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The finite-time ruin probabilities of a dependent bidimensional risk model with subexponential claims and Brownian perturbations*

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Abstract. The paper considers a dependent bidimensional risk model with stochastic return and Brownian perturbations in which the price processes of the investment portfolio of the two lines of business are two geometric Lévy processes, and the claim-number processes of the two lines of business follows two different stochastic processes, which can be dependent. When the two components of each pair of claims from the two lines of business are strongly asymptotically independent and have subexponential distributions, the asymptotics of the finite-time ruin probability are obtained. Numerical studies are carried out to check the accuracy of the asymptotics of the finite-time ruin probability for the claims having regularly varying tail distributions.

Keywords: subexponential distribution, finite-time ruin probability, stochastic return, Lévy process.

1 Introduction

Consider an insurance company operating two lines of business, and the operator invests its wealth in financial assets. Assume that the two lines of business with varying levels of claims are exposed to similar catastrophic environments like car accident, earthquakes, or terrorist attack. In this case, there may exist some dependence structure among the claims

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of two lines of business. For example, a serious car accident may produce two types ofclaims, the one is car insurance claim, and the other is personal accident insurance claim, which leads to some dependent claims between the two lines of business. To describe the above situation, we consider a bidimensional continuous-time risk model with stochastic return on investment, the discounted value of the surplus process at time $t \geqslant 0$ can be expressed by

$$\begin{pmatrix} U_1(t) \\ U_2(t) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \int_{0-}^t \mathrm{e}^{-R_1(s)}\, C_1(\mathrm{d}s) \\ \int_{0-}^t \mathrm{e}^{-R_2(s)}\, C_2(\mathrm{d}s) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{N_1(t)} X_i \mathrm{e}^{-R_1(\tau_i^{(1)})} \\ \sum_{i=1}^{N_2(t)} Y_i \mathrm{e}^{-R_2(\tau_i^{(2)})} \end{pmatrix} + \begin{pmatrix} \delta_1 \int_{0-}^t \mathrm{e}^{-\widetilde{R}_1(s)}\, B_1(\mathrm{d}s) \\ \delta_2 \int_{0-}^t \mathrm{e}^{-\widetilde{R}_2(s)}\, B_2(\mathrm{d}s) \end{pmatrix}, \tag{1}$$

where $(x, y)^{\mathrm{T}}$ is the vector of the initial surpluses, and for any $0 \leqslant a \leqslant b < \infty$, the integral symbols $\int_{a-}^b$ and $\int_{[a,b]}$ are understood as $\int_a^b$ and $\int_{(a,b]}$, respectively. For the kth ($k = 1, 2$) line of business, $C_k(t) = \int_0^t c_k(s)\,\mathrm{d}s$ denotes the premium accumulation up to time $t \geqslant 0$, here $c_k(t)$ is the density function of premium income of the kth line of business at time $t \geqslant 0$, and the price process of the investment portfolio is a geometric Lévy process $\{\mathrm{e}^{R_k}(t),\, t \geqslant 0\}$, where $R_k(t)$ is a nonnegative Lévy process, which starts from zero and has independent and stationary increments. For more details about Lévy process, see Applebaum [1], Cont and Tankov [6], and Sato [18]. For $i \geqslant 1$, the random vector $(X_i, Y_i)^{\mathrm{T}}$ denotes the ith pair of claims from the two lines of business. Assume that $\{(X_i, Y_i)^{\mathrm{T}},\, i \geqslant 1\}$ is a sequence of independent and identically distributed (i.i.d.) random vectors with a generic random vector $(X, Y)^{\mathrm{T}}$, here X and Y are nonnegative random variables with distributions F_k, $k = 1, 2$, respectively. $\{B_k(t),\, t \geqslant 0\}$ is the Brownian perturbation of the kth line of business, and $\delta_k \geqslant 0$ is the corresponding volatility factor, $k = 1, 2$. $\{\widetilde{R}_k(t),\, t \geqslant 0\}$, $k = 1, 2$, are other nonnegative Lévy processes. $\{\tau_i^{(k)},\, i \geqslant 0\}$ denote claim-arrival times of the kth line of business with $\tau_0^{(k)} = 0$, $k = 1, 2$. The claim-number process of the kth line of business is constituted by $\{\tau_i^{(k)},\, i \geqslant 0\}$ as

$$N_k(t) = \sup\bigl\{i \geqslant 0\colon \tau_i^{(k)} \leqslant t\bigr\}, \quad t \geqslant 0,$$

with finite mean function $\lambda_k(t) = \mathbf{E}[N_k(t)]$, $t \geqslant 0$, $k = 1, 2$, where $\sup \emptyset = 0$ by convention. The interarrival times of the kth ($k = 1, 2$) line of business, $\{T_i^{(k)} = \tau_i^{(k)} - \tau_{i-1}^{(k)},\, i \geqslant 1\}$ are supposed to be nonnegative random variables. Assume that $\{(T_i^{(1)}, T_i^{(2)})^{\mathrm{T}},\, i \geqslant 1\}$ is a sequence of i.i.d. random vectors.

A kind of finite-time ruin probability at time $t \geqslant 0$ for the above risk model is defined as

$$\psi(x, y, t) = \mathbf{P}\Bigl(\inf_{0 \leqslant s \leqslant t} U_1(s) < 0,\ \inf_{0 \leqslant s \leqslant t} U_2(s) < 0 \Bigm| U_1(0) = x,\ U_2(0) = y\Bigr).$$

In the past decades, more and more scholars paid their attention to investigating multi-dimensional dependent risk models, especially, bidimensional ones. One research direction is to consider that the claims of each line of business have a dependence structure, such as Yang et al. [24], Cheng [2], Cheng et al. [4], Wang et al. [21], and so on. The other research direction is to investigate the case that there exists a dependence structure between the claims and the corresponding interarrival times in each line of business. For example, Jiang et al. [10], Li [11], Guo et al. [9], Wang et al. [19], Liu et al. [16], and so on. Recently, researchers consider that there exists a dependence structure between the claims from the two lines of business. Yang and Li [22] considered the risk model (1) with a constant interest, i.e., $R_1(t) = R_2(t) = rt$, $t \geqslant 0$, for some constant $r \geqslant 0$, $\delta_1 = \delta_2 = 0$, and $(X, Y)^{\mathrm{T}}$ follows a bivariate Farlie–Gumbel–Morgenstern (FGM) distribution. Yang and Yuen [26] extended the results of Yang and Li [22] to the case where $(X, Y)^{\mathrm{T}}$ follows a bivariate Sarmanov distribution. Li and Yang [15] considered the risk model (1) with a constant interest and $(X, Y)^{\mathrm{T}}$ following a copula, which can include the FGM copula. Li [12] improved the results of Yang and Li [22], considering that X and Y are strongly asymptotically independent (SAI), i.e., there is some constant $\rho > 0$ such that

$$\mathbf{P}(X > x,\, Y > y) \sim \rho \overline{F_1}(x)\overline{F_2}(y) \quad \text{as } (x, y) \to (\infty, \infty). \tag{2}$$

The SAI structure was introduced by Li [13]. Li [12] pointed out that some commonly used copulas satisfy (2), for example, the FGM copula, the Frank copula, and the Ali–Mikhail–Haq copula. For this dependence structure, Cheng et al. [3] studied the risk model (1) with a constant interest and perturbations. Li [14] considered that the investment portfolios of the two lines of business have stochastic return, and the price processes of the investment portfolios of two lines of business are a same geometric Lévy process. Specifically, Li [14] investigated the risk model (1) for the case that $R_1(t) = R_2(t)$, $t \geqslant 0$, $C_k(t) = c_k t$, $t \geqslant 0$ for some constant $c_k > 0$, $k = 1, 2$, $N_1(t) = N_2(t)$, $t \geqslant 0$, and $\delta_1 = \delta_2 = 0$, and gave the asymptotics of a finite-time ruin probability for regularly varying claims. This paper still investigates X and Y are SAI and considers the two lines of business have different stochastic return and claim-number processes, which can be dependent. The paper also discusses the influence of the perturbations on the ruin probability and gives the asymptotics of the finite-time ruin probability for all subexponential claims.

As usual, assume that $\{(X, Y)^{\mathrm{T}}, (X_i, Y_i)^{\mathrm{T}}, i \geqslant 1\}$, $\{(T_i^{(1)}, T_i^{(2)})^{\mathrm{T}}, i \geqslant 1\}$, $\{R_1(t), t \geqslant 0\}$, $\{R_2(t), t \geqslant 0\}$, $\{\widetilde{R}_1(t), t \geqslant 0\}$, $\{\widetilde{R}_2(t), t \geqslant 0\}$, $\{B_1(t), t \geqslant 0\}$, and $\{B_2(t), t \geqslant 0\}$ are mutually independent, and $0 \leqslant c_k(t) \leqslant M_0$ for some constant $M_0 > 0$ and for all $t \geqslant 0$ and $k = 1, 2$.

The rest of the paper is organized as follows. Section 2 provides some preliminaries. The main results are presented in Section 3. Section 4 conducts some numerical simulations to check the accuracy of the main results. Section 5 gives some lemmas and the proofs of main results.

2 Preliminaries

Hereafter, all limit relationships hold as $(x, y)^{\mathrm{T}} \to (\infty, \infty)^{\mathrm{T}}$, unless otherwise stated. For two positive bivariate functions $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$, we write $a(x, y) \lesssim b(x, y)$ or

$b(x,y) \gtrsim a(x,y)$ if $\limsup a(x,y)/b(x,y) \leqslant 1$; $a(x,y) \sim b(x,y)$ if $\lim a(x,y)/b(x,y) = 1$; $a(x,y) = o(1)b(x,y)$ if $\lim a(x,y)/b(x,y) = 0$; and $a(x,y) = O(1)b(x,y)$ if $\limsup a(x,y)/b(x,y) < \infty$. For any real numbers x, y, define $x \wedge y = \min\{x,y\}$ and $x \vee y = \max\{x,y\}$.

In this paper, we mainly consider the claims with heavy-tailed distributions. We recall the definition of heavy-tailed distribution and introduce some distribution classes with heavy-tailed distributions. For a real-valued random variable ξ with a proper distribution V, say that V (or ξ) is heavy-tailed if $\mathbf{E}\mathrm{e}^{\gamma\xi} = \infty$ for all $\gamma > 0$.

For a distribution V on $(-\infty,\infty)$, say that V belongs to the dominated varying tail distribution class, denoted by $V \in \mathcal{D}$, if for any $0 < u < 1$,

$$\limsup_{x\to\infty} \frac{\overline{V}(xu)}{\overline{V}(x)} < \infty.$$

Associated with the class $\mathcal{D}$ is the long-tailed distribution class $\mathcal{L}$. Say that a distribution V on $(-\infty,\infty)$ belongs to the class $\mathcal{L}$ if for any $u > 0$,

$$\lim_{x\to\infty} \frac{\overline{V}(x+u)}{\overline{V}(x)} = 1.$$

Say that a distribution V on $(-\infty,\infty)$ belongs to the regularly varying tail distribution class $\mathcal{R}_{-\alpha}$ for some $\alpha \geqslant 0$ if

$$\lim_{x\to\infty} \frac{\overline{V}(xu)}{\overline{V}(x)} = u^{-\alpha}$$

holds for any fixed $u > 0$.

One of the most important heavy-tailed distribution classes is the subexponential distribution class $\mathcal{S}$. Say that a distribution V on $[0,\infty)$ belongs to the subexponential distribution class $\mathcal{S}$ if

$$\lim_{x\to\infty} \frac{\overline{V^{*n}}(x)}{\overline{V}(x)} = n$$

holds for some (or, equivalently, for all) $n \geqslant 2$, where V^{*n} represents the n-fold convolution of V, $n \geqslant 2$. Moreover, say that a distribution V on $(-\infty,\infty)$ belongs to the class $\mathcal{S}$ if $V(x)\mathbf{1}_{\{x\geqslant 0\}}$ belongs to the class $\mathcal{S}$, where $\mathbf{1}_{\{A\}}$ is the indicator function of the set A. If $V \in \mathcal{L}$, then for any $\epsilon > 0$, it holds that

$$\mathrm{e}^{-\epsilon x} = o(1)\overline{V}(x) \quad \text{as } x \to \infty \tag{3}$$

(see, e.g., Lemma 1.3.5(b) of Embrechts et al. [7]).

The above heavy-tailed distribution subclasses have the following relation:

$$\mathcal{R}_{-\alpha} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L}.$$

For more details about heavy-tailed distribution classes, one can see Embrechts et al. [7] and Foss et al. [8].

3 Main results

In the following, we present the main results of this paper in which the two claim-number processes of the two lines of business can be dependent, and the risk model has perturbations and subexponential claims.

Theorem 1. *Consider the risk model* (1). *Assume that* X *and* Y *are SAI with constant* $\rho > 0$ *and* $F_k \in \mathcal{S}$, $k = 1, 2$. *Assume also that* $T_1^{(1)}$ *and* $T_1^{(2)}$ *are independent. Then for any fixed* $t > 0$ *satisfying* $\mathbf{P}(T_1^{(k)} \leqslant t) > 0$, $k = 1, 2$, *it holds that*

$$\psi(x, y, t) \sim \int_{0^-}^{t}\int_{0^-}^{t} \mathbf{P}\bigl(X\mathrm{e}^{-R_1(s)} > x\bigr)\mathbf{P}\bigl(Y\mathrm{e}^{-R_2(p)} > y\bigr)\,\lambda(\mathrm{d}s, \mathrm{d}p) + (\rho - 1)\int_{0^-}^{t}\int_{0^-}^{t} \mathbf{P}\bigl(X\mathrm{e}^{-R_1(s)} > x\bigr)\mathbf{P}\bigl(Y\mathrm{e}^{-R_2(p)} > y\bigr)\tilde{\lambda}(\mathrm{d}s, \mathrm{d}p),$$

where

$$\lambda(s, p) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathbf{P}\bigl(\tau_i^{(1)} \leqslant s,\ \tau_j^{(2)} \leqslant p\bigr), \quad s \geqslant 0,\ p \geqslant 0,$$

and

$$\tilde{\lambda}(s, p) = \sum_{i=1}^{\infty} \mathbf{P}\bigl(\tau_i^{(1)} \leqslant s,\ \tau_i^{(2)} \leqslant p, \quad s \geqslant 0,\ p \geqslant 0.$$

Remark 1. It follows from Lemma 4.3 of Yang and Li [23] that for any $s \geqslant 0$ and $p \geqslant 0$,

$$\mathbf{E}\bigl[N_1(s)N_2(p)\bigr] = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathbf{P}\bigl(\tau_i^{(1)} \leqslant s,\ \tau_j^{(2)} \leqslant p\bigr).$$

Thus, if $\mathbf{P}(T_1^{(k)} \leqslant t) > 0$ for some $t \geqslant 0$ and each $k = 1, 2$, then

$$\mathbf{E}\bigl[N_1(t)N_2(t)\bigr] \geqslant \mathbf{P}\bigl(T_1^{(1)} \leqslant t,\ T_1^{(2)} \leqslant t\bigr) = \mathbf{P}\bigl(T_1^{(1)} \leqslant t\bigr)\mathbf{P}\bigl(T_1^{(2)} \leqslant t\bigr) > 0.$$

Remark 2. If the risk model (1) has a constant interest $r \geqslant 0$ and $\{X_i,\ i \geqslant 1\}$ and $\{Y_i,\ i \geqslant 1\}$ are independent, i.e., $R_1(t) = R_2(t) = rt$, $t \geqslant 0$, and $\rho = 1$, then from Theorem 1 it holds that

$$\psi(x, y, t) \sim \int_{0^-}^{t}\int_{0^-}^{t} \overline{F_1}\bigl(x\mathrm{e}^{rs}\bigr)\overline{F_2}\bigl(y\mathrm{e}^{rp}\bigr)\lambda(\mathrm{d}s, \mathrm{d}p),$$

which extends Theorem 2.1 of Yang and Li [23] to the risk model with Brownian perturbations.

Remark 3. If $R_1(t) = R_2(t) = \widetilde{R}_1(t) = \widetilde{R}_2(t) = rt$, $t \geqslant 0$, and $r \geqslant 0$ is a constant interest, the result of Theorem 1 is same as the finite-time ruin probability $\psi_{and}(x, y)$ in Cheng et al. [3] for a risk model with Brownian perturbations.

Remark 4. As pointed out by a referee, Cheng et al. [3] used a general stochastic process to describe the perturbation. Let the perturbations $\{B_k(t),\, t \geqslant 0\}$, $k = 1, 2$, be two general stochastic processes with $B_k(0) = 0$, $k = 1, 2$, in (1). If for any fixed $t > 0$, all $c > 0$, and each $k = 1, 2$,

$$\mathbf{P}\Biggl(\sup_{0\leqslant v\leqslant t} \delta_k \int_{0-}^{v} \mathrm{e}^{-\widetilde{R}_k(s)}\, B_k(\mathrm{d}s) > x\Biggr) = o(1)\overline{F_k}(cx), \tag{4}$$

then in the same way as in the proof of Theorem 1, the result of Theorem 1 still holds. In fact, the proof of Theorem 1 relies on (20) to deal with the Brownian perturbation. If $\{B_k(t),\, t \geqslant 0\}$, $k = 1, 2$, are two general stochastic processes, then (20) can be guaranteed by (4).

Particularly, if the claims have regularly varying tail distributions, then the following can be obtained from Theorem 1 immediately.

Corollary 1. *Under the conditions of Theorem* 1*, suppose that* $F_k \in \mathcal{R}_{-\alpha}$ *for some* $\alpha \geqslant 0$ *and* $k = 1, 2$*,* $R_1(t) = R_2(t) = rt$*,* $t \geqslant 0$*, for some constant* $r \geqslant 0$*. If for all* $t \geqslant 0$*,* $N_1(t) = N_2(t) = N(t)$ *with finite mean function* $\lambda(t) = \mathbf{E}[N(t)]$*, then*

$$\psi(x, y, t) \sim \overline{F_1}(x)\overline{F_2}(y)\Biggl(\int_{0-}^{t}\int_{0-}^{t-s} 2\mathrm{e}^{-2r\alpha s - r\alpha p}\,\lambda(\mathrm{d}p)\,\lambda(\mathrm{d}s) + \rho\int_{0-}^{t} \mathrm{e}^{-2r\alpha s}\,\lambda(\mathrm{d}s)\Biggr). \tag{5}$$

4 Numerical simulations

In this section, we use Monte Carlo method to verify the accuracy of asymptotic estimation of Corollary 1. We assume that X and Y have a common Pareto distribution

$$F(x) = \begin{cases} 1 - (\frac{a}{a+x})^b, & x > a,\\ 0, & x \leqslant a, \end{cases}$$

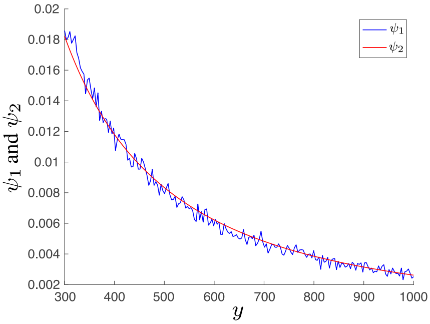
with the scale parameter $a = 1$ and the shape parameter $b = 1.32$. Therefore, $F_1 = F_2 = F \in \mathcal{R}_{-1.32}$. As noted in the introduction, the FGM copula satisfies SAI. We assume that $(X, Y)^{\mathrm{T}}$ has a FGM copula with the following joint distribution:

$$\pi(x, y) = F_1(x)F_2(y)\bigl(1 + \gamma\overline{F_1}(x)\overline{F_2}(y)\bigr), \quad \gamma \in [-1, 1],$$

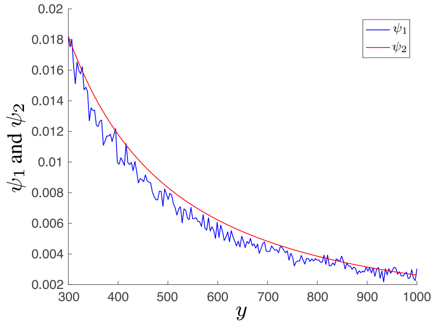
and the corresponding copula has the following form:

$$C(u, v) = uv\bigl(1 + \gamma(1 - u)(1 - v)\bigr), \quad u, v \in [0, 1], \tag{6}$$

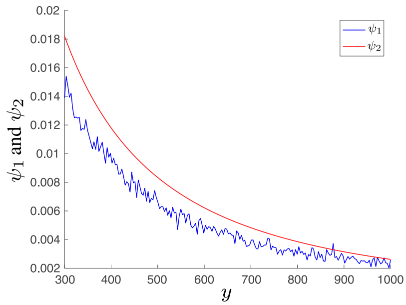
where γ is a dependence coefficient. From Example 2.1 of Li [14] we know that X and Y are SAI with a constant $\rho = 1+\gamma$. For more details about copula, one can see Nelsen [17]. In our simulation, we set $\gamma = 0.2$, $\gamma = 0.5$, and $\gamma = 0.8$, respectively. Suppose that $\{N(t),\, t \geqslant 0\}$ is a Poisson counting process generated by $\tau_i = \sum_{l=1}^{i} T_l$, $i \geqslant 1$, and with



(a) $t = 10,\ \gamma = 0.2,\ x = 300$



(b) $t = 10,\ \gamma = 0.5,\ x = 300$



(c) $t = 10,\ \gamma = 0.8,\ x = 300$

Figure 1. Accuracy of the asymptotic estimate.

$\mathbf{E}[N(t)] = \lambda t$, $t \geqslant 0$, where $\lambda = 5$. Set the premium $C_1(t) = C_2(t) = 1.8t$, $t \geqslant 0$, the rate $r = 0.03$, $n = 200$, $N = 50000$. Fix the initial capital $x = 300$, and let the initial capital y change from 300 to 1000 in a step 3.5. Divide $t = 10$ into n parts. The algorithm of the kth test can be written as follows.

1. Generate random pairs $(u, v)^{\mathrm{T}}$ by the FGM copula (6);

2. Generate X by $F^{-1}(u)$ from (u, v), and generate Y by $F^{-1}(v)$ from (u, v);

3. Generate T_l, where the distribution of T_l is an exponential distribution with a parameter of 0.2 and $\tau_i = \sum_{l=1}^{i} T_l$, $i = 1, 2, \dots, N$;

4. If $\tau_i > t$, $i = 1, 2, \dots, N$, then the test is terminated. Otherwise, calculate $\sum_{l=1}^{i} X_l \mathrm{e}^{-r\tau_l}$ and $\sum_{l=1}^{i} Y_l \mathrm{e}^{-r\tau_l}$, $i = 1, 2, \dots, N$;

5. Calculate $B_k := \int_{0-}^{t} \mathrm{e}^{-rs} C_k(\mathrm{d}s) = 1.8t\,(1/r - \mathrm{e}^{-rt}/r)$, where $k = 1, 2$. If $\min_i\{x - \sum_{l=1}^{i} X_l \mathrm{e}^{-r\tau_l} + B_1\} < 0$ and $\min_i\{y - \sum_{l=1}^{i} Y_l \mathrm{e}^{-r\tau_l} + B_2\} < 0$ both hold at the same time, then write $\pi_k = 1$, otherwise write $\pi_k = 0$;

6. Calculate the ruin probability $\psi_1(x, y, t) = \sum_{k=1}^{N} \pi_k / N$. $\psi_2(x, y, t)$ is the right side of (5). We can calculate it directly by the specific parameters. Based on the above algorithm, by changing the dependence coefficients γ we obtain the graphs of $\psi_1(x, y, t)$ and $\psi_2(x, y, t)$. Observing Figs. 1(a)–1(c), we can note that the values of ruin probability decrease with the increasing of initial capital y. Though Figs. 1(a)–1(c) have different

dependence coefficients, with the increasing of initial capital y, the values of $\psi_1(x,y,t)$ and $\psi_2(x,y,t)$ get closer in a certain range. Figures 1(a)–1(c) also show that the simulations fit well with the asymptotic function, which verifies the accuracy of the asymptotic estimation in a specific context.

5 Proofs of main results

5.1 Some lemmas

Before proving the main results, we first give some useful lemmas. By the proof of Lemma 3.4 of Cheng et al. [3] the following lemma can be obtained.

Lemma 1. *Let $\{(X,Y)^{\mathrm{T}},(X_i,Y_i)^{\mathrm{T}},\ i\geqslant 1\}$ be a sequence of i.i.d. nonnegative random vectors with generic marginal distributions $F\in\mathcal{S}$ and $G\in\mathcal{S}$, respectively. Assume that X and Y are SAI with constant $\rho>0$. Let ξ and η be two independent real-valued random variables with distributions F_ξ and F_η, respectively, and be independent of all other random sources. If*

$$\overline{F_\xi}(x)=o(1)\overline{F}\left(\frac{x}{a}\right)\quad \text{and}\quad \overline{F_\eta}(y)=o(1)\overline{G}\left(\frac{y}{a}\right)$$

for some $a>0$, then for any fixed $b\geqslant a$, $n\geqslant 1$, and $m\geqslant 1$,

$$\mathbf{P}\left(\sum_{i=1}^{n}c_iX_i+\xi>x,\ \sum_{j=1}^{m}d_jY_j+\eta>y\right)\sim\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{P}\left(X_i>\frac{x}{c_i},\ Y_j>\frac{y}{d_j}\right)$$

holds uniformly for all $(c_1,\dots,c_n)\in[a,b]^n$ and $(d_1,\dots,d_m)\in[a,b]^m$.

The next lemma follows from Corollary 2.5 of Cline and Samorodnitsky [5].

Lemma 2. *Let ξ and θ be two nonnegative independent random variables. If the distribution of ξ is subexponential and θ is a bounded random variable, then the distribution of $\theta\xi$ is subexponential.*

The next lemma is Lemma 3.3 of Cheng et al. [3].

Lemma 3. *Let $\{(X,Y)^{\mathrm{T}},(X_i,Y_i)^{\mathrm{T}},\ i\geqslant 1\}$ be a sequence of i.i.d. nonnegative random vectors with generic marginal distributions $F\in\mathcal{S}$ and $G\in\mathcal{S}$, respectively. Assume that ξ and η are two independent real-valued random variables with distributions F_ξ and F_η, respectively, and independent of all other random sources. If*

$$\mathbf{P}(X>x,\ Y>y)=O(1)\overline{F}(x)\overline{G}(y),$$

$\overline{F_\xi}(x)=O(1)\overline{F}(x)$, and $\overline{F_\eta}(y)=O(1)\overline{G}(y)$, then for any $\varepsilon>0$, there exists a positive constant $K(F,G,F_\xi,F_\eta)$ such that for all $x\geqslant 0$, $y\geqslant 0$, $n\geqslant 1$, and $m\geqslant 1$,

$$\mathbf{P}\left(\sum_{i=1}^{n}X_i+\xi>x,\ \sum_{j=1}^{m}Y_j+\eta>y\right)\leqslant K(1+\varepsilon)^{n+m}\overline{F}(x)\overline{G}(y).$$

Lemma 4. *Let $\{(X,Y)^{\mathrm{T}}, (X_i,Y_i)^{\mathrm{T}}, i \geqslant 1\}$ be a sequence of i.i.d. nonnegative random vectors with generic marginal distributions $F \in \mathcal{S}$ and $G \in \mathcal{S}$. Assume that X and Y are SAI with constant $\rho > 0$. Let $\{(\theta_i^{(1)}, \theta_i^{(2)})^{\mathrm{T}}, i \geqslant 1\}$ be a sequence of nonnegative and upper bounded random variables, but not degenerate at zero. Assume also that $\{(X,Y)^{\mathrm{T}}, (X_i,Y_i)^{\mathrm{T}}, i \geqslant 1\}$, $\{\theta_i^{(1)}, i \geqslant 1\}$, and $\{\theta_i^{(2)}, i \geqslant 1\}$ are mutually independent. Then for any fixed $n \geqslant 1$ and $m \geqslant 1$,*

$$\mathbf{P}\left(\sum_{i=1}^{n}\theta_i^{(1)}X_i > x,\ \sum_{j=1}^{m}\theta_j^{(2)}Y_j > y\right) \sim \sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{P}\big(\theta_i^{(1)}X_i > x,\ \theta_j^{(2)}Y_j > y\big).$$

Proof. We first prove that for any fixed $n \geqslant 1$ and $m \geqslant 1$,

$$\mathbf{P}\left(\sum_{i=1}^{n}\theta_i^{(1)}X_i > x,\ \sum_{j=1}^{m}\theta_j^{(2)}Y_j > y\right) \gtrsim \sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{P}\big(\theta_i^{(1)}X_i > x,\ \theta_j^{(2)}Y_j > y\big). \tag{7}$$

Without loss of generality, we assume that $\theta_i^{(k)} \leqslant 1$, $k = 1,2$, $i \geqslant 1$. Applying Bonferroni's inequality, it holds that for any fixed $n \geqslant 1$, $m \geqslant 1$, $x > 0$, and $y > 0$,

$$\begin{aligned}
&\mathbf{P}\left(\sum_{i=1}^{n}\theta_i^{(1)}X_i > x,\ \sum_{j=1}^{m}\theta_j^{(2)}Y_j > y\right)\\
&\quad\geqslant \sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{P}\big(\theta_i^{(1)}X_i > x,\ \theta_j^{(2)}Y_j > y\big)\\
&\qquad - \sum_{1\leqslant i\neq k\leqslant n}\sum_{j=1}^{m}\mathbf{P}\big(\theta_i^{(1)}X_i > x,\ \theta_k^{(1)}X_k > x,\ \theta_j^{(2)}Y_j > y\big)\\
&\qquad - \sum_{1\leqslant l\neq j\leqslant m}\sum_{i=1}^{n}\mathbf{P}\big(\theta_i^{(1)}X_i > x,\ \theta_l^{(2)}Y_l > y,\ \theta_j^{(2)}Y_j > y\big)\\
&\quad=: \sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{P}\big(\theta_i^{(1)}X_i > x,\ \theta_j^{(2)}Y_j > y\big) - I_1(x,y) - I_2(x,y).
\end{aligned} \tag{8}$$

In the following, we first deal with $I_1(x,y)$. For any $x > 0$ and $y > 0$,

$$\begin{aligned}
I_1(x,y) &= \left(\sum_{i=1}^{n}\sum_{\substack{k=1\\k\neq i}}^{n}\sum_{\substack{j=1\\j\neq i\\j\neq k}}^{m} + \sum_{i=1}^{n}\sum_{\substack{k=1\\k\neq i}}^{n}\sum_{1\leqslant j=k\leqslant m} + \sum_{i=1}^{n}\sum_{\substack{k=1\\k\neq i}}^{n}\sum_{1\leqslant j=i\leqslant m}\right)\\
&\qquad\times \mathbf{P}\big(\theta_i^{(1)}X_i > x,\ \theta_k^{(1)}X_k > x,\ \theta_j^{(2)}Y_j > y\big)\\
&=: I_{11}(x,y) + I_{12}(x,y) + I_{13}(x,y).
\end{aligned} \tag{9}$$

Since $\theta_k^{(1)} \leqslant 1$, $k \geqslant 1$, and $\{\theta_i^{(1)}, i \geqslant 1\}$, $\{\theta_j^{(2)}, j \geqslant 1\}$, and $\{(X_i, Y_i)^{\mathrm{T}}, i \geqslant 1\}$ are mutually independent, it holds that

$$\begin{aligned} I_{11}(x,y) &\leqslant \sum_{i=1}^{n} \sum_{\substack{k=1\\k\neq i}}^{n} \sum_{\substack{j=1\\j\neq i\\j\neq k}}^{m} \mathbf{P}\big(\theta_i^{(1)} X_i > x,\ X_k > x,\ \theta_j^{(2)} Y_j > y\big) \\ &\leqslant n\overline{F}(x) \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}\big(\theta_i^{(1)} X_i > x,\ \theta_j^{(2)} Y_j > y\big) \\ &= o(1) \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}\big(\theta_i^{(1)} X_i > x,\ \theta_j^{(2)} Y_j > y\big). \end{aligned} \tag{10}$$

For $I_{12}(x,y)$, if $n \leqslant m$, then by $\theta_i^{(1)} \leqslant 1$, $i \geqslant 1$, and the mutual independence of $\{\theta_i^{(1)}, i \geqslant 1\}$, $\{\theta_j^{(2)}, j \geqslant 1\}$, and $\{(X_i, Y_i)^{\mathrm{T}}, i \geqslant 1\}$

$$\begin{aligned} I_{12}(x,y) &\leqslant \sum_{i=1}^{n} \sum_{\substack{k=1\\k\neq i}}^{n} \mathbf{P}\big(X_i > x,\ \theta_k^{(1)} X_k > x,\ \theta_k^{(2)} Y_k > y\big) \\ &\leqslant n\overline{F}(x) \sum_{k=1}^{n} \mathbf{P}\big(\theta_k^{(1)} X_k > x,\ \theta_k^{(2)} Y_k > y\big) \\ &= o(1) \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}\big(\theta_i^{(1)} X_i > x,\ \theta_j^{(2)} Y_j > y\big). \end{aligned} \tag{11}$$

If $n > m$, then by $\theta_i^{(1)} \leqslant 1$, $i \geqslant 1$, and the mutual independence of $\{\theta_i^{(1)}, i \geqslant 1\}$, $\{\theta_j^{(2)}, j \geqslant 1\}$, and $\{(X_i, Y_i)^{\mathrm{T}}, i \geqslant 1\}$

$$\begin{aligned} I_{12}(x,y) &\leqslant n\overline{F}(x) \sum_{k=1}^{m} \mathbf{P}\big(\theta_k^{(1)} X_k > x,\ \theta_k^{(2)} Y_k > y\big) \\ &= o(1) \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}\big(\theta_i^{(1)} X_i > x,\ \theta_j^{(2)} Y_j > y\big). \end{aligned} \tag{12}$$

Similarly, it holds that

$$I_{13}(x,y) = o(1) \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}\big(\theta_i^{(1)} X_i > x,\ \theta_j^{(2)} Y_j > y\big). \tag{13}$$

By (9)–(13) it holds that

$$I_{1}(x,y) = o(1) \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}\big(\theta_i^{(1)} X_i > x,\ \theta_j^{(2)} Y_j > y\big). \tag{14}$$

It follows from the same way of the proof of $I_1(x,y)$ that

$$I_2(x,y) = o(1)\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{P}\bigl(\theta_i^{(1)}X_i > x,\ \theta_j^{(2)}Y_j > y\bigr),$$

which, combined with (8) and (14), gives that (7) holds.

Now we show that for any fixed $n \geqslant 1$ and $m \geqslant 1$,

$$\mathbf{P}\left(\sum_{i=1}^{n}\theta_i^{(1)}X_i > x,\ \sum_{j=1}^{m}\theta_j^{(2)}Y_j > y\right) \lesssim \sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{P}\bigl(\theta_i^{(1)}X_i > x,\ \theta_j^{(2)}Y_j > y\bigr). \quad (15)$$

We first assume that the random variables $\theta_i^{(1)}$, $\theta_j^{(2)}$, $i,j \geqslant 1$, are positive. For any subsets $I \subset \mathbb{I} = \{1,\dots,n\}$, $J \subset \mathbb{J} = \{1,\dots,m\}$ and any $0 < \varepsilon_1, \varepsilon_2 < 1$, write $I^{\mathrm{c}} = \mathbb{I}\setminus I,\ J^{\mathrm{c}} = \mathbb{J}\setminus J,$

$$\varOmega_I^{\varepsilon_1}\bigl(\theta^{(1)}\bigr) = \bigl\{\omega_1\colon\ \theta_i^{(1)} > \varepsilon_1 \text{ for } i \in I, \text{ and } \theta_q^{(1)} \leqslant \varepsilon_1 \text{ for } q \in I^{\mathrm{c}}\bigr\}$$

and

$$\varOmega_J^{\varepsilon_2}\bigl(\theta^{(2)}\bigr) = \bigl\{\omega_2\colon\ \theta_j^{(2)} > \varepsilon_2 \text{ for } j \in J, \text{ and } \theta_p^{(2)} \leqslant \varepsilon_2 \text{ for } p \in J^{\mathrm{c}}\bigr\}.$$

Since $\{(X_i,Y_i)^{\mathrm{T}},\ i \geqslant 1\}$, $\{\theta_i^{(1)},\ i \geqslant 1\}$, and $\{\theta_i^{(2)},\ i \geqslant 1\}$ are mutually independent, by Lemma 1 for $\xi = \eta \equiv 0$, it holds that

$$\begin{aligned}
&\mathbf{P}\left(\sum_{i=1}^{n}\theta_i^{(1)}X_i > x,\ \sum_{j=1}^{m}\theta_j^{(2)}Y_j > y\right)\\
&\quad\sim \sum_{I\subset\mathbb{I}}\sum_{J\subset\mathbb{J}}\Biggl[\sum_{i\in I}\sum_{j\in J}\mathbf{P}\bigl(\theta_i^{(1)}X_i > x,\ \varOmega_I^{\varepsilon_1}\bigl(\theta^{(1)}\bigr),\ \theta_j^{(2)}Y_j > y,\ \varOmega_J^{\varepsilon_2}\bigl(\theta^{(2)}\bigr)\bigr)\\
&\qquad+\sum_{i\in I}\sum_{p\in J^{\mathrm{c}}}\mathbf{P}\bigl(\theta_i^{(1)}X_i > x,\ \varepsilon_2 Y_p > y,\ \varOmega_I^{\varepsilon_1}\bigl(\theta^{(1)}\bigr),\ \varOmega_J^{\varepsilon_2}\bigl(\theta^{(2)}\bigr)\bigr)\\
&\qquad+\sum_{q\in I^{\mathrm{c}}}\sum_{j\in J}\mathbf{P}\bigl(\varepsilon_1 X_q > x,\ \theta_j^{(2)}Y_j > y,\ \varOmega_I^{\varepsilon_1}\bigl(\theta^{(1)}\bigr),\ \varOmega_J^{\varepsilon_2}\bigl(\theta^{(2)}\bigr)\bigr)\\
&\qquad+\sum_{q\in I^{\mathrm{c}}}\sum_{p\in J^{\mathrm{c}}}\mathbf{P}\bigl(\varepsilon_1 X_q > x,\ \varepsilon_2 Y_p > y,\ \varOmega_I^{\varepsilon_1}\bigl(\theta^{(1)}\bigr),\ \varOmega_J^{\varepsilon_2}\bigl(\theta^{(2)}\bigr)\bigr)\Biggr]\\
&\quad\leqslant \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{i\in I\subset\mathbb{I}}\sum_{j\in J\subset\mathbb{J}}\mathbf{P}\bigl(\theta_i^{(1)}X_i > x,\ \varOmega_I^{\varepsilon_1}\bigl(\theta^{(1)}\bigr), \theta_j^{(2)}Y_j > y,\ \varOmega_J^{\varepsilon_2}\bigl(\theta^{(2)}\bigr)\bigr)\\
&\qquad+\sum_{i=1}^{n}\sum_{p=1}^{m}\sum_{i\in I\subset\mathbb{I}}\sum_{p\notin J\subset\mathbb{J}}\mathbf{P}\bigl(\theta_i^{(1)}X_i > x,\ \theta_p^{(2)}Y_p > y,\ \varOmega_I^{\varepsilon_1}\bigl(\theta^{(1)}\bigr)\bigr)\frac{\mathbf{P}(\varOmega_J^{\varepsilon_2}(\theta^{(2)}))}{\mathbf{P}(\theta_p^{(2)} > \varepsilon_2)}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{q=1}^{n}\sum_{j=1}^{m}\sum_{q\notin I\subset\mathbb{I}}\sum_{j\in J\subset\mathbb{J}}\mathbf{P}\big(\theta_q^{(1)}X_q>x,\ \theta_j^{(2)}Y_j>y,\ \Omega_J^{\varepsilon_2}\big(\theta^{(2)}\big)\big)\frac{\mathbf{P}(\Omega_I^{\varepsilon_1}(\theta^{(1)}))}{\mathbf{P}(\theta_q^{(1)}>\varepsilon_1)}\\
&+\sum_{q=1}^{n}\sum_{p=1}^{m}\sum_{q\notin I\subset\mathbb{I}}\sum_{p\notin J\subset\mathbb{J}}\mathbf{P}\big(\theta_q^{(1)}X_q>x,\ \theta_p^{(2)}Y_p>y\big)\frac{\mathbf{P}(\Omega_I^{\varepsilon_1}(\theta^{(1)}))}{\mathbf{P}(\theta_q^{(1)}>\varepsilon_1)}\frac{\mathbf{P}(\Omega_J^{\varepsilon_2}(\theta^{(2)}))}{\mathbf{P}(\theta_p^{(2)}>\varepsilon_2)}\\
\leqslant{}&\left(1+\max_{1\leqslant p,q\leqslant n}\left\{\frac{\mathbf{P}(\theta_p^{(2)}\leqslant\varepsilon_2)}{\mathbf{P}(\theta_p^{(2)}>\varepsilon_2)},\frac{\mathbf{P}(\theta_q^{(1)}\leqslant\varepsilon_1)}{\mathbf{P}(\theta_q^{(1)}>\varepsilon_1)},\frac{\mathbf{P}(\theta_q^{(1)}\leqslant\varepsilon_1)}{\mathbf{P}(\theta_q^{(1)}>\varepsilon_1)}\frac{\mathbf{P}(\theta_p^{(2)}\leqslant\varepsilon_2)}{\mathbf{P}(\theta_p^{(2)}>\varepsilon_2)}\right\}\right)\\
&\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{P}\big(\theta_i^{(1)}X_i>x,\ \theta_j^{(2)}Y_j>y\big).
\end{aligned}
$$

Now let $\varepsilon_1,\varepsilon_2\downarrow 0$ in the above formula. Since $\theta_i^{(1)}$ and $\theta_j^{(2)}$, $i,j\geqslant 1$ are positive random variables, we get that (15) holds.

Now we prove (15) for the case where the random variables $\theta_i^{(1)}$ and $\theta_j^{(2)}$, $i,j\geqslant 1$, may take value 0 with a positive probability. For the above subsets I of $\mathbb{I}$ and J of $\mathbb{J}$, write

$$\Omega_I^0\big(\theta^{(1)}\big)=\big\{\omega_1\colon\ \theta_i^{(1)}>0\text{ for }i\in I,\ \text{and }\theta_q^{(1)}=0\text{ for }q\in I^{\mathrm{c}}\big\}$$

and

$$\Omega_J^0\big(\theta^{(2)}\big)=\big\{\omega_2\colon\ \theta_j^{(2)}>0\text{ for }j\in J,\ \text{and }\theta_p^{(2)}=0\text{ for }p\in J^{\mathrm{c}}\big\}.$$

Thus, for sufficiently large x and y, it holds that

$$
\begin{aligned}
&\mathbf{P}\left(\sum_{i=1}^{n}\theta_i^{(1)}X_i>x,\ \sum_{j=1}^{m}\theta_j^{(2)}Y_j>y\right)\\
&\quad\lesssim\sum_{\emptyset\neq I\subset\mathbb{I}}\sum_{\emptyset\neq J\subset\mathbb{J}}\sum_{i\in I}\sum_{j\in J}\mathbf{P}\big(\theta_i^{(1)}X_i>x,\ \theta_j^{(2)}Y_j>y,\ \Omega_I^0\big(\theta^{(1)}\big),\ \Omega_J^0\big(\theta^{(2)}\big)\big)\\
&\quad=\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{P}\big(\theta_i^{(1)}X_i>x,\ \theta_j^{(2)}Y_j>y\big),
\end{aligned}
$$

where the second step has used relation (15) for positive random variables $\theta_i^{(1)}$ and $\theta_j^{(2)}$, $i,j\geqslant 1$. This ends the proof of Lemma 4. □

The following lemma is an extension of Lemma 4. It plays an important role in the proofs of the main result. To prove this lemma, we will follow the line of the proof of Lemma 2.5 in Wang et al. [20].

Lemma 5. *Let $\{(X,Y)^{\mathrm{T}},(X_i,Y_i)^{\mathrm{T}},\ i\geqslant 1\}$ be a sequence of i.i.d. nonnegative random vectors with generic marginal distributions $F\in\mathcal{S}$ and $G\in\mathcal{S}$. Assume that X and Y are SAI with constant $\rho>0$. Let $\{(\theta_i^{(1)},\theta_i^{(2)})^{\mathrm{T}},\ i\geqslant 1\}$ be a sequence of nonnegative and upper bounded random variables, but not degenerate at zero. Assume also that $\{(X,Y)^{\mathrm{T}},(X_i,Y_i)^{\mathrm{T}},\ i\geqslant 1\}$, $\{\theta_i^{(1)},\ i\geqslant 1\}$, and $\{\theta_i^{(2)},\ i\geqslant 1\}$ are mutually independent. Let ξ and η be two independent real-valued random variables with distributions F_ξ and*

F_η, respectively, and be independent of all other random sources. If $\overline{F_\xi}(x) = o(1)\overline{F}(cx)$ and $\overline{F_\eta}(y) = o(1)\overline{G}(cy)$ for all $c > 0$, then for any fixed $n \geqslant 1$ and $m \geqslant 1$,

$$\mathbf{P}\left(\sum_{i=1}^{n}\theta_i^{(1)}X_i + \xi > x,\ \sum_{j=1}^{m}\theta_j^{(2)}Y_j + \eta > y\right) \sim \sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{P}\big(\theta_i^{(1)}X_i > x,\ \theta_j^{(2)}Y_j > y\big).$$

Proof. It suffices to prove for any fixed $n \geqslant 1$ and $m \geqslant 1$ that

$$\mathbf{P}\left(\sum_{i=1}^{n}\theta_i^{(1)}X_i + \xi > x,\ \sum_{j=1}^{m}\theta_j^{(2)}Y_j + \eta > y\right) \gtrsim \sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{P}\big(\theta_i^{(1)}X_i > x,\ \theta_j^{(2)}Y_j > y\big) \tag{16}$$

and

$$\mathbf{P}\left(\sum_{i=1}^{n}\theta_i^{(1)}X_i + \xi > x,\ \sum_{j=1}^{m}\theta_j^{(2)}Y_j + \eta > y\right) \lesssim \sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{P}\big(\theta_i^{(1)}X_i > x,\ \theta_j^{(2)}Y_j > y\big). \tag{17}$$

We first prove that for any $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, $u \geqslant 0$, and $v \geqslant 0$,

$$\lim_{(x,y)\to(\infty,\infty)} \frac{\mathbf{P}(\theta_i^{(1)}X_i > x+u,\ \theta_j^{(2)}Y_j > y+v)}{\mathbf{P}(\theta_i^{(1)}X_i > x,\ \theta_j^{(2)}Y_j > y)} = 1. \tag{18}$$

Since $\theta_i^{(1)}$ and $\theta_j^{(2)}$ are upper bounded, $F \in \mathcal{S} \subset \mathcal{L}$ and $G \in \mathcal{S} \subset \mathcal{L}$, by Lemma 2 we know that $\theta_i^{(1)}X_i$ and $\theta_j^{(2)}Y_j$ have subexponential distributions.

If $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, and $i \neq j$, since X_i, Y_j, $\theta_i^{(1)}$, and $\theta_j^{(2)}$ are mutually independent, it holds that

$$\lim_{(x,y)\to(\infty,\infty)} \frac{\mathbf{P}(\theta_i^{(1)}X_i > x+u,\ \theta_j^{(2)}Y_j > y+v)}{\mathbf{P}(\theta_i^{(1)}X_i > x,\ \theta_j^{(2)}Y_j > y)} = 1. \tag{19}$$

If $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, and $i = j$, let $0 \leqslant \theta_i^{(1)} \leqslant M$ and $0 \leqslant \theta_i^{(2)} \leqslant M$ for some constant $M > 0$. Since X_i and Y_i are SAI with constant $\rho > 0$ and $\theta_i^{(1)}X_i$ and $\theta_i^{(2)}Y_i$

have long-tailed distributions, it holds that

$$\begin{aligned}
&\lim_{(x,y)\to(\infty,\infty)} \frac{\mathbf{P}(\theta_i^{(1)}X_i > x+u,\ \theta_i^{(2)}Y_i > y+v)}{\mathbf{P}(\theta_i^{(1)}X_i > x,\ \theta_i^{(2)}Y_i > y)}\\
&\quad= \lim_{(x,y)\to(\infty,\infty)} \frac{\rho\int_{0-}^{M}\int_{0-}^{M}\mathbf{P}(X_i > \frac{x+u}{s})\,\mathbf{P}(Y_i > \frac{y+v}{t})\,\mathbf{P}(\theta_i^{(1)}\in \mathrm{d}s)\,\mathbf{P}(\theta_i^{(2)}\in \mathrm{d}t)}{\rho\int_{0-}^{M}\int_{0-}^{M}\mathbf{P}(X_i > \frac{x}{s})\mathbf{P}(Y_i > \frac{y}{t})\,\mathbf{P}(\theta_i^{(1)}\in \mathrm{d}s)\,\mathbf{P}(\theta_i^{(2)}\in \mathrm{d}t)}\\
&\quad= \lim_{(x,y)\to(\infty,\infty)} \frac{\mathbf{P}(\theta_i^{(1)}X_i > x+u)}{\mathbf{P}(\theta_i^{(1)}X_i > x)}\,\frac{\mathbf{P}(\theta_i^{(2)}Y_i > y+v)}{\mathbf{P}(\theta_i^{(2)}Y_i > y)} = 1,
\end{aligned}$$

which, combined with (19), gives that (18) holds.

We first prove (16). By Fatou's lemma, Lemma 4, and (18) it holds that

$$\begin{aligned}
&\liminf_{(x,y)\to(\infty,\infty)} \frac{\mathbf{P}(\sum_{i=1}^{n}\theta_i^{(1)}X_i + \xi > x,\ \sum_{j=1}^{m}\theta_j^{(2)}Y_j + \eta > y)}{\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{P}(\theta_i^{(1)}X_i > x,\ \theta_j^{(2)}Y_j > y)}\\
&\quad\geqslant \int_{0^-}^{\infty}\int_{0^-}^{\infty} \liminf_{(x,y)\to(\infty,\infty)} \frac{\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{P}(\theta_i^{(1)}X_i > x+u,\ \theta_j^{(2)}Y_j > y+v)}{\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{P}(\theta_i^{(1)}X_i > x,\ \theta_j^{(2)}Y_j > y)}\\
&\qquad\quad \mathbf{P}\big(\xi^- \in \mathrm{d}u\big)\,\mathbf{P}\big(\eta^- \in \mathrm{d}v\big)\\
&\quad\geqslant \int_{0^-}^{\infty}\int_{0^-}^{\infty} \liminf_{(x,y)\to(\infty,\infty)} \min_{1\leqslant i,j\leqslant n} \frac{\mathbf{P}(\theta_i^{(1)}X_i > x+u,\ \theta_j^{(2)}Y_j > y+v)}{\mathbf{P}(\theta_i^{(1)}X_i > x,\ \theta_j^{(2)}Y_j > y)}\\
&\qquad\quad \mathbf{P}\big(\xi^- \in \mathrm{d}u\big)\,\mathbf{P}\big(\eta^- \in \mathrm{d}v\big) = 1,
\end{aligned}$$

which shows that (16) holds.

Relation (17) can be proved by first considering that the random variables $\theta_i^{(1)}$ and $\theta_j^{(2)}$, $i,j \geqslant 1$, are positive, and then considering that the random variables $\theta_i^{(1)}$ and $\theta_j^{(2)}$, $i,j \geqslant 1$, may take value 0 with a positive probability. The way is similar to that of the proof of (15). We omit the details. This completes the proof of Lemma 5. □

The following lemma can be obtained from Theorem 1 of Li [13] because the random variables $\{(\theta_i^{(1)}, \theta_i^{(2)})^{\mathrm{T}},\ i \geqslant 1\}$ are upper bounded.

Lemma 6. *Let $\{(X,Y)^{\mathrm{T}}, (X_i,Y_i)^{\mathrm{T}},\ i\geqslant 1\}$ be a sequence of i.i.d. nonnegative random vectors with generic marginal distributions $F \in \mathcal{L}\cap\mathcal{D}$ and $G \in \mathcal{L}\cap\mathcal{D}$. Assume that X and Y are SAI with constant $\rho > 0$. Let $\{(\theta_i^{(1)}, \theta_i^{(2)})^{\mathrm{T}},\ i \geqslant 1\}$ be a sequence of nonnegative and upper bounded random variables, but not degenerate at zero. Assume also that $\{(X,Y)^{\mathrm{T}}, (X_i,Y_i)^{\mathrm{T}},\ i\geqslant 1\}$ and $\{(\theta_i^{(1)}, \theta_i^{(2)})^{\mathrm{T}},\ i \geqslant 1\}$ are independent, and $\{\theta_i^{(1)},\ i\geqslant 1\}$ and $\{\theta_i^{(2)},\ i\geqslant 1\}$ can be arbitrary dependent. Then for any fixed $n \geqslant 1$ and $m \geqslant 1$,*

$$\mathbf{P}\left(\sum_{i=1}^{n}\theta_i^{(1)}X_i > x,\ \sum_{j=1}^{m}\theta_j^{(2)}Y_j > y\right) \sim \sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{P}\big(\theta_i^{(1)}X_i > x,\ \theta_j^{(2)}Y_j > y\big).$$

In the risk model (1), for any $t \geqslant 0$ and $k = 1, 2$, set

$$p_k(t) = \int_{0^-}^{t} \mathrm{e}^{-\widetilde{R}_k(s)}\, B_k(\mathrm{d}s)$$

and denote

$$\overline{p}_k(t) = \sup_{0 \leqslant s \leqslant t} p_k(s) \geqslant 0 \quad \text{and} \quad \underline{p}_k(t) = \inf_{0 \leqslant s \leqslant t} p_k(s) \leqslant 0.$$

The result of the following lemma is (19) of Yang et al. [25].

Lemma 7. *Consider the risk model* (1) *with nonnegative Lévy processes* $\{\widetilde{R}_k(t),\ t \geqslant 0\}$. *Then for any fixed* $t > 0$ *and any* $x > 0$,

$$\mathbf{P}\big(\delta_k \overline{p}_k(t) > x\big) = \mathbf{P}\big(\delta_k \underline{p}_k(t) < -x\big) \leqslant 2\overline{\Phi}\left(\frac{x}{\delta_k \sqrt{t}}\right), \quad k = 1, 2,$$

where Φ *is the standard Gaussian distribution.*

Remark 5. Under the conditions of Lemma 7, if $F_k \in \mathcal{S}$, $k = 1, 2$, by Lemma 7 and (3) it holds for all $c > 0$ and $t > 0$ that

$$\mathbf{P}\big(\delta_1 \overline{p}_1(t) > x\big) = o(1)\overline{F_1}(cx) \quad \text{and} \quad \mathbf{P}\big(\delta_2 \overline{p}_2(t) > y\big) = o(1)\overline{F_2}(cy). \tag{20}$$

5.2 Proof of Theorem 1

Firstly, we deal with the upper bound for $\psi(x, y, t)$. Choosing some large N_0, by Lemma 7 it holds for any fixed $t > 0$, all $x > 0$, and $y > 0$ that

$$\begin{aligned}
&\psi(x, y, t) \\
&\quad \leqslant \mathbf{P}\left(\sum_{i=1}^{N_1(t)} X_i \mathrm{e}^{-R_1(\tau_i^{(1)})} + \delta_1 \overline{p}_1(t) > x,\ \sum_{j=1}^{N_2(t)} Y_j \mathrm{e}^{-R_2(\tau_j^{(2)})} + \delta_2 \overline{p}_2(t) > y\right) \\
&\quad = \left(\sum_{m=0}^{N_0} \sum_{n=0}^{N_0} + \sum_{m=0}^{N_0} \sum_{n=N_0+1}^{\infty} + \sum_{m=N_0+1}^{\infty} \sum_{n=0}^{\infty}\right) \\
&\qquad\quad \mathbf{P}\left(\sum_{i=1}^{m} X_i \mathrm{e}^{-R_1(\tau_i^{(1)})} + \delta_1 \overline{p}_1(t) > x,\ \sum_{j=1}^{n} Y_j \mathrm{e}^{-R_2(\tau_j^{(2)})} + \delta_2 \overline{p}_2(t) > y,\right. \\
&\qquad\qquad\quad \left. N_1(t) = m,\ N_2(t) = n\right) \\
&\quad =: I_1(x, y, t) + I_2(x, y, t) + I_3(x, y, t).
\end{aligned}$$

For $I_1(x,y,t)$,

$$\begin{aligned}
&I_1(x,y,t)\\
&\quad=\left(\sum_{m=1}^{N_0}\sum_{n=1}^{N_0}+\sum_{m=0}^{0}\sum_{n=1}^{N_0}+\sum_{n=0}^{0}\sum_{m=1}^{N_0}+\sum_{m=0}^{0}\sum_{n=0}^{0}\right)\\
&\qquad\mathbf{P}\Biggl(\sum_{i=1}^{m}X_i\mathrm{e}^{-R_1(\tau_i^{(1)})}+\delta_1\overline{p}_1(t)>x,\ \sum_{j=1}^{n}Y_j\mathrm{e}^{-R_2(\tau_j^{(2)})}+\delta_2\overline{p}_2(t)>y,\\
&\qquad\quad N_1(t)=m,\ N_2(t)=n\Biggr)\\
&\quad=:I_{11}(x,y,t)+I_{12}(x,y,t)+I_{13}(x,y,t)+I_{14}(x,y,t). \qquad (21)
\end{aligned}$$

For any $t\geqslant 0$, $n\geqslant 1$, and $m\geqslant 1$, set

$$\Omega_m^1=\bigl\{(z_1,z_2,\dots,z_{m+1})\colon\ 0\leqslant z_1\leqslant z_2\leqslant\cdots\leqslant z_m\leqslant t<z_{m+1}\bigr\}$$

and

$$\Omega_n^2=\bigl\{(q_1,q_2,\dots,q_{n+1})\colon\ 0\leqslant q_1\leqslant q_2\leqslant\cdots\leqslant q_n\leqslant t<q_{n+1}\bigr\}.$$

By Lemma 5 and (20) it holds that

$$\begin{aligned}
&I_{11}(x,y,t)\\
&\sim\sum_{m=1}^{N_0}\sum_{n=1}^{N_0}\sum_{i=1}^{m}\sum_{j=1}^{n}\int_{\Omega_m^1}\int_{\Omega_n^2}\mathbf{P}\bigl(X_i\mathrm{e}^{-R_1(z_i)}>x,\ Y_j\mathrm{e}^{-R_2(q_j)}>y\bigr)\\
&\qquad\mathbf{P}\bigl(\tau_1^{(1)}\in\mathrm{d}z_1,\ \dots,\ \tau_{m+1}^{(1)}\in\mathrm{d}z_{m+1},\ \tau_1^{(2)}\in\mathrm{d}q_1,\ \dots,\\
&\qquad\tau_{n+1}^{(2)}\in\mathrm{d}q_{n+1}\bigr)\\
&\leqslant\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{0^-}^{t}\int_{0^-}^{t}\mathbf{P}\bigl(X_i>x\mathrm{e}^{R_1(s)},\ Y_j>y\mathrm{e}^{R_2(p)}\bigr)\,\mathbf{P}\bigl(\tau_i^{(1)}\in\mathrm{d}s,\ \tau_j^{(2)}\in\mathrm{d}p\bigr)\\
&\sim\sum_{i=1}^{\infty}\sum_{1\leqslant j\neq i<\infty}\int_{0^-}^{t}\int_{0^-}^{t}\mathbf{P}\bigl(X_i\mathrm{e}^{-R_1(s)}>x\bigr)\,\mathbf{P}(Y_j\mathrm{e}^{-R_2(p)}>y)\,\mathbf{P}(\tau_i^{(1)}\in\ \mathrm{d}s,\ \tau_j^{(2)}\in\mathrm{d}p)\\
&\quad+\rho\sum_{i=1}^{\infty}\int_{0^-}^{t}\int_{0^-}^{t}\mathbf{P}\bigl(X_i\mathrm{e}^{-R_1(s)}>x\bigr)\,\mathbf{P}(Y_i\mathrm{e}^{-R_2(p)}>y)\,\mathbf{P}\bigl(\tau_i^{(1)}\in\mathrm{d}s,\ \tau_i^{(2)}\in\mathrm{d}p\bigr)\\
&=\int_{0^-}^{t}\int_{0^-}^{t}\mathbf{P}\bigl(X\mathrm{e}^{-R_1(s)}>x\bigr)\mathbf{P}\bigl(Y\mathrm{e}^{-R_2(p)}>y\bigr)\,\lambda(\mathrm{d}s,\mathrm{d}p)\\
&\quad+(\rho-1)\int_{0^-}^{t}\int_{0^-}^{t}\mathbf{P}\bigl(X\mathrm{e}^{-R_1(s)}>x\bigr)\,\mathbf{P}\bigl(Y\mathrm{e}^{-R_2(p)}>y\bigr)\,\tilde{\lambda}(\mathrm{d}s,\mathrm{d}p)\\
&=:\phi(x,y,t). \qquad (22)
\end{aligned}$$

For $I_{12}(x,y,t)$, since $F_1 \in \mathcal{S}$ and $\mathrm{e}^{-R_1(t)} \leqslant 1$, a.s., $t \geqslant 0$, by Lemma 2 it holds that $X_1\mathrm{e}^{-R_1(t)}$ has a subexponential distribution. Thus by Lemma 7 we get that for any $t > 0$,

$$\mathbf{P}\big(\delta_1\overline{p}_1(t) > x\big) = o(1)\mathbf{P}\big(X\mathrm{e}^{-R_1(t)} > x\big). \tag{23}$$

Since $\{Y_j,\ j \geqslant 1\}$ are i.i.d. and $\mathrm{e}^{-R_2(\tau_j^{(2)})} \leqslant 1$ a.s., $j \geqslant 1$, by Lemma 2.5 of Wang et al. [20] and (20) it holds that

$$\begin{aligned}\sum_{n=1}^{N_0}&\mathbf{P}\Bigg(\sum_{j=1}^{n} Y_j\mathrm{e}^{-R_2(\tau_j^{(2)})} + \delta_2\overline{p}_2(t) > y,\ N_2(t) = n\Bigg)\\ &\lesssim \int_{0-}^{t} \mathbf{P}\big(Y\mathrm{e}^{-R_2(p)} > y\big)\,\lambda_2(\mathrm{d}p).\end{aligned} \tag{24}$$

Thus by (23) and (24) it holds that

$$\begin{aligned}I_{12}(x,y,t) &\leqslant \mathbf{P}\big(\delta_1\overline{p}_1(t) > x\big)\sum_{n=1}^{N_0}\mathbf{P}\Bigg(\sum_{j=1}^{n} Y_j\mathrm{e}^{-R_2(\tau_j^{(2)})} + \delta_2\overline{p}_2(t) > y,\ N_2(t) = n\Bigg)\\ &= o(1)\mathbf{P}\big(X\mathrm{e}^{-R_1(t)} > x\big)\int_{0^-}^{t} \mathbf{P}\big(Y\mathrm{e}^{-R_2(p)} > y\big)\,\lambda_2(\mathrm{d}p).\end{aligned} \tag{25}$$

From the definition of $\phi(x,y,t)$, since $T_1^{(1)}$ is independent of $\{T_k^{(2)},\ k \geqslant 1\}$, it holds that all $t > 0$, $x > 0$, and $y > 0$ for

$$\phi(x,y,t) \geqslant (\rho \wedge 1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{0^-}^{t}\int_{0^-}^{t} \mathbf{P}\big(X\mathrm{e}^{-R_1(s)} > x\big)\mathbf{P}\big(Y\mathrm{e}^{-R_2(p)} > y\big)\,\mathbf{P}\big(\tau_i^{(1)} \in \mathrm{d}s,\ \tau_j^{(2)} \in \mathrm{d}p\big) \tag{26}$$

$$\geqslant \big(\rho \wedge 1\big)\mathbf{P}\big(X\mathrm{e}^{-R_1(t)} > x\big)\,\mathbf{P}\big(T_1^{(1)} \leqslant t\big)\int_{0^-}^{t} \mathbf{P}\big(Y\mathrm{e}^{-R_2(p)} > y\big)\,\lambda_2(\mathrm{d}p) \tag{27}$$

$$\geqslant \big(\rho \wedge 1\big)\mathbf{P}\big(T_1^{(1)} \leqslant t\big)\lambda_2(t)\mathbf{P}\big(X\mathrm{e}^{-R_1(t)} > x\big)\mathbf{P}\big(Y\mathrm{e}^{-R_2(t)} > y\big). \tag{28}$$

Similarly, since $T_1^{(2)}$ is independent of $\{T_k^{(1)},\ k \geqslant 1\}$, it holds for all $t > 0$, $x > 0$, and $y > 0$ that

$$\begin{aligned}\phi(x,y,t) \geqslant (\rho \wedge 1)&\mathbf{P}\big(Y\mathrm{e}^{-R_2(t)} > y\big)\mathbf{P}\big(T_1^{(2)} \leqslant t\big)\\ &\times \int_{0^-}^{t} \mathbf{P}\big(X\mathrm{e}^{-R_1(s)} > x\big)\,\lambda_1(\mathrm{d}s).\end{aligned} \tag{29}$$

Since $\mathbf{P}(T_1^{(1)} \leqslant t) > 0$, by (25) and (27) it holds that

$$I_{12}(x,y,t) = o(1)\phi(x,y,t). \tag{30}$$

Similarly, by $\mathbf{P}(T_1^{(2)} \leqslant t) > 0$ and (29) we can obtain that

$$I_{13}(x,y,t) = o(1)\phi(x,y,t). \tag{31}$$

For $I_{14}(x,y,t)$, similar to (23), it holds that for any $t > 0$,

$$\mathbf{P}\big(\delta_2\overline{p}_2(t) > y\big) = o(1)\mathbf{P}\big(Y\mathrm{e}^{-R_2(t)} > y\big). \tag{32}$$

By (23), (32), (28), $\mathbf{P}(T_1^{(1)} \leqslant t) > 0$, and $\lambda_2(t) \geqslant \mathbf{P}(T_1^{(2)} \leqslant t) > 0$ it holds that

$$I_{14}(x,y,t) \leqslant \mathbf{P}\big(\delta_1\overline{p}_1(t) > x\big)\mathbf{P}\big(\delta_2\overline{p}_2(t) > y\big) = o(1)\phi(x,y,t). \tag{33}$$

Thus, by (21), (22), (30), (31), and (33) it holds that

$$I_1(x,y,t) \lesssim \phi(x,y,t). \tag{34}$$

As for $I_2(x,y,t)$, it holds for any $t > 0$, $x > 0$, and $y > 0$ that

$$\begin{aligned} &I_2(x,y,t) \\ &\quad\leqslant \sum_{m=1}^{N_0}\sum_{n=N_0+1}^{\infty}\mathbf{P}\Bigg(\sum_{i=1}^{m}X_i\mathrm{e}^{-R_1(\tau_i^{(1)})} + \delta_1\overline{p}_1(t) > x, \\ &\qquad\qquad \sum_{j=1}^{n}Y_j\mathrm{e}^{-R_2(\tau_j^{(2)})} + \delta_2\overline{p}_2(t) > y,\ N_1(t) = m,\ N_2(t) = n\Bigg) \\ &\qquad + \sum_{n=N_0+1}^{\infty}\mathbf{P}\Bigg(\delta_1\overline{p}_1(t) > x,\ \sum_{j=1}^{n}Y_j\mathrm{e}^{-R_2(\tau_j^{(2)})} + \delta_2\overline{p}_2(t) > y,\ N_2(t) = n\Bigg) \\ &\quad =: I_{21}(x,y,t) + I_{22}(x,y,t). \end{aligned} \tag{35}$$

We first deal with $I_{21}(x,y,t)$. Since $\{(X_i,Y_i)^{\mathrm{T}},\ i \geqslant 1\}$ are i.i.d. and $\{(X_i,Y_i)^{\mathrm{T}},\ i \geqslant 1\}$, $\{R_1(t),\ t \geqslant 0\}$, $\{R_2(t),\ t \geqslant 0\}$, $T_1^{(1)}$, and $T_1^{(2)}$ are independent, it holds for any $t \geqslant 0$ that $\{(X_i\mathrm{e}^{-R_1(T_1^{(1)})}\mathbf{1}_{\{T_1^{(1)} \leqslant t\}}, Y_i\mathrm{e}^{-R_2(T_1^{(2)})}\mathbf{1}_{\{T_1^{(2)} \leqslant t\}})^{\mathrm{T}},\ i \geqslant 1\}$ are i.i.d. Since $0 \leqslant \mathrm{e}^{-R_k(T_1^{(k)})}\mathbf{1}_{\{T_1^{(k)} \leqslant t\}} \leqslant 1$ and $F_k \in \mathcal{S}$, $k = 1,2$, it follows from Lemma 2 that $X\mathrm{e}^{-R_1(T_1^{1})}\mathbf{1}_{\{T_1^{(1)} \leqslant t\}}$ and $Y\mathrm{e}^{-R_2(T_1^{(2)})}\mathbf{1}_{\{T_1^{(2)} \leqslant t\}}$ have subexponential distributions. Because X and Y are SAI with constant $\rho > 0$, we know that for any $t > 0$ satisfying $\mathbf{P}(T_1^{(k)} \leqslant t) > 0$, $k = 1,2$,

$$\begin{aligned} &\mathbf{P}\big(X\mathrm{e}^{-R_1(T_1^{(1)})}\mathbf{1}_{\{T_1^{(1)} \leqslant t\}} > x,\ Y\mathrm{e}^{-R_2(T_1^{(2)})}\mathbf{1}_{\{T_1^{(2)} \leqslant t\}} > y\big) \\ &\quad \sim \rho\mathbf{P}\big(X\mathrm{e}^{-R_1(T_1^{(1)})}\mathbf{1}_{\{T_1^{(1)} \leqslant t\}} > x\big)\mathbf{P}\big(Y\mathrm{e}^{-R_2(T_1^{(2)})}\mathbf{1}_{\{T_1^{(2)} \leqslant t\}} > y\big), \end{aligned} \tag{36}$$

and, similarly to (20), we get that

$$\mathbf{P}\big(\delta_1\overline{p}_1(t) > x\big) = o(1)\mathbf{P}\big(X\mathrm{e}^{-R_1(T_1^{(1)})}\mathbf{1}_{\{T_1^{(1)}\leqslant t\}} > x\big)$$

and

$$\mathbf{P}\big(\delta_2\overline{p}_2(t) > y\big) = o(1)\mathbf{P}\big(Y\mathrm{e}^{-R_2(T_1^{(2)})}\mathbf{1}_{\{T_1^{(2)}\leqslant t\}} > y\big).$$

Thus $\{(X_i\mathrm{e}^{-R_1(T_1^{(1)})}\mathbf{1}_{\{T_1^{(1)}\leqslant t\}}, Y_i\mathrm{e}^{-R_2(T_1^{(2)})}\mathbf{1}_{\{T_1^{(2)}\leqslant t\}})^{\mathrm{T}},\ i \geqslant 1\}$, $\xi = \delta_1\overline{p}_1(t)$, and $\eta = \delta_2\overline{p}_2(t)$ satisfy the conditions of Lemma 3. Therefore, by Lemma 3, (36), and (26), for any $\varepsilon_0 > 0$, there exists a constant $K > 0$ such that for all $x > 0$ and $y > 0$,

$$\begin{aligned}
I_{21}(x,y,t) \leqslant{}& \sum_{m=1}^{N_0}\sum_{n=N_0+1}^{\infty}\mathbf{P}\Bigg(\sum_{i=1}^{m}X_i\mathrm{e}^{-R_1(T_1^{(1)})}\mathbf{1}_{\{T_1^{(1)}\leqslant t\}} + \delta_1\overline{p}_1(t) > x,\\
&\qquad\sum_{j=1}^{n}Y_j\mathrm{e}^{-R_2(T_1^{(2)})}\mathbf{1}_{\{T_1^{(2)}\leqslant t\}} + \delta_2\overline{p}_2(t) > y\Bigg)\\
&\qquad\times\mathbf{P}\big(N_1(t)\geqslant m-1,\ N_2(t)\geqslant n-1\big)\\
\leqslant{}& K\sum_{m=1}^{N_0}\sum_{n=N_0+1}^{\infty}(1+\varepsilon_0)^{m+n}\mathbf{P}\big(N_1(t)\geqslant m-1,\ N_2(t)\geqslant n-1\big)\\
&\times\mathbf{P}\big(X_1\mathrm{e}^{-R_1(T_1^{(1)})}\mathbf{1}_{\{T_1^{(1)}\leqslant t\}} > x\big)\mathbf{P}\big(Y_1\mathrm{e}^{-R_2(T_1^{(2)})}\mathbf{1}_{\{T_1^{(2)}\leqslant t\}} > y\big)\\
\leqslant{}& 2\rho^{-1}K\mathbf{P}\big(X_1\mathrm{e}^{-R_1(T_1^{(1)})}\mathbf{1}_{\{T_1^{(1)}\leqslant t\}} > x,\ Y_1\mathrm{e}^{-R_2(T_1^{(2)})}\mathbf{1}_{\{T_1^{(2)}\leqslant t\}} > y\big)\\
&\times\sum_{m=1}^{N_0}\sum_{n=N_0+1}^{\infty}(1+\varepsilon_0)^{m+n}\mathbf{P}\big(N_1(t)\geqslant m-1,\ N_2(t)\geqslant n-1\big)\\
\leqslant{}& 2\rho^{-1}(\rho\wedge 1)^{-1}K\phi(x,y,t)\\
&\times\sum_{m=1}^{N_0}\sum_{n=N_0+1}^{\infty}(1+\varepsilon_0)^{m+n}\mathbf{P}\big(N_1(t)\geqslant m-1,\ N_2(t)\geqslant n-1\big).
\end{aligned}$$

Choose some small enough $\varepsilon_0 > 0$ such that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(1+\varepsilon_0)^{m+n}\mathbf{P}\big(N_1(t)\geqslant m-1,\ N_2(t)\geqslant n-1\big) < \infty.$$

Thus

$$\lim_{N_0\to\infty}\limsup\frac{I_{21}(x,y,t)}{\phi(x,y,t)} = 0. \tag{37}$$

For $I_{22}(x,y,t)$,

$$\begin{aligned}
I_{22}(x,y,t) ={}& \mathbf{P}\big(\delta_1\overline{p}_1(t) > x\big)\\
&\times\sum_{n=N_0+1}^{\infty}\mathbf{P}\Bigg(\sum_{j=1}^{\infty}Y_j\mathrm{e}^{-R_2(\tau_j^{(2)})} + \delta_2\overline{p}_2(t) > y,\ N_2(t) = n\Bigg).
\end{aligned}$$

Using the same proof of (37), by Lemma 2.6 of Wang et al. [20], (23), and (27) it holds that

$$\lim_{N_0\to\infty} \limsup \frac{I_{22}(x,y,t)}{\phi(x,y,t)} = 0. \tag{38}$$

By (35), (37), and (38) it holds that

$$\lim_{N_0\to\infty} \limsup \frac{I_2(x,y,t)}{\phi(x,y,t)} = 0. \tag{39}$$

Using the similar proof of (39), we can get

$$\lim_{N_0\to\infty} \limsup \frac{I_3(x,y,t)}{\phi(x,y,t)} = 0. \tag{40}$$

Thus, by (18), (34), (39), and (40) we get the upper bound for $\psi(x,y,t)$.

Next, we prove the lower bound for $\psi(x,y,t)$. By $0 \leqslant c_k(t) \leqslant M_0$ and $R_k(t) \geqslant 0$, for any $t \geqslant 0$ and $k = 1,2$, it holds for any $t > 0$, $x > 0$, and $y > 0$ that

$$\begin{aligned}
\psi(x,y,t) \geqslant \mathbf{P}\Bigg(&\sum_{i=1}^{N_1(t)} X_i \mathrm{e}^{-R_1(\tau_i^{(1)})} - \delta_1 \overline{p}_1(t) - M_0 t > x, \\
&\sum_{j=1}^{N_2(t)} Y_j \mathrm{e}^{-R_2(\tau_j^{(2)})} - \delta_2 \overline{p}_2(t) - M_0 t > y \Bigg) \\
=:\ & J(x,y,t).
\end{aligned}$$

Since $-\delta_k \overline{p}_k(t) - M_0 t \leqslant 0$, $k = 1,2$, it holds that for all $c_k > 0$, $k = 1,2$,

$$\mathbf{P}\big(-\delta_1 \overline{p}_1(t) - M_0 t > x\big) = o\left(\overline{F_1}\left(\frac{x}{c_1}\right)\right)$$

and

$$\mathbf{P}\big(-\delta_2 \overline{p}_2(t) - M_0 t > y\big) = o\left(\overline{F_2}\left(\frac{y}{c_2}\right)\right).$$

Thus, by Lemma 5 it holds that

$$\begin{aligned}
&J(x,y,t) \\
&\quad \geqslant \sum_{m=1}^{N_0} \sum_{n=1}^{N_0} \int_{\Omega_m^1}\int_{\Omega_n^2} \mathbf{P}\Bigg(\sum_{i=1}^{m} X_i \mathrm{e}^{-R_1(z_i)} - \delta_1 \overline{p}_1(t) - M_0 t > x, \\
&\qquad\qquad \sum_{j=1}^{n} Y_j \mathrm{e}^{-R_2(q_j)} - \delta_2 \overline{p}_2(t) - M_0 t > y \Bigg) \\
&\qquad\qquad \mathbf{P}\big(\tau_1^{(1)} \in \mathrm{d}z_1,\ \ldots,\ \tau_{m+1}^{(1)} \in \mathrm{d}z_{m+1},\ \tau_1^{(2)} \in \mathrm{d}q_1,\ \ldots,\ \tau_1^{(2)} \in \mathrm{d}q_{n+1}\big)
\end{aligned}$$

$$
\begin{aligned}
&\sim \Biggl(\sum_{m=1}^{N_0}\sum_{n=1}^{N_0}\sum_{i=1}^{m}\sum_{j=1}^{n}\Biggr) \\
&\qquad \int\limits_{\Omega_m^1}\int\limits_{\Omega_n^2} \mathbf{P}\bigl(X_i \mathrm{e}^{-R_1(z_i)} > x,\ Y_j \mathrm{e}^{-R_2(q_j)} > y\bigr) \\
&\qquad\qquad \mathbf{P}\bigl(\tau_1^{(1)} \in \mathrm{d}z_1,\ \ldots,\ \tau_{m+1}^{(1)} \in \mathrm{d}z_{m+1},\ \tau_1^{(2)} \in \mathrm{d}q_1,\ \ldots,\ \tau_1^{(2)} \in \mathrm{d}q_{n+1}\bigr) \\
&= \Biggl(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} - \sum_{m=1}^{\infty}\sum_{n=N_0+1}^{\infty} - \sum_{m=N_0+1}^{\infty}\sum_{n=1}^{N_0}\Biggr)\sum_{i=1}^{m}\sum_{j=1}^{n} \\
&\qquad \mathbf{P}\bigl(X_i \mathrm{e}^{-R_1(\tau_i^{(1)})} > x,\ Y_j \mathrm{e}^{-R_2(\tau_j^{(2)})} > y,\ N_1(t) = m,\ N_2(t) = n\bigr) \\
&=: J_1(x,y,t) - J_2(x,y,t) - J_3(x,y,t).
\end{aligned} \tag{41}
$$

By the proof of (22) we get that

$$
J_1(x,y,t) \sim \phi(x,y,t). \tag{42}
$$

For $J_2(x,y,t)$, by (26)

$$
\begin{aligned}
&J_2(x,y,t) \\
&\quad \lesssim (\rho \vee 1) \sum_{m=1}^{\infty}\sum_{n=N_0+1}^{\infty} \int\limits_{0-}^{t}\int\limits_{0-}^{t} mn\mathbf{P}\bigl(N_1(t-s) = m-1,\ N_2(t-p) = n-1\bigr) \\
&\qquad\qquad \times \mathbf{P}\bigl(X\mathrm{e}^{-R_1(s)} > x\bigr)\mathbf{P}\bigl(Y\mathrm{e}^{-R_2(p)} > y\bigr) \\
&\qquad\qquad \mathbf{P}\bigl(T_1^{(1)} \in \mathrm{d}s,\ T_1^{(2)} \in \mathrm{d}p\bigr) \\
&\quad \leqslant (\rho \vee 1)(\rho \wedge 1)^{-1}\mathbf{E}\bigl[N_1(t)N_2(t)\mathbf{1}_{\{N_2(t) \geqslant N_0\}}\bigr]\phi(x,y,t),
\end{aligned}
$$

which, combined with $E[N_1(t)N_2(t)] < \infty$, yields

$$
\lim_{N_0\to\infty} \limsup \frac{J_2(x,y,t)}{\phi(x,y,t)} = 0. \tag{43}
$$

Similarly, we can get

$$
\lim_{N_0\to\infty} \limsup \frac{J_3(x,y,t)}{\phi(x,y,t)} = 0. \tag{44}
$$

By (41)–(44) we get the lower bound for $\psi(x,y,t)$. It completes the proof of Theorem 1.

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References

1. D. Applebaum, *Lévy Processes and Stochastic Calculus*, Cambridge Univ. Press, Cambridge, 2004, https://doi.org/10.1017/CBO9780511755323.
2. D. Cheng, Uniform asymptotics for the finite-time ruin probability of a generalized bidimensional risk model with Brownian perturbations, *Stochastics*, **93**(1):56–71, 2019, https://doi.org/10.1080/17442508.2019.1708362.
3. D. Cheng, Y. Yang, X. Wang, Asymptotic finite-time ruin probabilities in a dependent bidimensional renewal risk model with subexponential claims, *Japan J. Ind. Appl. Math.*, **37**: 657–675, 2020, https://doi.org/10.1007/s13160-020-00418-y.
4. F. Cheng, D. Cheng, Z. Chen, Asymptotic behavior for finite-time ruin probabilities in a generalized bidimensional risk model with subexponential claims, *Japan J. Ind. Appl. Math.*, **38**(3): 947–963, 2021, https://doi.org/10.1007/s13160-021-00472-0.
5. D.B.H. Cline, G. Samorodnitsky, Subexponentiality of the product of independent random variables, *Stochastic Processes Appl.*, **49**(1):75–98, 1994, https://doi.org/10.1016/0304-4149(94)90113-9.
6. R. Cont, P. Tankov, *Financial Modelling with Jump Processes*, Chapman & Hall/CRC, New York, 2004, https://doi.org/10.1201/9780203485217.
7. P. Embrechts, C. Klüppelberg, T. Mikosch, *Modelling Extremal Events for Insurance and Finance*, Springer, Berlin, Heidelberg, 1997, https://doi.org/10.1007/978-3-642-33483-2.
8. S. Foss, D. Korshunov, S. Zachary, *An Introduction to Heavy-Tailed and Subexponential Distributions*, Springer, New York, 2013, https://doi.org/10.1007/978-1-4614-7101-1.
9. F. Guo, D. Wang, H. Yang, Asymptotic results for ruin probability in a two-dimensional risk model with stochastic investment returns, *J. Comput. Appl. Math.*, **325**:198–221, 2017, https://doi.org/10.1016/j.cam.2017.04.049.
10. T. Jiang, Y. Wang, Y. Chen, H. Xu, Uniform asymptotic estimate for finite-time ruin probabilities of a time-dependent bidimensional renewal model, *Insur. Math. Econ.*, **64**:45–53, 2015, https://doi.org/10.1016/j.insmatheco.2015.04.006.
11. J. Li, Uniform asymptotics for a multi-dimensional time-dependent risk model with multivariate regularly varying claims and stochastic return, *Insur. Math. Econ.*, **71**:195–204, 2016, https://doi.org/10.1016/j.insmatheco.2016.09.003.
12. J. Li, A revisit to asymptotic ruin probabilities for a bidimensional renewal risk model, *Stat. Probab. Lett.*, **140**:23–32, 2018, https://doi.org/10.1016/j.spl.2018.04.003.
13. J. Li, On the joint tail behavior of randomly weighted sums of heavy-tailed random variables, *J. Multivariate Anal.*, **164**:40–53, 2018, https://doi.org/10.1016/j.jmva.2017.10.008.

14. J. Li, Asymptotic ruin probabilities for a two-dimensional risk model with dependent claims and stochastic return, *Commun. Stat., Theory Methods*, **53**(16):5773–5784, 2024, https://doi.org/10.1080/03610926.2023.2232906.
15. J. Li, H. Yang, Asymptotic ruin probabilities for a bidimensional renewal risk model with constant interest rate and dependent claims, *J. Math. Anal. Appl.*, **426**(1):247–266, 2015, https://doi.org/10.1016/j.jmaa.2015.01.047.
16. Z. Liu, B. Geng, X. Man, X. Liu, Uniform asymptotics for ruin probabilities of a time-dependent bidimensional renewal risk model with dependent subexponential claims, *Stochastics*, **95**(7):1147–1169, 2023, https://doi.org/10.1080/17442508.2023.2165397.
17. R.B. Nelsen, *An Introduction to Copulas*, Springer, New York, 2006, https://doi.org/10.1007/0-387-28678-0.
18. K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Univ. Press, Cambridge, 1999.
19. B. Wang, J. Yan, D. Cheng, Asymptotic infinite-time ruin probabilities for a bidimensional time-dependence risk model with heavy-tailed claims, *Japan J. Ind. Appl. Math.*, pp. 177–194, 2022, https://doi.org/10.1007/s13160-021-00487-7.
20. K. Wang, L. Chen, Y. Yang, M. Gao, The finite-time ruin probability of a risk model with stochastic return and Brownian perturbation, *Japan J. Ind. Appl. Math.*, **35**:1173–1189, 2018, https://doi.org/10.1007/s13160-018-0321-0.
21. S. Wang, H. Qian, H. Sun, B. Geng, Uniform asymptotics for ruin probabilities of a non standard bidimensional perturbed risk model with subexponential claims, *Commun. Stat., Theory Methods*, **51**(22):7871–7884, 2021, https://doi.org/10.1080/03610926.2021.1882496.
22. H. Yang, J. Li, Asymptotic finite-time ruin probability for a bidimensional renewal risk model with constant interest force and dependent subexponential claims, *Insur. Math. Econ.*, **58**:185–192, 2014, https://doi.org/10.1016/j.insmatheco.2014.07.007.
23. H. Yang, J. Li, Asymptotic ruin probabilities for a bidimensional renewal risk model, *Stochastics*, **89**(5):687–708, 2017, https://doi.org/10.1080/17442508.2016.1276909.
24. Y. Yang, K. Wang, J. Liu, Z. Zhang, Asymptotics for a bidimensional risk model with two geometric Lévy price processes, *J. Ind. Manag. Optim.*, **15**(2):481–505, 2019, https://doi.org/10.3934/jimo.2018053.
25. Y. Yang, X. Wang, Z. Zhang, Finite-time ruin probability of a perturbed risk model with dependent main and delayed claims, *Nonlinear Anal. Model. Control*, **26**(5):801–820, 2021, https://doi.org/10.15388/NAMC.2021.26.23963.
26. Y. Yang, K.C. Yuen, Finite-time and infinite-time ruin probabilities in a two-dimensional delayed renewal risk model with Sarmanov dependent claims, *J. Math. Anal. Appl.*, **442**(2):600–626, 2016, https://doi.org/10.1016/j.jmaa.2016.04.068.