsup_{n\to\infty}$ in the last inequality and using

$$d\big(x^{(n)}, x^{(n+1)}\big) \to 0 \quad \text{and} \quad d\big(x^{(n)}, x^*\big) \to 0,$$

we obtain

$$\limsup_{n\to\infty} d\big(x^{(n+1)}, Tx^*\big) \leqslant 0,$$

hence $\lim_{n\to\infty} d(x^{(n+1)}, Tx^*) = 0$. Then the b-triangle inequality implies

$$d(x^*, Tx^*) \leqslant s\big(d\big(x^*, x^{(n+1)}\big) + d\big(x^{(n+1)}, Tx^*\big)\big) \underset{n\to\infty}{\longrightarrow} 0,$$

so $Tx^* = x^*$.

For uniqueness, let y^* be another fixed point and assume $\alpha(x^*, y^*) \geqslant 1$. Then (4) yields

$$d(x^*, y^*) \leqslant \psi\big(\mu d(x^*, y^*)\big) \leqslant \mu d(x^*, y^*),$$

and since $\mu < 1$, we obtain $d(x^*, y^*) = 0$, hence $x^* = y^*$. □

Remark 6. If $\alpha(x_e, y_u) \equiv 1$ for all $x_e, y_u \in SP(X)$ and $\psi(t) \leqslant t$ for all $t \geqslant 0$, then (4) reduces to the rational contractive condition (2). Consequently, Theorem 4 extends Theorem 3.

Remark 7. It is worth noting that many fixed point results in b-metric spaces, including Nadler-type theorems for multivalued mappings (see, for example, Mitrović et al. [16]), are established purely by metric arguments based on completeness and contractive inequalities. No compactness or ball-generated topological assumptions are required. This observation is consistent with the results obtained in this section: although the induced topology of a soft b-metric space may exhibit nonclassical features, contraction-based fixed point principles remain valid as long as completeness is preserved.

5 Compactness and structural divergence

In classical metric spaces, compactness admits several equivalent characterizations, including compactness by open covers, sequential compactness, and total boundedness together with completeness. Moreover, compact subsets of Hausdorff spaces are closed. In soft b-metric spaces, part of this picture remains valid once one works with the induced topology τ_d. By Theorem 1, the space $(SP(X), \tau_d)$ is Hausdorff and first countable, and hence compact sets are τ_d-closed and sequential arguments can be formulated within the induced topology.

The main divergence appears when τ_d-open covers are replaced by covers consisting only of soft open balls. Since soft open balls need not be τ_d-open when $s > 1$, ball-based compactness and topological compactness are no longer interchangeable. This distinction is important for fixed point arguments in which ball covers are implicitly treated as open covers.

5.1 Ball-compactness and its limitations

We recall Definition 3. The purpose of this subsection is to clarify the relationship between ball-compactness and compactness in the induced topology τ_d.

Proposition 4. *Every ball-compact subset $K \subseteq SP(X)$ is τ_d-compact.*

Proof. Let $\{U_i\}_{i\in I}$ be a τ_d-open cover of K. For each $x_e \in K$, choose $i(x_e) \in I$ such that $x_e \in U_{i(x_e)}$. Since $U_{i(x_e)}$ is τ_d-open, there exists $r(x_e) > 0$ such that $B(x_e, r(x_e)) \subseteq U_{i(x_e)}$. Thus

$$\big\{B\big(x_e, r(x_e)\big),\ x_e \in K\big\}$$

is a cover of K by soft open balls. By ball-compactness, there are $x_{e_1}, \dots, x_{e_m} \in K$ such that

$$K \subseteq \bigcup_{j=1}^{m} B\big(x_{e_j}, r(x_{e_j})\big).$$

Consequently,

$$K \subseteq \bigcup_{j=1}^{m} U_{i(x_{e_j})},$$

and therefore the original τ_d-open cover admits a finite subcover. □

The converse need not hold when $s > 1$. The following example shows that the failure is not only formal; it is a genuine compactness gap between the ball structure and the induced topology.

Proposition 5. *In general, τ_d-compactness does not imply ball-compactness in soft b-metric spaces with $s > 1$.*

Proof. Consider the soft b-metric space constructed in Proposition 1. Thus $X = \{a, b\} \cup \{c_n,\ n \in \mathbb{N}\}$, $S = \{e\}$, and the distance satisfies

$$d(a_e, b_e) = 1, \qquad d\big(a_e, (c_n)_e\big) = 2, \qquad d\big(b_e, (c_n)_e\big) = \frac{1}{n}.$$

Let

$$K = \{b_e\} \cup \big\{(c_n)_e,\ n \in \mathbb{N}\big\}.$$

We first show that K is τ_d-compact. Let $\{U_i\}_{i\in I}$ be a τ_d-open cover of K, and choose U_{i_0} with $b_e \in U_{i_0}$. Since U_{i_0} is τ_d-open, there exists $\delta > 0$ such that $B(b_e, \delta) \subseteq U_{i_0}$. Choose $N \in \mathbb{N}$ such that $1/n < \delta$ for all $n \geqslant N$. Then $(c_n)_e \in B(b_e, \delta) \subseteq U_{i_0}$ for all $n \geqslant N$. The finitely many remaining points $(c_1)_e, \dots, (c_{N-1})_e$ are covered by finitely many members of the cover. Hence K is τ_d-compact.

We now show that K is not ball-compact. Consider the family

$$\mathcal{C} = \left\{B\left(a_e, \frac{3}{2}\right)\right\} \cup \left\{B\left((c_n)_e, \frac{1}{2n}\right),\ n \in \mathbb{N}\right\}.$$

Since $b_e \in B(a_e, 3/2)$ and each $(c_n)_e$ belongs to $B((c_n)_e, 1/(2n))$, the family $\mathcal{C}$ is a cover of K by soft open balls. However, $B(a_e, 3/2)$ contains b_e but none of the points $(c_n)_e$. Moreover,

$$B\left((c_n)_e, \frac{1}{2n}\right) = \big\{(c_n)_e\big\}$$

for every $n \in \mathbb{N}$. Thus any finite subfamily of $\mathcal{C}$ covers only finitely many of the points $(c_n)_e$. Therefore $\mathcal{C}$ has no finite subcover, and K is not ball-compact. □

Remark 8. Propositions 4 and 5 show that ball-compactness is strictly stronger than τ_d-compactness in general. The reason is that a cover by soft open balls need not be a τ_d-open cover when $s > 1$. Consequently, ball-compactness is not a purely topological notion in $(SP(X), \tau_d)$.

Corollary 3. *If every soft open ball is τ_d-open, then for every $K \subseteq SP(X)$, ball-compactness is equivalent to τ_d-compactness. In particular, the equivalence holds when $s = 1$.*

Proof. The implication from ball-compactness to τ_d-compactness follows from Proposition 4. Conversely, if every soft open ball is τ_d-open and K is τ_d-compact, then every cover of K by soft open balls is a τ_d-open cover. Hence it has a finite subcover. □

5.2 Strong balls and τ_d-compactness

The obstruction described above comes from the fact that a soft open ball need not be τ_d-open. However, its τ_d-interior is always open and provides a canonical topological substitute for ball-based arguments.

Definition 7. Let (X, d, S, s) be a soft b-metric space. For $x_e \in SP(X)$ and $r > 0$, the *strong ball* centred at x_e with radius r is defined by

$$B^{\circ}(x_e, r) := \operatorname{Int}_{\tau_d}\big(B(x_e, r)\big).$$

Proposition 6. *The family*

$$\mathcal{B}^{\circ} = \big\{B^{\circ}(x_e, r),\ x_e \in SP(X),\ r > 0\big\}$$

forms a base for the topology τ_d.

Proof. Let $U \in \tau_d$ and let $x_e \in U$. By Definition 2, there exists $r > 0$ such that $B(x_e, r) \subseteq U$. By Lemma 1, the centre x_e belongs to $\operatorname{Int}_{\tau_d} B(x_e, r)$. Therefore

$$x_e \in B^{\circ}(x_e, r) \subseteq B(x_e, r) \subseteq U.$$

Thus every τ_d-open set is a union of strong balls, and $\mathcal{B}^{\circ}$ is a base for τ_d. □

Theorem 5. *A subset $K \subseteq SP(X)$ is τ_d-compact if and only if every cover of K by strong balls admits a finite subcover.*

Proof. If K is τ_d-compact, then every strong-ball cover of K is a τ_d-open cover. Hence it admits a finite subcover.

Conversely, assume that every strong-ball cover of K has a finite subcover. Let $\{U_i\}_{i \in I}$ be a τ_d-open cover of K. For each $x_e \in K$, choose $i(x_e) \in I$ such that $x_e \in U_{i(x_e)}$. By Proposition 6, there exists $r(x_e) > 0$ such that

$$x_e \in B^{\circ}\big(x_e, r(x_e)\big) \subseteq U_{i(x_e)}.$$

The family $\{B^{\circ}(x_e, r(x_e)),\ x_e \in K\}$ is a strong-ball cover of K, and therefore it admits a finite subcover. It follows that the original cover $\{U_i\}_{i \in I}$ also has a finite subcover. Thus K is τ_d-compact. □

Remark 9. Since $(SP(X), \tau_d)$ is Hausdorff, every τ_d-compact set is τ_d-closed. The real divergence in soft b-metric spaces lies not in separation properties, but in the mismatch between the ball structure and the induced topology. In particular, fixed point principles relying on topological compactness, such as Schauder-type theorems, cannot be transferred to the soft b-metric framework through ball-based arguments alone. They require either additional assumptions ensuring that balls are τ_d-open or a reformulation in terms of the topology τ_d.

Remark 10. These observations also explain why the fixed point results established in this paper are formulated through metric contractive conditions rather than compactness assumptions. Although compactness plays a central role in classical fixed point theory, the soft b-metric framework requires careful separation between metric balls and topological open sets whenever $s > 1$.

6 Algorithms and numerical verification

In this section, we complement Theorem 3 with an explicit Picard-type iteration and a short computational illustration. The purpose of this section is not to replace the theoretical convergence proof, but to show how the fixed point guaranteed by the rational contractive principle can be approximated in practice. We also compare the resulting Picard iterates with those generated by a standard linear contraction.

6.1 The iterative algorithm

Algorithm 1 summarizes the Picard iteration and the stopping criterion for approximating the unique soft fixed point ensured by Theorem 3. Iterative procedures of this type are also common in the study of nonlinear integral and fractional integral equations; see, for example, [9].

Algorithm 1. Fixed point iteration for soft rational contractions

Input:
- Soft mapping $T : SP(X) \to SP(X)$
- Initial guess $x^{(0)} \in SP(X)$
- Tolerance $\varepsilon > 0$
- Maximum number of iterations $N_{\max}$
- Soft b-metric d with coefficient $s \geqslant 1$

Output:
- Approximate fixed point x^* or failure message

Set $n \leftarrow 0$
while $n < N_{\max}$ **do**
  Compute $x^{(n+1)} = T(x^{(n)})$
  Compute error $E_n = d(x^{(n)}, x^{(n+1)})$
  if $E_n < \varepsilon$ **then**
    return $x^{(n+1)}$
  end if
  $n \leftarrow n + 1$
end while
return "Convergence not reached within $N_{\max}$ iterations"

Under the hypotheses of Theorem 3, the Picard orbit $\{x^{(n)}\}$ produced by the iteration scheme in Algorithm 1 converges in the soft b-metric to the unique fixed point.

6.2 Numerical example and comparative analysis

We consider a concrete instance in order to illustrate the convergence behaviour of the iteration and to compare it with a standard linear contraction.

Example 2. Let $X = [0, 1]$ and let the parameter set be $S = [0, 1]$. Equip $SP(X)$ with the soft b-metric $d(x_e, y_u) = |x - y|^2 + |e - u|^2$ with coefficient $s = 2$. In the numerical illustration, we fix a parameter $e \in S$ and work on the slice $SP_e(X) = \{x_e,\, x \in X\}$. On this slice, where $e = u$, the metric reduces to $d(x_e, y_e) = |x - y|^2$.

Consider the operator $T : SP_e(X) \to SP_e(X)$ defined by

$$T(x_e) = \left(\frac{x}{2 + x}\right)_e.$$

For $x, y \in [0, 1]$, we have

$$\left|\frac{x}{2 + x} - \frac{y}{2 + y}\right| = \frac{2|x - y|}{(2 + x)(2 + y)} \leqslant \frac{1}{2}|x - y|.$$

Therefore

$$d(Tx_e, Ty_e) \leqslant \frac{1}{4} d(x_e, y_e).$$

Thus T satisfies the contractive framework of Theorem 3 on the fixed-parameter slice with $\lambda = 0$ and $\mu = 1/4$. Since $s\mu = 2 \cdot (1/4) < 1$, the theorem guarantees a unique fixed point. We perform the iteration starting from $x^{(0)} = 1.0_e$.

Table 1 reports the Picard iterates $\{x^{(n)}\}$ generated by Algorithm 1 together with the successive errors $d(x^{(n)}, x^{(n+1)})$. For comparison, we also include the corresponding data for the linear Banach contraction $L(x) = x/2$.

Table 1 shows that both iterations converge to the fixed point 0 on the fixed-parameter slice. The nonlinear iteration reaches a smaller error at the same iteration level in this example. The reported values therefore provide a computational illustration of the Picard mechanism behind Theorem 3, rather than an additional proof of convergence.

Table 1. Convergence profile: nonlinear contraction vs. linear contraction.

n	$x^{(n)}$ (nonlinear)	$d(x^{(n)}, x^{(n+1)})$ (nonlinear)	$y^{(n)}$ (linear)	$d(y^{(n)}, y^{(n+1)})$ (linear)
1	0.333333	0.036281	0.500000	0.062500
2	0.142857	0.005805	0.250000	0.015625
3	0.066667	0.001184	0.125000	0.003906
4	0.032258	0.000268	0.062500	0.000976
5	0.015873	0.000064	0.031250	0.000244
⋮	⋮	⋮	⋮	⋮
10	0.000489	$< 10^{-7}$	0.000976	$< 10^{-6}$

7 Application

7.1 Application to soft integral equations

In this subsection we apply Theorem 3 to establish existence and uniqueness of solutions for a class of soft Fredholm integral equations. The example is designed to show how residual terms of the form $d(x, Tx)$ and $d(y, Ty)$ can be incorporated naturally into nonlinear integral operators. Thus the rational contractive framework is useful in situations where a direct Banach-type estimate may be too restrictive.

Let $X = C([a, b], \mathbb{R})$ be the space of real-valued continuous functions on $[a, b]$, and let S be a parameter set. A soft point $x_\lambda \in SP(X)$ is identified with the function $x \in X$ associated with the parameter λ. Define a soft b-metric d on $SP(X)$ by

$$d(x_\lambda, y_\mu) = \sup_{t\in[a,b]} \big|x(t) - y(t)\big|^2 + \mathbf{1}_{\lambda\neq\mu}.$$

Fix $\lambda_0 \in S$ and consider the fixed-parameter slice $SP_{\lambda_0}(X)$; in the sequel, we work on this slice. Here $\lambda = \mu = \lambda_0$. It is well known that $(SP(X), d)$ is a complete soft b-metric space with coefficient $s = 2$.

We consider the soft integral equation

$$x(t) = g(t) + \int_a^b K\big(t, r, x(r)\big)\,\mathrm{d}r, \quad t \in [a, b], \tag{5}$$

where $g : [a, b] \to \mathbb{R}$ and $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous.

Theorem 6. *Assume that there exist constants $\xi, \eta \geqslant 0$ with $\xi + 2\eta < 1$ such that, for all $x, y \in X$ and all $t, r \in [a, b]$,*

$$\begin{aligned}&\big|K\big(t, r, x(r)\big) - K\big(t, r, y(r)\big)\big|^2\\ &\quad\leqslant \frac{1}{(b-a)^2}\left(\xi\frac{D(x, Tx)\,D(y, Ty)}{1 + D(x, y)} + \eta D(x, y)\right),\end{aligned} \tag{6}$$

where

$$D(x, y) = \sup_{t\in[a,b]} \big|x(t) - y(t)\big|^2, \qquad (Tx)(t) = g(t) + \int_a^b K\big(t, r, x(r)\big)\,\mathrm{d}r.$$

Then the integral equation (5) *admits a unique solution in X, equivalently, a unique soft solution on the fixed-parameter slice $SP_{\lambda_0}(X)$.*

Proof. Define $T : X \to X$ by

$$(Tx)(t) = g(t) + \int_a^b K\big(t, r, x(r)\big)\,\mathrm{d}r.$$

For $\lambda \in S$, define the induced soft mapping on $SP(X)$ by $T(x_\lambda) = (Tx)_\lambda$.

Fix $x, y \in X$ and $t \in [a, b]$. Then

$$\begin{aligned}\big|Tx(t) - Ty(t)\big|^2 &= \left| \int_a^b \big(K\big(t, r, x(r)\big) - K\big(t, r, y(r)\big)\big)\,\mathrm{d}r \right|^2 \\ &\leqslant (b-a) \int_a^b \big|K\big(t, r, x(r)\big) - K\big(t, r, y(r)\big)\big|^2\,\mathrm{d}r.\end{aligned}$$

Using (6) and integrating with respect to r, we obtain

$$\big|Tx(t) - Ty(t)\big|^2 \leqslant \xi \frac{D(x, Tx)D(y, Ty)}{1 + D(x, y)} + \eta D(x, y).$$

Taking the supremum over $t \in [a, b]$ yields

$$D(Tx, Ty) \leqslant \xi \frac{D(x, Tx)D(y, Ty)}{1 + D(x, y)} + \eta D(x, y).$$

If $\lambda = \mu$, then $d(x_\lambda, y_\lambda) = D(x, y)$ and $d(x_\lambda, Tx_\lambda) = D(x, Tx)$. Hence

$$d(Tx_\lambda, Ty_\lambda) \leqslant \xi \frac{d(x_\lambda, Tx_\lambda)\, d(y_\lambda, Ty_\lambda)}{1 + d(x_\lambda, y_\lambda)} + \eta d(x_\lambda, y_\lambda).$$

This inequality is exactly of the form (2) on the fixed-parameter slice. Since $s = 2$ and $\xi + 2\eta < 1$, Theorem 3 guarantees the existence and uniqueness of a soft fixed point in $SP_{\lambda_0}(X)$. This fixed point is precisely a unique solution of (5) in X. □

Remark 11. The above result illustrates why rational contractive conditions are particularly well suited for nonlinear integral equations. In many cases, a global Banach-type estimate of the form $d(Tx, Ty) \leqslant kd(x, y)$ cannot be obtained with $sk < 1$ because kernel estimates naturally depend on the residuals $d(x, Tx)$ and $d(y, Ty)$ rather than on $d(x, y)$ alone. The rational inequality incorporates these residual terms and becomes increasingly effective along Picard iterates as they approach a solution. When $\xi = 0$, the condition reduces to a Banach-type contraction with constant η, showing that the rational framework genuinely extends the classical principle rather than replacing it.

7.2 Application to Caputo fractional differential equations

We also indicate how Theorem 3 can be used for an integral operator associated with a nonlinear Caputo fractional boundary value problem. This application uses the theorem through its Banach-type special case $\lambda = 0$, and it complements the preceding Fredholm integral equation where the rational residual term is retained explicitly.

We consider a nonlinear boundary value problem involving the Caputo fractional derivative of order $\beta \in (1, 2]$:

$$^{C}D^{\beta}x(t) = f\big(t, x(t)\big), \quad t \in [0, 1], \qquad x(0) = 0, \qquad x(1) = \int_0^1 x(s)\,\mathrm{d}s. \tag{7}$$

It is well known that (7) can be rewritten as an equivalent integral equation by means of the associated Green function $G(t,s)$. Let $X = C([0,1],\mathbb{R})$ and fix $\lambda_0 \in S$. On the fixed-parameter slice $SP_{\lambda_0}(X)$, we use the soft b-metric

$$d(x_{\lambda_0}, y_{\lambda_0}) = \sup_{t\in[0,1]} \big|x(t) - y(t)\big|^2, \quad s = 2.$$

Thus, throughout this application, all soft points are taken with the same parameter λ_0.

Theorem 7. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous and suppose that there exist constants $\xi, \eta \geqslant 0$ such that, for all $t \in [0,1]$ and $u, v \in \mathbb{R}$,*

$$\big|f(t,u) - f(t,v)\big|^2 \leqslant \xi \frac{|u-v|^2}{1+|u-v|^2} + \eta|u-v|^2.$$

Assume further that the Green function G associated with (7) *satisfies*

$$\sup_{t\in[0,1]} \int_0^1 \big|G(t,s)\big|^2 \,\mathrm{d}s \leqslant \Lambda$$

and that the parameters fulfill $2\Lambda(\xi+\eta) < 1$. Then the boundary value problem (7) *admits a unique solution in X, equivalently, a unique soft solution on the fixed-parameter slice $SP_{\lambda_0}(X)$.*

Proof. Fix $\lambda_0 \in S$ and restrict the discussion to the fixed-parameter slice $SP_{\lambda_0}(X)$. Define the operator $\widetilde{T} : SP_{\lambda_0}(X) \to SP_{\lambda_0}(X)$ by

$$\widetilde{T}(x_{\lambda_0}) = (Tx)_{\lambda_0}, \qquad (Tx)(t) = \int_0^1 G(t,s) f\big(s, x(s)\big)\,\mathrm{d}s, \quad t \in [0,1].$$

Let $x, y \in X = C([0,1],\mathbb{R})$ be arbitrary. Using the Cauchy–Schwarz inequality, for each fixed $t \in [0,1]$, we obtain

$$\begin{aligned}
\big|Tx(t) - Ty(t)\big|^2 &= \left| \int_0^1 G(t,s)\big(f\big(s,x(s)\big) - f\big(s,y(s)\big)\big)\,\mathrm{d}s \right|^2 \\
&\leqslant \left(\int_0^1 \big|G(t,s)\big|^2 \,\mathrm{d}s \right) \left(\int_0^1 \big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|^2 \,\mathrm{d}s \right) \\
&\leqslant \Lambda \int_0^1 \big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|^2 \,\mathrm{d}s.
\end{aligned}$$

Since $r/(1+r) \leqslant r$ for $r \geqslant 0$, the hypothesis on f implies that, for all $s \in [0,1]$ and $u, v \in \mathbb{R}$,

$$\big|f(s,u) - f(s,v)\big|^2 \leqslant (\xi+\eta)|u-v|^2.$$

Applying this pointwise estimate with $u = x(s)$ and $v = y(s)$ yields

$$\big|Tx(t) - Ty(t)\big|^2 \leqslant \Lambda(\xi+\eta) \int_0^1 \big|x(s) - y(s)\big|^2 \,\mathrm{d}s \leqslant \Lambda(\xi+\eta)\|x-y\|_\infty^2.$$

Taking the supremum over $t \in [0,1]$, we obtain

$$\begin{aligned} d(\widetilde{T}x_{\lambda_0}, \widetilde{T}y_{\lambda_0}) &= \sup_{t\in[0,1]} \big|Tx(t) - Ty(t)\big|^2 \leqslant \mu \sup_{t\in[0,1]} \big|x(t) - y(t)\big|^2 \\ &= \mu d(x_{\lambda_0}, y_{\lambda_0}), \end{aligned}$$

where $\mu = \Lambda(\xi+\eta)$. Since $s = 2$ and $2\mu < 1$, Theorem 3 applies (with $\lambda = 0$), and consequently, (7) has a unique solution in X, equivalently, a unique soft solution in $SP_{\lambda_0}(X)$. □

8 Conclusion

We studied the interplay between topology and fixed point theory in soft b-metric spaces. From the topological point of view, we showed that when $s > 1$, a soft open ball may fail to be open in the induced topology τ_d. This shows that ball-based arguments cannot be used as a substitute for τ_d-topological reasoning. It also explains why compactness-type statements formulated through ball covers may differ from compactness in the induced topology. In this direction, strong balls provide a natural τ_d-based substitute for ordinary balls and restore a workable cover formulation of compactness.

On the fixed point side, we established existence and uniqueness results for Reich–Ćirić-type contractions and for a rational contractive condition in complete soft b-metric spaces. We also obtained an α–ψ extension of the rational principle. These results go beyond the Banach-type setting by allowing contractive inequalities involving defect terms such as $d(x,Tx)$ and $d(y,Ty)$. This flexibility is useful in applications to nonlinear soft integral equations, where residual-type estimates arise naturally. The theoretical results were complemented by an explicit Picard-type iterative scheme and a computational illustration of the convergence behaviour.

Overall, the results indicate that the case $s = 1$ behaves in the familiar soft metric manner, whereas the regime $s > 1$ exhibits genuinely different topological features. This distinction should be taken into account in future work on compactness-based fixed point principles, multivalued mappings, and nonlinear integral or fractional models in soft b-metric spaces. Further directions include metrization questions related to τ_d, compactness notions compatible with both the induced topology and the ball structure, and extensions of rational contractive methods to broader soft metric-type settings.

Remark 12 [Future work]. Mitrović et al. [16] introduced a generalized Nadler-type contraction for multivalued mappings in classical b-metric spaces. A natural direction for further research is to develop a suitable notion of a soft Hausdorff b-metric and to investigate whether the topological phenomena highlighted in the present paper, especially the possible failure of openness of balls when $s > 1$, affect the extension of Nadler-type fixed point results to the soft setting.

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