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Stochastic stability of fractional-order quaternion-valued neural networks on time scales with general probabilistic bounded Markovian switching and neutral delays*

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Abstract. This paper investigates the stochastic stability problem of fractional-order quaternion-valued neural networks (FOQVNNs) on time scales with general probabilistic bounded Markovian switching and neutral delays. Owing to the noncommutative nature of quaternion algebra and the coexistence of fractional dynamics, time-scale calculus, Markovian switching, and neutral delays, stability analysis becomes highly challenging, especially when conventional decomposition methods are employed. To overcome these difficulties, the considered system is treated directly in the quaternion domain without decomposition, thereby preserving the intrinsic algebraic structure and avoiding unnecessary dimensional expansion. By constructing an appropriate Lyapunov–Krasovskii functional and combining the free-weighting matrix technique with matrix inequality approaches, a sufficient condition guaranteeing stochastic stability for the considered FOQVNNs is derived and expressed as quaternion-valued linear matrix inequalities. The obtained criteria are general and can be efficiently verified by using a numerical example.

Keywords: time scales, neutral delay, fractional order, general probabilistic bounded Markovian switching, quaternion-valued neural networks.

1 Introduction

In recent decades, neural networks have been extensively investigated due to their remarkable performance in a wide range of applications, including signal processing, pattern recognition, associative memory, intelligent control, and image analysis [5]. The

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successful implementation of these applications relies fundamentally on the stability of the underlying neural network models. It has been well recognized that various forms of time delays inevitably arise in practical neural networks as a result of finite signal transmission speeds, information processing times, and hardware constraints [20, 23]. Such delays may cause oscillations or even instability of the system, thereby significantly degrading its performance [32]. Consequently, the stability analysis of neural networks with time delays has attracted sustained research attention [2, 8, 30].

In many practical systems, delays affect not only the state variables but also their derivatives, giving rise to neutral-type delays. Neutral delays commonly arise in engineering systems such as lossless transmission lines, population dynamics with gestation effects, and distributed networks involving electromagnetic wave propagation, where the current state dynamics depend on the rate of change of past states. Compared with retarded delays, the presence of neutral delays introduces additional analytical difficulties: the system dynamics become implicit, the characteristic equation becomes transcendental, and Lyapunov-based methods must handle derivative terms within the functional, all of which significantly complicate the stability analysis [7, 29]. In recent years, neural networks with neutral delays have been widely studied, and a variety of effective stability criteria have been reported [1, 6, 14, 24]. Nevertheless, most existing results are restricted to integer-order systems or real-valued frameworks, which may not be sufficient to capture the intrinsic characteristics of complex dynamical processes.

On the other hand, quaternion-valued neural networks (QVNNs), whose states, connection weights, and activation functions are all defined in the quaternion domain, have received increasing attention in recent years [3, 17, 19]. Owing to their capability of naturally representing multidimensional information, QVNNs have demonstrated superior performance over real-valued and complex-valued neural networks in applications such as color image processing, three-dimensional signal representation, and robotic motion control [21]. However, the noncommutative nature of quaternion multiplication poses substantial difficulties for both theoretical analysis and numerical computation. Conventional decomposition methods, which transform quaternion-valued systems into real-valued or complex-valued counterparts, often lead to a significant increase in system dimension and may destroy the inherent algebraic structure of quaternions, resulting in conservative stability conditions [10].

Meanwhile, fractional-order calculus has proven to be an effective tool for modeling systems with memory and hereditary properties. Compared with integer-order models, fractional-order neural networks can more accurately describe many real-world phenomena. Motivated by these advantages, fractional-order neural networks have been intensively studied, and a large number of stability results have been obtained [9, 22, 31]. However, most of the existing works focus on either continuous-time or discrete-time systems separately. In reality, many practical systems exhibit hybrid dynamics involving both continuous and discrete behaviors. To address this issue, the theory of time scales was introduced, providing a unified framework that encompasses both continuous and discrete cases as special instances [11, 16, 25]. Despite its significance, the stability analysis of fractional-order quaternion-valued neural networks on time scales remains relatively underexplored, especially in the presence of neutral delays.

Furthermore, neural networks implemented in real environments are often subject to abrupt structural changes caused by component failures, environmental disturbances, or switching control strategies. These phenomena can be effectively modeled by Markovian switching mechanisms [4, 12, 27]. In practice, however, obtaining exact transition probabilities of Markov processes is usually difficult or even impossible. Therefore, it is more realistic and meaningful to consider Markovian switching with general probabilistic bounded transition rates, which includes fully known, partially known, and completely unknown transition probabilities as special cases [13, 18, 28]. Incorporating such switching mechanisms into fractional-order quaternion-valued neural networks on time scales with neutral delays further complicates the stability analysis and poses new theoretical challenges.

Based on the above analysis, simultaneously addressing fractional-order dynamics, the noncommutative algebra of quaternions, the unified framework of time scales, neutral delays, and generally probabilistically bounded Markovian switching renders the stability analysis of such systems highly challenging. In particular, conventional decomposition methods can destroy the intrinsic algebraic structure of quaternions, leading to dimensional expansion and conservative results, which makes it difficult to accurately capture the true dynamical behavior of such hybrid stochastic systems. Moreover, characterizing the stability of fractional-order neural networks on time scales remains a relatively underexplored area of research, and no systematic analytical framework has yet been established for systems that simultaneously incorporate neutral delays and probabilistically bounded Markovian switching. Therefore, this paper aims to establish a theoretical and methodological framework for the stochastic stability analysis of fractional-order quaternion-valued neural networks on time scales with generally probabilistically bounded Markovian switching and neutral delays directly within the quaternion domain.

The main contributions of this paper can be summarized as follows:

(i) The stochastic stability problem of fractional-order quaternion-valued neural networks on time scales with both neutral delays and general probabilistic bounded Markovian switching is investigated for the first time.

(ii) A direct analytical framework is developed in the quaternion domain without resorting to decomposition, which preserves the intrinsic algebraic structure of quaternions and reduces computational complexity.

(iii) The derived stability criteria are expressed as quaternion-valued linear matrix inequalities, which can be efficiently verified using standard numerical tools.

Notations. Let $\mathbb{Q}$ denote the skew field of quaternions. The spaces of n-dimensional quaternion-valued vectors and $n \times m$ quaternion-valued matrices are denoted by $\mathbb{Q}^n$ and $\mathbb{Q}^{n\times m}$, respectively. For a quaternion-valued matrix H, its conjugate transpose is written as H^*. A matrix $H \in \mathbb{Q}^{n\times n}$ is said to be positive definite, denoted by $H > 0$, if it satisfies $H^* = H$ and is Hermitian positive definite. The maximum eigenvalue of a quaternion Hermitian matrix H is denoted by $\lambda_{\max}(H)$. For any vector $x \in \mathbb{Q}^n$ and matrix $H \in \mathbb{Q}^{n\times n}$, the corresponding vector norm and matrix norm are denoted by $\|x\|$ and $\|H\|$, respectively. For a time scale $\mathbb{T}$, the closed interval $[e, g]_{\mathbb{T}}$ is defined as $[e, g]_{\mathbb{T}} = \{t \in \mathbb{T} \mid e \leqslant t \leqslant g\}$. $\mathbf{E}\{\cdot\}$ indicates the mathematical expectation.

2 Model description and preliminaries

A time scale, denoted as $\mathbb{T}$, is defined as an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The topological structure of $\mathbb{T}$ is characterized using the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, which are respectively formulated as

$$\sigma(t) = \inf\{s \in \mathbb{T}\colon\ s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T}\colon\ s < t\}.$$

Here we adopt the convention that $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. The distance between a point and its successor, known as the graininess function $\theta : \mathbb{T} \to [0, \infty)$, is defined by $\theta(t) = \sigma(t) - t$.

Based on these operators, the points in $\mathbb{T}$ are classified as follows: a point $t \in \mathbb{T}$ is termed right-dense if $\sigma(t) = t$ and right-scattered if $\sigma(t) > t$. Conversely, t is called left-dense provided $\rho(t) = t$, whereas it is left-scattered if $\rho(t) < t$. In this work, it is assumed that the graininess function is bounded uniformly, specifically,

$$\sup_{t \in \mathbb{T}} \theta(t) \leqslant \theta_0 < \infty.$$

A vector-valued function $g : \mathbb{T} \to \mathbb{Q}^n$ is classified as rd-continuous (denoted by $g \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{Q}^n)$) if it satisfies two conditions: it is continuous at all right-dense points and possesses finite left-sided limits at all left-dense points. Because each component of a quaternion-valued function can be viewed as a real-valued mapping, the standard calculus on time scales applies componentwise to the real and imaginary parts of g.

For a real-valued function $g : \mathbb{T} \to \mathbb{R}$, the delta derivative $g^{\Delta}(t)$ at $t \in \mathbb{T}$ exists if there is a number $g^{\Delta}(t)$ such that for every $\varepsilon > 0$, there exists a neighbourhood U of t with

$$\left|g\big(\sigma(t)\big) - g(s) - g^{\Delta}(t)\big(\sigma(t) - s\big)\right| \leqslant \varepsilon\left|\sigma(t) - s\right| \quad \forall s \in U.$$

If g is delta differentiable on $\mathbb{T}$, then its delta integral is defined via

$$\int_a^b g(s)\Delta s = G(b) - G(a), \quad \text{where } G^{\Delta}(t) = g(t).$$

We now introduce the fractional order delta derivative on time scales. Let $q \in (0, 1)$, and let $\mathbb{T}$ be a time scale with $[t_0, a] \subseteq \mathbb{T}$. For an rd-continuous function $\psi : \mathbb{T} \to \mathbb{Q}^n$, the Caputo-type fractional delta derivative of order q is defined as

$$\Delta^q \psi(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} \psi^{\Delta}(s)\Delta s.$$

When $\mathbb{T} = \mathbb{R}$, this definition coincides with the classical Caputo fractional derivative.

Fractional-order quaternion-valued neural networks naturally arise in real-world scenarios involving multidimensional, long-memory, and hybrid dynamic processes. Moreover, Markovian switching with generally bounded transition probabilities accounts for abrupt environmental changes or component failures where exact transition rates are inaccessible. Therefore, providing a unified framework for such complex systems and investigating their stability are of great practical significance. Motivated by this, we consider the following class of fractional-order quaternion-valued neural networks subject to both discrete and neutral delays and governed by a Markovian switching mechanism with generally bounded transition probabilities:

$$\begin{aligned}\varDelta^q \breve{\varphi}(t) = {}& -G_{r_t}\breve{\varphi}(t) + F_{r_t} g\big(\breve{\varphi}(t)\big) + D_{r_t} g\big(\breve{\varphi}\big(t-\varrho(t)\big)\big) \\ & + E_{r_t}\varDelta^q \breve{\varphi}\big(t-\varkappa(t)\big)\end{aligned} \tag{1}$$

for $t \geqslant 0$. The system parameters are defined as follows: $\breve{\varphi}(t) = (\breve{\varphi}_1(t), \dots, \breve{\varphi}_n(t))^T \in \mathbb{Q}^n$ represents the state vector of the neural network. The matrix $G_{r_t} \in \mathbb{R}^{n\times n}$ denotes the self-feedback weight matrix, while $F_{r_t}, D_{r_t}, E_{r_t} \in \mathbb{Q}^{n\times n}$ correspond to the connection weight matrices. The neuron activation functions are described by the vector-valued map $g(\breve{\varphi}(t)) = (g_1(\breve{\varphi}_1(t)), \dots, g_n(\breve{\varphi}_n(t)))^T \in \mathbb{Q}^n$. Furthermore, $\varrho(t)$ and $\varkappa(t)$ represent the time-varying discrete and neutral delays, respectively, satisfying the conditions $0 \leqslant \varrho(t) \leqslant \varrho$ and $0 \leqslant \varkappa(t) \leqslant \varkappa$, where ϱ and $\varkappa$ are constant upper bounds.

The dynamics described in (1) are subject to the following initial condition:

$$\breve{\varphi}(s) = \chi(s) \quad \forall s \in [-\omega, 0],$$

where the delay horizon is defined as $\omega = \max\{\varrho, \varkappa\}$, and the function $\chi(\cdot)$ is assumed to be continuously differentiable on the interval $[-\omega, 0]$.

(H1) For each neuron $i = 1, \dots, n$, the activation function $g_i : \mathbb{Q} \to \mathbb{Q}$ is continuous and satisfies the Lipschitz condition

$$\big|g_i(m_1) - g_i(m_2)\big| \leqslant \ell_i |m_1 - m_2| \quad \forall m_1, m_2 \in \mathbb{Q},$$

where $\ell_i > 0$. Define the diagonal matrix $L = \operatorname{diag}(\ell_1, \dots, \ell_n)$.

(H2) The Markov process $\{r_t,\ t \geqslant 0\}$ has a transition rate matrix $\varGamma = [\gamma_{ij}]_{n\times n}$, whose entries are only partially known. Specifically, each entry γ_{ij} $(i \neq j)$ is either completely unknown or known to lie in a given interval $[\gamma_{\alpha ij}, \gamma_{\beta ij}]$, where $\gamma_{\alpha ij} \leqslant \gamma_{\beta ij}$. The diagonal entries are determined by $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. For each mode i, we partition the index set N into

$$N = N^i_{uk} \cup N^i_k,$$

where N^i_{uk} contains the indices for which the corresponding transition rates are completely unknown, and N^i_k contains those for which at least an interval bound is available.

When the system operates in a fixed mode $i \in N$, it can be written as

$$\begin{aligned}\varDelta^q \breve{\varphi}(t) = {}& -G_i\breve{\varphi}(t) + F_i g\big(\breve{\varphi}(t)\big) + D_i g\big(\breve{\varphi}\big(t-\varrho(t)\big)\big) \\ & + E_i\varDelta^q \breve{\varphi}\big(t-\varkappa(t)\big).\end{aligned} \tag{2}$$

Throughout this paper, $\{r_t,\, t \geqslant 0\}$ is a continuous time Markov chain taking values in the finite state space N. Its transition probabilities over an infinitesimal interval $\eta t > 0$ are given by

$$\mathbf{P}\{r_{t+\eta t} = j \mid r_t = i\} = \begin{cases} \gamma_{ij}\, \eta t + o(\eta t), & j \neq i, \\ 1 + \gamma_{ii}\, \eta t + o(\eta t), & j = i, \end{cases}$$

where $\gamma_{ij} \geqslant 0$ ($j \neq i$) denotes the transition rate from mode i to mode j, and $o(\eta t)$ satisfies $\lim_{\eta t \to 0} o(\eta t)/\eta t = 0$. The diagonal entries are determined by the conservative property $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$.

The following lemmas will be used in the subsequent stability analysis.

Definition 1. (See [18].) System (1) is regarded as stochastically stable if, for every initial mode r_0 and initial condition $\varsigma(0)$, the following criterion is satisfied:

$$\mathbf{E}\left\{ \int_0^\infty \left\|\varsigma(t)\right\|^2 \Delta t \;\middle|\; \varsigma(0), r_0 \right\} < +\infty.$$

Lemma 1. (*See* [26].) *Suppose $Z \in \mathbb{Q}^{n\times n}$ is a Hermitian matrix that is positive definite, and let $\psi : [a,b]_{\mathbb{T}} \to \mathbb{Q}^n$ denote a vector-valued function. Then*

$$\left(\int_a^b \psi(\varrho)\Delta\varrho \right)^* Z \left(\int_a^b \psi(\varrho)\Delta\varrho \right) \leqslant (b-a) \int_a^b \psi^*(\varrho) Z \psi(\varrho) \Delta\varrho.$$

Lemma 2. (*See* [15].) *Given any positive definite Hermitian matrix $M \in \mathbb{Q}^{n\times n}$ and vectors $u_1, u_2 \in \mathbb{Q}^n$, the following holds*:

$$u_1^* u_2 + u_2^* u_1 \leqslant u_1^* M u_1 + u_2^* M^{-1} u_2.$$

3 Main results

Theorem 1. *Under assumption* (H1)–(H2) *and for a given positive constant θ, the equilibrium point of system* (1) *is stochastically stable on the time scale $\mathbb{T}$, provided that there exist ten quaternion-valued matrices $U_{11}, U_{12}, U_{13}, U_{22}, U_{23}, U_{33}, W_1, W_2, Z_1, Z_2 \in \mathbb{Q}^{n\times n}$ and five real positive-definite diagonal matrices $S_1, S_2, P_j, H_i, Q_{ij} \in \mathbb{R}^{n\times n}$* (*where $i, j \in N$*) *satisfying the following conditions*:

$$U = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{12}^* & U_{22} & U_{23} \\ U_{13}^* & U_{23}^* & U_{33} \end{pmatrix} > 0, \tag{3}$$

$$\Lambda = (\Lambda_{xy})_{7\times 7} < 0, \tag{4}$$

$$P_j - H_i \leqslant 0, \quad j \in N_{uk}^i,\ j \neq i, \tag{5}$$

$$P_j - H_i \geqslant 0, \quad j \in N_{uk}^i,\ j = i, \tag{6}$$

$$P_j - H_i - Q_{ij} \leqslant 0, \quad y \in N_k^i, \tag{7}$$

where $\Lambda_{yx} = \Lambda_{xy} \in \mathbb{Q}^{2\times 2}$, and the nonzero entries of Λ are given by $\Lambda_{yx} = \Lambda^*_{xy} \in \mathbb{Q}^{2\times 2}$,

$$\Lambda_{11} = -Z_1^* G_i - G_i^* Z_1 + P_i + \varrho U_{11} + U_{13} + U_{13}^* + \varrho W_1 + LS_1 L,$$
$$\Lambda_{12} = -Z_1^* - G_i^* Z_2, \qquad \Lambda_{13} = P_i,$$
$$\Lambda_{14} = \varrho U_{12} - U_{13} + U_{23}^* - \theta P_i, \qquad \Lambda_{15} = Z_1^* F_i,$$
$$\Lambda_{16} = Z_1^* D_i, \qquad \Lambda_{17} = Z_1^* E_i, \qquad \Lambda_{22} = W_2 + \varrho U_{33} - Z_2^* - Z_2,$$
$$\Lambda_{25} = Z_2^* F_i, \qquad \Lambda_{26} = Z_2^* D_i, \qquad \Lambda_{27} = Z_2^* E_i,$$
$$\Lambda_{33} = -\frac{1}{\varrho} W_1 + \sum_{j \in N_k^i} \gamma_{a_{ij}} (P_j - H_i) + \sum_{j \in N_k^i} \varepsilon_{ij} Q_{ij},$$
$$\Lambda_{34} = -P_i, \quad \Lambda_{44} = \varrho U_{22} - U_{23} - U_{23}^* + \theta P_i + LS_2 L,$$
$$\Lambda_{55} = -S_1, \qquad \Lambda_{66} = -S_2, \qquad \Lambda_{77} = -W_2,$$

while all other entries Λ_{xy} are zero.

Proof. To facilitate the stability analysis, we construct a candidate Lyapunov–Krasovskii functional as follows:

$$\begin{aligned} V\bigl(\breve{\varphi}(t), i\bigr) &= V_1\bigl(\breve{\varphi}(t), i\bigr) + V_2\bigl(\breve{\varphi}(t), i\bigr) + V_3\bigl(\breve{\varphi}(t), i\bigr) \\ &\quad + V_4\bigl(\breve{\varphi}(t), i\bigr) + V_5\bigl(\breve{\varphi}(t), i\bigr), \end{aligned}$$

where

$$V_1\bigl(\breve{\varphi}(t), i\bigr) = \Biggl(\int_{t-\varrho}^{t} \breve{\varphi}(\zeta)\Delta\zeta\Biggr)^* P_i \Biggl(\int_{t-\varrho}^{t} \breve{\varphi}(\zeta)\Delta\zeta\Biggr),$$

$$V_2\bigl(\breve{\varphi}(t), i\bigr) = \int_{-\varrho}^{0}\int_{t+\nu}^{t} \breve{\varphi}^*(\zeta) W_1 \breve{\varphi}(\zeta)\Delta\zeta\Delta\nu,$$

$$V_3\bigl(\breve{\varphi}(t), i\bigr) = \int_{t-\varkappa(t)}^{t} \bigl(\Delta^q \breve{\varphi}(\zeta)\bigr)^* W_2 \Delta^q \breve{\varphi}(\zeta)\Delta\zeta,$$

$$V_4\bigl(\breve{\varphi}(t), i\bigr) = \int_{-\varrho}^{0}\int_{t+\nu}^{t} \bigl(\Delta^q \breve{\varphi}(\zeta)\bigr)^* U_{33} \Delta^q \breve{\varphi}(\zeta)\Delta\zeta\Delta\nu,$$

$$V_5\bigl(\breve{\varphi}(t), i\bigr) = \int_{0}^{t}\int_{\nu-\varrho(\nu)}^{\nu} \Theta^*(\nu, \zeta) U \Theta(\nu, \zeta)\Delta\zeta\Delta\nu.$$

Here

$$\Theta(\nu, \zeta) = \bigl(\breve{\varphi}^*(\nu),\ \breve{\varphi}^*\bigl(\nu - \varrho(\nu)\bigr), \bigl(\Delta^q \breve{\varphi}(\zeta)\bigr)^*\bigr)^*.$$

Evaluating the derivative of $V_1(t)$ leads to the following expression:

$$
\begin{aligned}
&V_1^{\Delta}\big(\breve{\varphi}(t), r_t\big)\\
&\quad= \lim_{\eta t \to 0^+} \frac{\mathbf{E}[V_1(\breve{\varphi}(t+\eta t), r_{t+\eta t}) \mid \breve{\varphi}(t),\, r_t = i] - V_1(\breve{\varphi}(t), i)}{\eta t}\\
&\quad= \lim_{\eta t \to 0^+} \frac{1}{\eta t}\Bigg\{\sum_{j=1}^{N} \mathbf{P}\{r_{t+\eta t} = j \mid r_t = i\} V_1(t+\eta t,\, j) - V_1\big(\breve{\varphi}(t), i\big)\Bigg\}\\
&\quad= \lim_{\eta t \to 0^+} \frac{1}{\eta t}\Bigg\{\sum_{j=1,\, j\neq i}^{N} \mathbf{P}\{r_{t+\eta t} = j \mid r_t = i\} V_1(t+\eta t,\, j)\\
&\qquad+ \mathbf{P}\{r_{t+\eta t} = i \mid r_t = i\} V_1(t+\eta t,\, i) - V_1\big(\breve{\varphi}(t), i\big)\Bigg\}\\
&\quad= \lim_{\eta t \to 0^+} \frac{1}{\eta t}\Bigg[(1+\gamma_{ii}\eta t) V_1(t+\eta t,\, i) + \sum_{j=1,\, j\neq i}^{N} (\gamma_{ij}\eta t) V_1(t+\eta t,\, j)\\
&\qquad+ o(\eta t) - V_1(t, i)\Bigg]\\
&\quad= \lim_{\eta t \to 0^+} \Bigg[\frac{V_1(t+\eta t,\, i) - V_1(t, i)}{\eta t} + \gamma_{ii} V_1(t+\eta t,\, i)\\
&\qquad+ \sum_{j=1,\, j\neq i}^{N} \gamma_{ij} V_1(t+\eta t,\, j) + \frac{o(\eta t)}{\eta t}\Bigg]\\
&\quad= V_1^{\Delta}(t, i) + \sum_{j=1}^{N} \gamma_{ij} V_1(t, j).
\end{aligned} \tag{8}
$$

According to assumption (H2), we can get that

$$
\sum_{j=1}^{N} \gamma_{ij} P_j = \sum_{j\in N_{uk}^i} \gamma_{ij} P_j + \sum_{j\in N_k^i} \gamma_{ij} P_j.
$$

For the known part, we use the median $\gamma_{a_{ij}} = (\gamma_{\alpha ij} + \gamma_{\beta ij})/2$ and the radius $\varepsilon_{ij} = (\gamma_{\beta ij} - \gamma_{\alpha ij})/2$ to express $\gamma_{ij} = \gamma_{a_{ij}} + \Delta\gamma_{ij}$, where $|\Delta\gamma_{ij}| \leqslant \varepsilon_{ij}$. For any $H_i, Q_{ij} > 0$, we have that

$$
\begin{aligned}
\sum_{j=1}^{N} \gamma_{ij} V_1(t, j) &= \Bigg(\int_{t-\varrho}^{t} \breve{\varphi}(\zeta)\Delta\zeta\Bigg)^{*} \sum_{j=1}^{N} \gamma_{ij} P_j \Bigg(\int_{t-\varrho}^{t} \breve{\varphi}(\zeta)\Delta\zeta\Bigg)\\
&= \Bigg(\int_{t-\varrho}^{t} \breve{\varphi}(\zeta)\Delta\zeta\Bigg)^{*} \Bigg(\sum_{j\in N_{uk}^i} \gamma_{ij} P_j + \sum_{j\in N_k^i} \gamma_{ij} P_j\Bigg) \Bigg(\int_{t-\varrho}^{t} \breve{\varphi}(\zeta)\Delta\zeta\Bigg).
\end{aligned}
$$

Note that since $\sum_{j=1}^{N}\gamma_{ij}=0$, it follows that $\sum_{j=1}^{N}\gamma_{ij}F_i=0$. When $j\in N_{uk}^i$ and $j\neq i$, we have from (5) that

$$\sum_{j\in N_{uk}^i}\gamma_{ij}(P_j-H_i)\leqslant 0.$$

If $j=i$, then we have from (6) that

$$\sum_{j\in N_{uk}^i}\gamma_{ij}(P_j-H_i)\geqslant 0.$$

For the known part, we employ

$$\gamma_{ij}(P_j-H_i)=\gamma_{a_{ij}}(P_j-H_i)+\Delta\gamma_{ij}(P_j-H_i).$$

Additionally, under the conditions $P_j-H_i-Q_{ij}\leqslant 0$ and $Q_{ij}>0$, the following is obtained:

$$\Delta\gamma_{ij}(P_j-H_i)\leqslant\varepsilon_{ij}Q_{ij}.$$

Thus

$$\sum_{j\in N_k^i}^{N}\gamma_{ij}(P_j-H_i)\leqslant\sum_{j\in N_k^i}\bigl(\gamma_{a_{ij}}(P_j-H_i)+\varepsilon_{ij}Q_{ij}\bigr).$$

Therefore, the following inequality holds:

$$\begin{aligned}
&\sum_{j=1}^{N}\gamma_{ij}V_1(t,j)\\
&\quad\leqslant\left(\int_{t-\varrho}^{t}\breve{\varphi}(\zeta)\Delta\zeta\right)^{*}\sum_{j\in N_k^i}\bigl(\gamma_{a_{ij}}(P_j-H_i)+\varepsilon_{ij}Q_{ij}\bigr)\left(\int_{t-\varrho}^{t}\breve{\varphi}(\zeta)\Delta\zeta\right)\\
&\quad\leqslant\left(\int_{t-\varrho}^{t}\breve{\varphi}(\zeta)\Delta\zeta\right)^{*}\left(\sum_{j\in N_k^i}\gamma_{a_{ij}}(P_j-H_i)+\sum_{j\in N_k^i}\varepsilon_{ij}Q_{ij}\right)\left(\int_{t-\varrho}^{t}\breve{\varphi}(\zeta)\Delta\zeta\right).
\end{aligned}\tag{9}$$

By computing the Δ-derivative of the functional $V(t)$ along the trajectory of system (1), we can obtain

$$\begin{aligned}
V_1^{\Delta}\bigl(\breve{\varphi}(t),i\bigr)&=\bigl(\breve{\varphi}(t)-\breve{\varphi}(t-\varrho)\bigr)^{*}P_i\int_{t-\varrho}^{t}\breve{\varphi}(\zeta)\Delta\zeta+\left(\int_{t-\varrho}^{t}\breve{\varphi}(\zeta)\Delta\zeta\right)^{*}P_i\bigl(\breve{\varphi}(t)-\breve{\varphi}(t-\varrho)\bigr)\\
&\quad+\theta(t)\bigl(\breve{\varphi}(t)-\breve{\varphi}(t-\varrho)\bigr)^{*}P_i\bigl(\breve{\varphi}(t)-\breve{\varphi}(t-\varrho)\bigr)\\
&\leqslant\theta\breve{\varphi}^{*}(t)P_i\breve{\varphi}+\breve{\varphi}^{*}(t)P_i\int_{t-\varrho}^{t}\breve{\varphi}(\zeta)\Delta\zeta-\theta\breve{\varphi}^{*}(t)P_i\breve{\varphi}(t-\varrho)\\
&\quad-\theta\breve{\varphi}^{*}(t-\varrho)P_i\breve{\varphi}(t)
\end{aligned}\tag{10}$$

$$+\left(\int_{t-\varrho}^{t}\breve{\varphi}(\zeta)\Delta\zeta\right)^{*}P_i\breve{\varphi}(t)-\left(\int_{t-\varrho}^{t}\breve{\varphi}(\zeta)\Delta\zeta\right)^{*}P_i\breve{\varphi}(t-\varrho)$$
$$-\breve{\varphi}^{*}(t-\varrho)P_i\int_{t-\varrho}^{t}\breve{\varphi}(\zeta)\Delta\zeta+\theta\breve{\varphi}^{*}(t-\varrho)P_i\breve{\varphi}(t-\varrho), \tag{11}$$

$$\begin{aligned}V_2^{\Delta}\big(\breve{\varphi}(t),i\big)&=\varrho\breve{\varphi}^{*}(t)W_1\breve{\varphi}(t)-\int_{-\varrho}^{0}\breve{\varphi}^{*}(t+\nu)W_1\breve{\varphi}(t+\nu)\Delta\nu\\&\leqslant\varrho\breve{\varphi}^{*}(t)W_1\breve{\varphi}(t)-\int_{t-\varrho}^{t}\breve{\varphi}^{*}(\zeta)W_1\breve{\varphi}(\zeta)\Delta\zeta\\&\leqslant\varrho\breve{\varphi}^{*}(t)W_1\breve{\varphi}(t)-\frac{1}{\varrho}\left(\int_{t-\varrho}^{t}\breve{\varphi}(\zeta)\Delta\zeta\right)^{*}W_1\left(\int_{t-\varrho}^{t}\breve{\varphi}(\zeta)\Delta\zeta\right),\end{aligned} \tag{12}$$

$$V_3^{\Delta}\big(\breve{\varphi}(t),i\big)=\big(\Delta^q\breve{\varphi}(t)\big)^{*}W_2\Delta^q\breve{\varphi}(t)-\big(\Delta^q\breve{\varphi}\big(t-\varkappa(t)\big)\big)^{*}W_2\Delta^q\breve{\varphi}\big(t-\varkappa(t)\big), \tag{13}$$

$$V_4^{\Delta}\big(\breve{\varphi}(t),i\big)=\varrho\big(\Delta^q\breve{\varphi}(t)\big)^{*}U_{33}\Delta^q\breve{\varphi}(t)-\int_{t-\varrho}^{t}\big(\Delta^q\breve{\varphi}(\zeta)\big)^{*}U_{33}\Delta^q\breve{\varphi}(\zeta)\Delta\zeta, \tag{14}$$

$$\begin{aligned}V_5^{\Delta}\big(\breve{\varphi}(t),i\big)&=\int_{t-\varrho(t)}^{t}\Theta^{*}(t,\zeta)U\Theta(t,\zeta)\Delta\zeta\\&\leqslant\breve{\varphi}^{*}(t)(\varrho U_{11}+U_{13}+U_{13}^{*})\breve{\varphi}(t)+\breve{\varphi}^{*}(t)(\varrho U_{12}-U_{13}+U_{23}^{*})\breve{\varphi}\big(t-\varrho(t)\big)\\&\quad+\breve{\varphi}^{*}\big(t-\varrho(t)\big)(\varrho U_{12}^{*}-U_{13}^{*}+U_{23})\breve{\varphi}(t)\\&\quad+\breve{\varphi}^{*}\big(t-\varrho(t)\big)(\varrho U_{22}-U_{23}-U_{23}^{*})\breve{\varphi}\big(t-\varrho(t)\big)\\&\quad+\int_{t-\varrho}^{t}\big(\Delta^q\breve{\varphi}(\zeta)\big)^{*}U_{33}\Delta^q\breve{\varphi}(\zeta)\Delta\zeta.\end{aligned} \tag{15}$$

By combining (8), (9), and (11), we have

$$\begin{aligned}V_1^{\Delta}\big(\breve{\varphi}(t),i\big)&\leqslant\theta\breve{\varphi}^{*}(t)P_i\breve{\varphi}+\breve{\varphi}^{*}(t)P_i\int_{t-\varrho}^{t}\breve{\varphi}(\zeta)\Delta\zeta-\theta\breve{\varphi}^{*}(t)P_i\breve{\varphi}(t-\varrho)\\&\quad-\theta\breve{\varphi}^{*}(t-\varrho)P_i\breve{\varphi}(t)\\&\quad+\left(\int_{t-\varrho}^{t}\breve{\varphi}(\zeta)\Delta\zeta\right)^{*}P_i\breve{\varphi}(t)-\left(\int_{t-\varrho}^{t}\breve{\varphi}(\zeta)\Delta\zeta\right)^{*}P_i\breve{\varphi}(t-\varrho)\\&\quad-\breve{\varphi}^{*}(t-\varrho)P_i\int_{t-\varrho}^{t}\breve{\varphi}(\zeta)\Delta\zeta+\theta\breve{\varphi}^{*}(t-\varrho)P_i\breve{\varphi}(t-\varrho)\end{aligned}$$

$$
\begin{aligned}
&+ \left(\int_{t-\varrho}^{t} \breve{\varphi}(\zeta) \Delta\zeta \right)^* \left(\sum_{j \in N_k^i} \gamma_{a_{ij}} (P_j - H_i) + \sum_{j \in N_k^i} \varepsilon_{ij} Q_{ij} \right) \\
&\times \left(\int_{t-\varrho}^{t} \breve{\varphi}(\zeta) \Delta\zeta \right).
\end{aligned} \tag{16}
$$

Lemma 1 is applied in obtaining inequalities (12) and (15) for $V_2^\Delta(\breve{\varphi}(t), i)$ and $V_5^\Delta(\breve{\varphi}(t), i)$. Furthermore, under condition (H1) and noting that S_1 and S_2 are real positive diagonal matrices, we obtain

$$
\begin{gathered}
0 \leqslant \breve{\varphi}^*(t) L S_1 L \breve{\varphi}(t) - g^*\big(\breve{\varphi}(t)\big) S_1 g\big(\breve{\varphi}(t)\big), \\
0 \leqslant \breve{\varphi}^*\big(t - \varrho(t)\big) L S_2 L \breve{\varphi}\big(t - \varrho(t)\big) - g^*\big(\breve{\varphi}\big(t - \varrho(t)\big)\big) S_2 g\big(\breve{\varphi}\big(t - \varrho(t)\big)\big).
\end{gathered} \tag{17}
$$

From the system model (2) it follows that the following equation must be identically satisfied:

$$
\begin{aligned}
0 = {} & \big(Z_1 \breve{\varphi}(t) + Z_2 \Delta^q \breve{\varphi}(t)\big)^* \big(-\Delta^q \breve{\varphi}(t) - G_i \breve{\varphi}(t) + F_i g\big(\breve{\varphi}(t)\big) \\
& + D_i g\big(\breve{\varphi}\big(t - \varrho(t)\big)\big) + E_i \Delta^q \breve{\varphi}\big(t - \varkappa(t)\big)\big) \\
& + \big(-\Delta^q \breve{\varphi}(t) - G_i \breve{\varphi}(t) + F_i g\big(\breve{\varphi}(t)\big) + D_i g\big(\breve{\varphi}\big(t - \varrho(t)\big)\big) \\
& + E_i \Delta^q \breve{\varphi}\big(t - \varkappa(t)\big)\big)^* \big(Z_1 \breve{\varphi}(t) + Z_2 \Delta^q \breve{\varphi}(t)\big).
\end{aligned} \tag{18}
$$

The derivatives presented in (12)–(16), together with the algebraic identities in (17)–(18), lead to

$$
\begin{aligned}
& V^\Delta\big(\breve{\varphi}(t), i\big) \\
& \quad \leqslant \breve{\varphi}^*(t)(-Z_1^* G_i - G_i^* Z_1 + P_i + \varrho U_{11} + U_{13} + U_{13}^* + \varrho W_1 + L S_1 L) \breve{\varphi}(t) \\
& \qquad + \breve{\varphi}^*(t)(-Z_1^* - G_x^* Z_2) \Delta^q \breve{\varphi}(t) + \breve{\varphi}^*(t) P_i \int_{t-\varrho}^{t} \breve{\varphi}(\zeta) \Delta\zeta + \big(\Delta^q \breve{\varphi}(t)\big)^* Z_2^* F_i g\big(\breve{\varphi}(t)\big) \\
& \qquad + \breve{\varphi}^*(t) Z_1^* F_i g\big(\breve{\varphi}(t)\big) + \breve{\varphi}^*(t) Z_1^* D_i g\big(\breve{\varphi}(t - \varrho)\big) + \breve{\varphi}^*(t) Z_1^* E_x \Delta^q \breve{\varphi}\big(t - \varkappa(t)\big) \\
& \qquad + \big(\Delta^q \breve{\varphi}(t)\big)^* (-Z_1 - Z_2^* G_i) \breve{\varphi}(t) + \big(\Delta^q \breve{\varphi}(t)\big)^* (W_2 + \varrho U_{33} - Z_2^* - Z_2) \Delta^q \breve{\varphi}(t) \\
& \qquad + \big(\Delta^q \breve{\varphi}(t)\big)^* Z_2^* D_i g\big(\breve{\varphi}(t - \varrho)\big) + \big(\Delta^q \breve{\varphi}(t)\big)^* Z_2^* E_i \Delta^q \breve{\varphi}\big(t - \varkappa(t)\big) \\
& \qquad + \left(\int_{t-\varrho}^{t} \breve{\varphi}(\zeta) \Delta\zeta \right)^* P_i \breve{\varphi}(t) + \breve{\varphi}^*(t)(\varrho U_{12} - U_{13} + U_{23}^* - \theta P_i) \breve{\varphi}(t - \varrho) \\
& \qquad + \left(\int_{t-\varrho}^{t} \breve{\varphi}(\zeta) \Delta\zeta \right)^* \left(-\frac{1}{\varrho} W_1 + \sum_{y \in N_k^i} \gamma_{a_{ij}} (P_j - H_i) + \sum_{j \in N_k^i} \varepsilon_{ij} Q_{ij} \right) \left(\int_{t-\vartheta}^{t} \breve{\varphi}(\zeta) \Delta\zeta \right)
\end{aligned}
$$

$$
\begin{aligned}
&- \left(\int\limits_{t-\varrho}^{t} \breve{\varphi}(\zeta)\Delta\zeta \right)^{*} P_i \breve{\varphi}(t-\varrho) + \breve{\varphi}(t-\varrho)^{*}(\varrho U_{12}^{*} - U_{13}^{*} + U_{23} - \theta P_i)\breve{\varphi}(t) \\
&- \breve{\varphi}^{*}(t-\varrho) P_i \left(\int\limits_{t-\varrho}^{t} \breve{\varphi}(\zeta)\Delta\zeta \right) \\
&+ \breve{\varphi}^{*}(t-\varrho)(\varrho U_{22} - U_{23} - U_{23}^{*} + \theta P_i + L S_2 L)\breve{\varphi}(t-\varrho) \\
&+ g^{*}\big(\breve{\varphi}(t)\big) F_i^{*} Z_1 \breve{\varphi}(t) + g^{*}\big(\breve{\varphi}(t)\big) F_i^{*} Z_2 \Delta^{q}\breve{\varphi}(t) - g^{*}\big(\breve{\varphi}(t)\big) S_1 g\big(\breve{\varphi}(t)\big) \\
&+ g^{*}\big(\breve{\varphi}\big(t-\varrho(t)\big)\big) D_i^{*} Z_1 \breve{\varphi}(t) + g^{*}\big(\breve{\varphi}(t-\varrho(t))\big) D_i^{*} Z_2 \Delta^{q}\breve{\varphi}(t) \\
&- g^{*}\big(\breve{\varphi}\big(t-\varrho(t)\big)\big) S_2 g\big(\breve{\varphi}\big(t-\varrho(t)\big)\big) + \big(\Delta^{q}\breve{\varphi}\big(t-\varkappa(t)\big)\big)^{*} E_i^{*} Z_1 \breve{\varphi}(t) \\
&+ \big(\Delta^{q}\breve{\varphi}\big(t-\varkappa(t)\big)\big)^{*} E_i^{*} Z_2 \Delta^{q}\breve{\varphi}(t) - \big(\Delta^{q}\breve{\varphi}\big(t-\varkappa(t)\big)\big)^{*} W_2 \Delta^{q}\breve{\varphi}\big(t-\varkappa(t)\big) \\
\leqslant{}& \phi^{*}(t) \Lambda \phi(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\phi(t) = \Bigg(&\breve{\varphi}^{*}(t),\ \big(\Delta^{q}\breve{\varphi}(t)\big)^{*},\ \left(\int\limits_{t-\varrho}^{t} \breve{\varphi}(\zeta)\Delta\zeta \right)^{*},\ \breve{\varphi}^{*}\big(t-\varrho(t)\big),\ g^{*}\big(\breve{\varphi}(t)\big), \\
&g^{*}\big(\breve{\varphi}\big(t-\varrho(t)\big)\big),\ \big(\Delta^{q}\breve{\varphi}\big(t-\varkappa(t)\big)\big)^{*} \Bigg).
\end{aligned}
$$

Given condition (4), which implies $V^{\Delta}(\breve{\varphi}(t), p) \leqslant 0$. Accordingly, one can identify a sufficiently small positive scalar c such that

$$
V^{\Delta}\big(\breve{\varphi}(t), p\big) \leqslant -c\big\|\breve{\varphi}(t)\big\|^{2}. \tag{19}
$$

Applying the Dynkin formula together with (19), we obtain

$$
\begin{aligned}
&\mathbf{E}\big\{V\big(\breve{\varphi}(r), r_t\big)\big\} - \mathbf{E}\big\{V\big(\breve{\varphi}(0), r_0\big)\big\} \\
&\quad = \mathbf{E}\Big\{ \int\limits_{0}^{t} V^{\Delta}\big(\breve{\varphi}(s), r_s\big)\,\Delta s \Big\} \leqslant -c\mathbf{E}\left\{ \int\limits_{0}^{t} \big\|\breve{\varphi}(s)\big\|^{2}\,\Delta s \right\}.
\end{aligned} \tag{20}
$$

From (20) we derive the inequality

$$
\begin{aligned}
c\mathbf{E}\left\{ \int\limits_{0}^{t} \big\|\breve{\varphi}(s)\big\|^{2}\,\Delta s \right\} &\leqslant \mathbf{E}\big\{V\big(\breve{\varphi}(0), r_0\big)\big\} - \mathbf{E}\big\{V\big(\breve{\varphi}(r), r_t\big)\big\} \\
&\leqslant \mathbf{E}\big\{V\big(\breve{\varphi}(0), r_0\big)\big\},
\end{aligned}
$$

which implies

$$
\mathbf{E}\left\{ \int\limits_{0}^{t} \big\|\breve{\varphi}(s)\big\|^{2}\,\Delta s \right\} \leqslant \frac{\mathbf{E}\{V(\breve{\varphi}(0), r_0)\}}{c} < \infty.
$$

By Definition 1, system (1) is stochastically stable. □

4 Example

To demonstrate the effectiveness and applicability of the derived theoretical results, we consider the following fractional-order quaternion-valued neural network with neutral delays and Markovian switching on the time scale $\mathbb{T} = \bigcup_{x=0}^{\infty}[2x,\, 2x+1]$:

$$\begin{aligned} \Delta^q \breve{\varphi}(t) &= -G_{r_t}\breve{\varphi}(t) + F_{r_t} g\bigl(\breve{\varphi}(t)\bigr) + D_{r_t} g\bigl(\breve{\varphi}\bigl(t-\varrho(t)\bigr)\bigr) \\ &\quad + E_{r_t}\Delta^q \breve{\varphi}\bigl(t-\varkappa(t)\bigr), \end{aligned} \tag{21}$$

where

$$\varrho(t) = 0.2\bigl|\sin(3t)\bigr|, \qquad \varkappa(t) = 0.04 + 0.02\bigl|\cos(4t)\bigr|,$$

$$g\bigl(\breve{\varphi}(t)\bigr) = 0.1\cdot\bigl(\bigl|\breve{\varphi}_a(t)\bigr| + \bigl|\breve{\varphi}_b(t)\bigr|i + \bigl|\breve{\varphi}_c(t)\bigr|j + \bigl|\breve{\varphi}_d(t)\bigr|k\bigr),$$

$$G_1 = G_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 0.0681{+}0.1480i{-}0.1151j{-}0.0812k & -0.1232{+}0.0801i{+}0.0841j{-}0.0962k \\ 0.0937{-}0.1578i{+}0.0968j{+}0.0925k & -0.1135{+}0.0909i{-}0.1062j{+}0.0621k \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.0123{+}0.0925i{-}0.0715j{-}0.0437k & 0.1024{-}0.0631i{-}0.0734j{+}0.0524k \\ 0.0928{-}0.0708i{+}0.0729j{+}0.0412k & 0.0941{-}0.0621i{+}0.0712j{-}0.0436k \end{bmatrix}$$

$$E_1 = \begin{bmatrix} -0.0711{+}0.0741i{+}0.0532j{+}0.0367k & -0.0532{+}0.0741i{+}0.0336j{+}0.0539k \\ -0.0527{+}0.0552i{-}0.0312j{-}0.0336k & 0.0534{-}0.0341i{-}0.0567j{-}0.0311k \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 0.0623{+}0.0328i{-}0.0516j{+}0.0320k & 0.1196{+}0.0640i{-}0.0742j{+}0.0402k \\ 0.0951{+}0.0643i{-}0.0703j{+}0.0419k & 0.0540{-}0.0318i{-}0.0891j{-}0.0311k \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.0307{+}0.0522i{+}0.0597j{-}0.0393k & 0.0802{-}0.0401i{+}0.0421j{+}0.0428k \\ 0.0623{+}0.0498i{-}0.0139j{-}0.0442k & -0.0506{-}0.0496i{+}0.0415j{-}0.0419k \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0.0317{-}0.0423i{+}0.0115j{-}0.0124k & -0.0108{+}0.0328i{+}0.0223j{-}0.0134k \\ 0.0320{-}0.0340i{+}0.0116j{-}0.0142k & -0.0218{-}0.0231i{-}0.0121j{-}0.0118k \end{bmatrix},$$

and the transition rate matrix $\varGamma = \left[\begin{smallmatrix} ? & ? \\ ? & [0.2,0.5] \end{smallmatrix}\right]$.

The Lipschitz constant matrix $L = \operatorname{diag}\{1,1\}$ clearly satisfies hypothesis (H1). Moreover, the computed values are $\gamma_{a_{22}} = 0.35$ and $\varepsilon_{22} = 0.15$ for the partially known transition rate.

By employing the MATLAB environment equipped with the YALMIP optimization toolbox and the SDPT3 numerical solver, we obtain the feasible solutions for linear matrix inequalities (3)–(7). These solutions are presented as follows:

$$P_1 = \begin{bmatrix} 121.8346 & 0 \\ 0 & 117.6154 \end{bmatrix}, \qquad H_1 = \begin{bmatrix} 65.8247 & 0 \\ 0 & 72.6381 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 72.4187 & 0 \\ 0 & 68.9471 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 339.5472 & 0 \\ 0 & 351.1724 \end{bmatrix}.$$

$$W_1 = \begin{bmatrix} 32.2361 & -1.8634{-}0.0289i{-}0.0452j{+}0.0354k \\ -1.8634{+}0.0289i{+}0.0452j{-}0.0354k & 34.8745 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 0.3093 & 0.0735{+}0.0386i{-}0.0821j{+}0.0624k \\ 0.0735{-}0.0386i{+}0.0821j{-}0.0624k & 2.2072 \end{bmatrix}$$

$$Z_1 = \begin{bmatrix} 2.3853{+}0.0165i{-}0.0028j{-}0.0251k & 0.6393{+}0.0143i{-}0.0274j{-}0.0105k \\ 0.6968{+}0.0352i{-}0.0021j{-}0.0063k & 10.944{+}0.0032i{-}0.0137j{-}0.0352k \end{bmatrix}$$

$$Z_2 = \begin{bmatrix} 2.3319{+}0.0384i{-}0.0028j{-}0.0467k & 0.6894{+}0.2251i{-}0.0274j{-}0.0337k \\ 0.7298{+}0.0481i{-}0.0137j{-}0.0190k & 11.3091{+}0.0032i{-}0.0137j{-}0.0164k \end{bmatrix},$$

$$U_{11} = \begin{bmatrix} 109.1346 & -3.5709{+}0.0241i{-}0.0042j{-}0.0172k \\ -3.5709{+}0.0037i{-}0.0132j{-}0.0011k & 96.9563 \end{bmatrix},$$

$$U_{12} = \begin{bmatrix} -18.3292{+}0.0015i{-}0.0036j{-}0.0021k & -2.2866{-}0.0024i{+}0.0325j{+}0.0015k \\ -5.6095{+}0.0028i{-}0.0382j{-}0.1253k & -29.1455{-}0.0024i{+}0.0054j{+}0.0032k \end{bmatrix},$$

$$U_{13} = \begin{bmatrix} 3.3086{-}0.0062i{+}0.0015j{+}0.0021k & -0.1777{-}0.0425i{+}0.0031j{+}0.0284k \\ -0.4887{+}0.0021i{-}0.0013j{-}0.0241k & 1.5384{+}0.0014i{-}0.0028j{-}0.0034k \end{bmatrix},$$

$$U_{22} = \begin{bmatrix} 136.0727 & -0.5522{-}0.0023i{+}0.0054j{+}0.0034k \\ -0.5522{+}0.0023i{-}0.0054j{-}0.0034k & 129.9018 \end{bmatrix},$$

$$U_{23} = \begin{bmatrix} 21.4328{+}0.0031i{-}0.0024j{-}0.0017k & 0.6414{-}0.0251i{+}0.0072j{+}0.0325k \\ 0.5421{+}0.0621i{-}0.0015j{-}0.0241k & 22.3472{-}0.0112i{+}0.0012j{+}0.0031k \end{bmatrix},$$

$$U_{33} = \begin{bmatrix} 4.0798 & 0.4523{-}0.1024i{+}0.0025j{+}0.0281k \\ 0.4523{+}0.1024i{-}0.0025j{-}0.0281k & -5.1036 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 32.4862 & 0 \\ 0 & 35.8324 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 21.1124 & 0 \\ 0 & 18.3697 \end{bmatrix}.$$

In light of the criteria established in Theorem 1, it can be inferred that system (21) achieves stochastic stability. With the prescribed initial conditions $\chi_1 = 1.2 + 0.8i - 0.8j + 0.6k$ and $\chi_2 = -0.8 - 0.6i + 0.6j - 0.4k$, the trajectories of the four state components and the corresponding modal switching sequences are visualized in Figs. 1 and 2, respectively. The simulation outcomes effectively confirm the rigor and practical applicability of the stability conditions derived in Theorem 1.

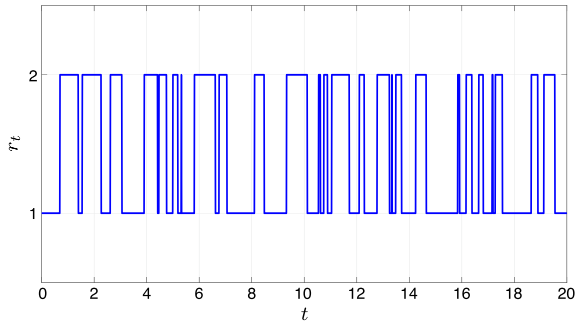

Figure 1. State trajectories of the four components of system (21).

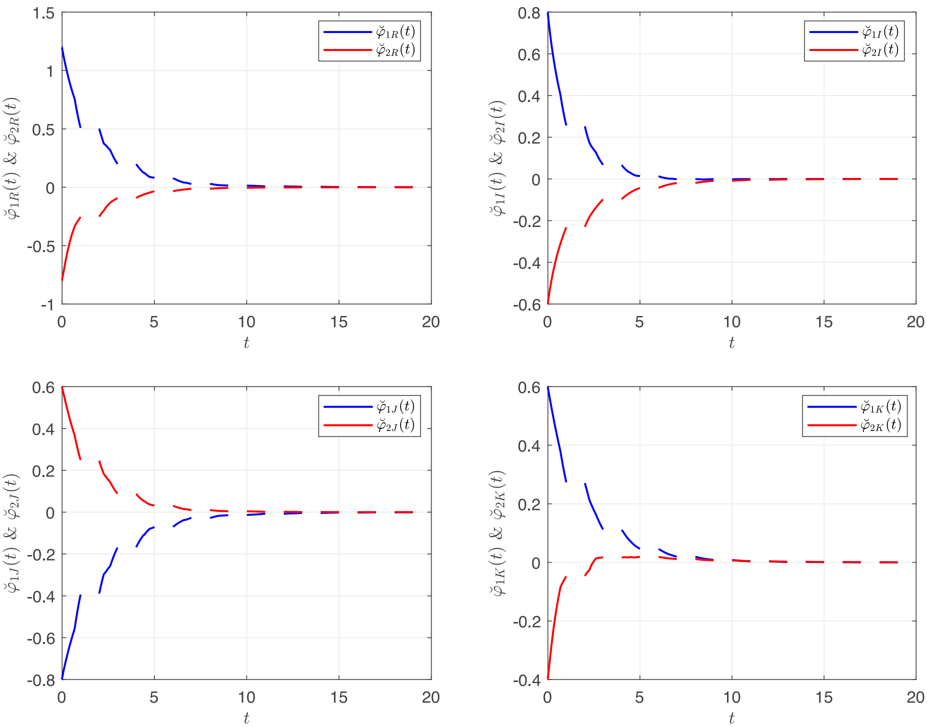

Figure 2. The modal switching trajectories for system (21).

5 Conclusions

In this paper, the stochastic stability problem of fractional-order quaternion-valued neural networks on time scales with general probabilistic bounded Markovian switching and neutral delays has been investigated. Unlike conventional decomposition methods, the proposed approach analyzes the system directly within the quaternion algebra, thereby preserving its intrinsic algebraic structure. Constructing a suitable Lyapunov–Krasovskii functional and employing the free-weighting matrix technique, sufficient conditions for stochastic stability have been derived in the form of quaternion-valued linear matrix inequalities. A numerical example has been provided to demonstrate the effectiveness of the theoretical results. Future research will extend this framework to synchronization and control problems, as well as applications in color image encryption.

Author contributions. All authors (C.H. and Q.S.) have contributed as follows: methodology, Q.S.; formal analysis, C.H.; software, C.H.; validation, C.H.; writing – original draft preparation, C.H. and Q.S.; writing – review & editing, C.H. and Q.S. All authors have read and approved the published version of the manuscript.

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