Nonlinear Analysis: Modelling and Control, Vol. 29, No. 3, 543–561
https://doi.org/10.15388/namc.2024.29.35028

A simulation function approach for optimization by approximate solutions with an application to fractional differential equation

Parvaneh Lo′lo′[a], **Maryam Shams**[b,1], **Stojan Radenović**[c,2]

[a]Department of Mathematics,
Behbahan Khatam Alanbia University of Technology,
Behbahan 6361663973, Iran
lolo@bkatu.ac.ir

[b]Department of Pure Mathematics,
Faculty of Mathematical Sciences, Shahrekord University,
Shahrekord 88186-34141, Iran
maryam.shams@sku.ac.ir

[c]Faculty of Mechanical Engineering, University of Belgrade,
Kraljice Marije 16, Belgrad 11120, Serbia
sradenovic@mas.bg.ac.rs

Received: December 16, 2022 / **Revised:** December 17, 2023 / **Published online:** April 5, 2024

Abstract. In this work, we study the existence and uniqueness of a common best proximity point for a pair of nonself functions that are not necessarily continuous using the simulation function. In the following, we state important common best proximity point theorems as results of the main theorems of this article. This achievement allows us to have an example that covers our main theorem but does not apply to the Banach contraction principle. Finally, an application of a nonlinear fractional differential equation to support the obtained conclusions.

Keywords: simulation functions, common best proximity point, $(\mathcal{Z}_d, T)$-contraction, P-property, commute proximally, fractional differential equation.

1 Introduction

Let A and B be nonempty subsets of the metric space (X, d). Also, suppose that $f\colon A \to B$ is nonself mapping. If $d(A, B) = \inf\{d(a, b),\ a \in A,\ b \in B\}$ and $d(a, fa) = d(A, B)$, then a is called the best proximity point. When a mapping does not have a fixed point, studying the best proximity point theory is a suitable way to obtain optimal approximate solutions. Therefore, optimization theory is developed with the theory of the best proximity points. If the mapping under study is selfmapping, the best proximity point

[1]Corresponding author.
[2]The author was supported by Basque Government grant No. 1555-22.

© 2024 The Author(s). Published by Vilnius University Press
This is an Open Access article distributed under the terms of the Creative Commons Attribution Licence, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.

is the fixed point. Therefore, the best proximity point theorems also act as a natural extension of fixed point theorems. Interesting best proximity point theorems with different control functions and various spaces can be seen in [1, 5, 7, 8, 12, 14, 17, 18, 21–24, 26–31, 33]. On the other hand, Khojasteh et al. introduced the meaning of $\mathcal{Z}$-contraction by using a concept of simulation function. Then fixed point consequences involving a $\mathcal{Z}$-contraction are appointed in [20]. In the following, Karapınar et al. [19] offered the best proximity point results using the simulation functions. Until now, several papers have been published in this field; see [6, 9, 10, 15, 16, 25]. In this work, using a contraction function via simulation function defined by Roldán et al. [32], we prove the existence and uniqueness of a common best proximity point for a pair of nonself functions that are not necessarily continuous. We show that important common best proximity point theorems can be stated as results. This achievement allows us to have one example that covers our main theorem but does not apply to the Banach contraction principle. In the end, an application of a nonlinear fractional differential equation to support the obtained conclusions.

2 Preliminaries

Let A and B be two nonempty subsets of a metric space (X, d); the following notions will be used all over the article:

$$\begin{aligned} d(A,B) &= \inf\bigl\{d(a,b),\ a \in A,\ b \in B\bigr\}, \\ A_0 &= \bigl\{a \in A\colon d(a,b) = d(A,B) \text{ for some } b \in B\bigr\}, \\ B_0 &= \bigl\{b \in B\colon d(a,b) = d(A,B) \text{ for some } a \in A\bigr\}. \end{aligned}$$

Definition 1. An element $a \in A$ is said to be a common best proximity point of the nonself mappings $S, T : A \to B$ if it satisfies the condition that

$$d(a, Sa) = d(a, Ta) = d(A, B).$$

Definition 2. The nonself mappings $S, T : A \to B$ are said to commute proximally if they satisfy the condition that

$$\bigl[d(u, Sx) = d(v, Tx) = d(A, B)\bigr] \quad \text{implies} \quad Sv = Tu.$$

Definition 3. If $A_0 \neq \emptyset$, then the pair (A, B) is said to have P-property if and only if for any $a_1, a_2 \in A_0$ and $b_2, b_2 \in B_0$,

$$d(a_1, b_1) = d(a_2, b_2) = d(A, B) \quad \text{implies} \quad d(a_1, a_2) = d(b_1, b_2).$$

Khojasteh et al. [20] gave the following definition of simulation function.

Definition 4. Let (X, d) be a metric space. A simulation function is a function $\zeta : [0, +\infty[\times [0, +\infty[\to \mathbb{R}$ satisfying the following conditions:

(ζ_1') $\zeta(0, 0) = 0$;

(ζ_2') $\zeta(p, q) < q - p$ for all $p, q > 0$;
(ζ_3') $\{p_n\}$ and $\{q_n\}$ are sequences in $(0, +\infty)$, and $\lim_{n\to+\infty} p_n = \lim_{n\to+\infty} q_n = k > 0$, then
$$\limsup_{n\to+\infty} \zeta(p_n, q_n) < 0.$$

The set of all simulation functions is denoted by $\mathcal{Z}$.

Then they proved the existence and uniqueness of fixed points for a selfmapping defined in a complete metric space.

Theorem 1. (*See* [20].) *Let (X, d) be a complete metric space, and let $f : X \to X$ be a $\mathcal{Z}$-contraction with respect to ζ, that is,*
$$\zeta\bigl(d(fx, fy), d(x, y)\bigr) \geqslant 0 \quad \textit{for all } x, y \in X.$$
Then f has a unique fixed point. Moreover, for every $x_0 \in X$, the Picard sequence $\{f^n x_0\}$ converges to this fixed point.

In the following, Argoubi et al. [11] point out the verity that condition (ζ_1') is not said in the proof of Theorem 1. Also, Roldán et al. [32] revised (ζ_3') by taking $p_n < q_n$.

Therefore, we use the following definition in this article.

Definition 5. Let (X, d) be a metric space. A simulation function is a function $\zeta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(p, q) < q - p$ for all $p, q > 0$;
(ζ_2) $\{p_n\}$ and $\{q_n\}$ are sequences in $(0, +\infty)$, and $\lim_{n\to+\infty} p_n = \lim_{n\to+\infty} q_n = k > 0$ and $p_n < q_n$ for all $n \in \mathbb{N}$, then
$$\limsup_{n\to+\infty} \zeta(p_n, q_n) < 0.$$

Example 1. If $\zeta_\lambda : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ be the function defined by $\zeta_\lambda(p, q) = \lambda q - p$, where $\lambda \in]0, 1[$. Then ζ_λ is a simulation function.

Then Roldán et al. [32] extended the definition of $\mathcal{Z}$-contraction with respect to ζ of Khojasteh et al. [20] for two functions.

Definition 6. Let A and B be two nonempty subsets of a metric space (X, d), and let $S, T : A \to B$ be nonself mappings. We say that S is a $(\mathcal{Z}_d, T)$-contraction if there exists a simulation function $\zeta \in \mathcal{Z}$ such that
$$\zeta\bigl(d(Sx, Sy), d(Tx, Ty)\bigr) \geqslant 0 \quad \text{for all } x, y \in A\text{: } Tx \neq Ty. \tag{1}$$

In the preceding definitions, if T is the identity mapping on X, the notions of $(\mathcal{Z}_d, T)$-contraction reduced to $\mathcal{Z}$-contraction with respect to ζ of Khojasteh et al. [20].

Remark 1.

(i) By axiom (ζ_1), it is clear that a simulation function must verify $\zeta(r, r) < 0$ for all $r > 0$.
(ii) Furthermore, if S is a $(\mathcal{Z}_d, T)$-contraction, then $d(Sx, Sy) < d(Tx, Ty)$ for all $x, y \in X$ such that $Tx \neq Ty$.

3 Main result

We begin our study with the following lemmas. In the first lemma, we state sufficient conditions for the uniqueness of the common best proximity point.

Lemma 1. *Let A and B be nonempty subsets of a metric space (X, d). Let also the nonself mappings $S, T : A \to B$ satisfy the following conditions*:

(i) *S and T commute proximally*;
(ii) *pair (A, B) has the P-property*;
(iii) *S is a $(\mathcal{Z}_d, T)$-contraction*;
(iv) *v is a common best proximity point of S and T.*

Then v is unique.

Proof. Since v is a common best proximity point of the mappings S and T, then

$$\begin{aligned} d(v, Sv) &= d(A, B), \\ d(v, Tv) &= d(A, B). \end{aligned} \tag{2}$$

Suppose that v' is another common best proximity point of the mappings S and T so that

$$\begin{aligned} d(v', Sv') &= d(A, B), \\ d(v', Tv') &= d(A, B). \end{aligned} \tag{3}$$

As the mapping S and T commute proximally, $Sv = Tv$ and $Sv' = Tv'$, therefore

$$d(Sv, Sv') = d(Tv, Tv').$$

If $d(Sv, Sv') = d(Tv, Tv') > 0$, since S is a $(\mathcal{Z}_d, T)$-contraction and by (ζ_1) we have

$$\begin{aligned} 0 &\leqslant \zeta\big(d(Sv, Sv'), d(Tv, Tv')\big) \\ &< d(Tv, Tv') - d(Sv, Sv') = 0, \end{aligned}$$

which is a contradiction. Therefore, $d(Sv, Sv') = d(Tv, Tv') = 0$ or $Sv = Sv'$ and $Tv = Tv'$. Since pair (A, B) has the P-property, then by (2) and (3) we have $v = v'$. □

In the following lemma, we state the sufficient conditions for the existence of the common best proximity point.

Lemma 2. *Let A and B be nonempty subsets of a metric space (X, d), and let A_0 be nonempty. Let also the nonself mappings $S, T : A \to B$ satisfy the following conditions*:

(i) *S and T commute proximally*;
(ii) *$S(A_0) \subset B_0$ (or $T(A_0) \subset B_0$)*;
(iii) *S is a $(\mathcal{Z}_d, T)$-contraction*;
(iv) *$u \in A_0$ is a coincidence point of S and T, or $Su = Tu$.*

Then the functions S and T have at least a common best proximity point.

Proof. Let $u \in A_0$ and $Su = Tu$. Since SA_0 is contained in B_0 (or $T(A_0) \subseteq B_0$), there exists an element $v \in A_0$ such that

$$d(v, Tu) = d(v, Su) = d(A, B).$$

Since S and T commute proximally, $Tv = Sv$. Therefore, $d(Su, Sv) = d(Tu, Tv)$. If $d(Su, Sv) = d(Tu, Tv) > 0$, since S is a $(\mathcal{Z}_d, T)$-contraction and by (ζ_1) we have

$$\begin{aligned} 0 &\leqslant \zeta\big(d(Su, Sv), d(Tu, Tv)\big) \\ &< d(Tu, Tv) - d(Su, Sv) = 0, \end{aligned}$$

which is a contradiction. Therefore, $d(Su, Sv) = d(Tu, Tv) = 0$ or $Sv = Su$ and $Tv = Tu$.

So, it follows that

$$\begin{aligned} d(v, Sv) = d(v, Su) = d(A, B), \\ d(v, Tv) = d(v, Tu) = d(A, B). \end{aligned}$$

Therefore, v is a common best proximity point of the mappings S and T. □

Now, using the above basic lemmas, we can state one of the main results of this article.

Theorem 2. *Let A and B be nonempty subsets of a complete metric space (X, d). Moreover, assume that A_0 is nonempty and closed. Let also the nonself mappings $S, T : A \to B$ satisfy the following conditions*:

(i) *S and T commute proximally*;
(ii) *pair (A, B) has the P-property*;
(iii) *S is a $(\mathcal{Z}_d, T)$-contraction*;
(iv) *$SA_0 \subseteq TA_0$ and $SA_0 \subseteq B_0$ (or $TA_0 \subseteq B_0$)*;
(v) *S and T are continuous.*

Then S and T have a unique common best proximity point.

Proof. Let a_0 be an element in A_0. Since $S(A_0) \subseteq T(A_0)$, there exists an element a_1 in A_0 such that $Sa_0 = Ta_1$. Proceeding inductively, it can be shown that there exists a sequence $\{a_n\}$ in A_0 such that

$$Sa_{n-1} = Ta_n.$$

Let there exist $n_0 \in \mathbb{N}$ such that $Sa_{n_0-1} = Sa_{n_0}$. By $S(\{a_n\}) \subseteq B_0$ (or $T(\{a_n\}) \subseteq B_0$) then there exists $u \in A_0$ such that

$$d(u, Ta_{n_0}) = d(u, Sa_{n_0-1}) = d(u, Sa_{n_0}) = d(A, B). \tag{4}$$

Since S and T commute proximally, $Su = Tu$. Again, since $Su \in B_0$ (or $Tu \in B_0$), there exists $v \in A_0$ such that

$$d(v, Tu) = d(v, Su) = d(A, B). \tag{5}$$

Let $d(u, v) > 0$. Because pair (A, B) has the P-property, by (4) and (5) we have

$$d(Su, Sa_{n_0}) = d(Tu, Ta_{n_0}) = d(u, v) > 0.$$

Since S is a $(\mathcal{Z}_d, T)$-contraction, then by (ζ_1) we have

$$\begin{aligned} 0 &\leqslant \zeta\bigl(d(Su, Sa_{n_0}), d(Tu, Ta_{n_0})\bigr), \\ &< d(Tu, Ta_{n_0}) - d(Su, Sa_{n_0}) = 0, \end{aligned}$$

which is a contradiction. Therefore, $d(u, v) = 0$, and by (5) u is a common best proximity point of S and T. In the case the proof is finalized.

So, we can suppose that $Sa_{n-1} \neq Sa_n$ for every $n \in \mathbb{N}$. Therefore, $d(Ta_{n-1}, Ta_n)$ and $d(Sa_{n-1}, Sa_n)$ both have positive values for every $n \in \mathbb{N}$. In view of the fact that $S(A_0)$ is contained in B_0, there exists a sequence $\{u_n\}$ in A_0 such that

$$d(u_n, Sa_n) = d(A, B) \tag{6}$$

for every nonnegative integer n. So, it follows from the choice of $\{a_n\}$ that

$$d(u_{n-1}, Ta_n) = d(u_{n-1}, Sa_{n-1}) = d(A, B)$$

for every positive integer n. Because of this fact the mappings S and T are commuting proximally

$$Su_{n-1} = Tu_n.$$

Moreover, (A, B) have P-property, therefore,

$$\begin{aligned} d(u_{n-1}, u_n) &= d(Sa_{n-1}, Sa_n) \\ d(u_{n-2}, u_{n-1}) &= d(Ta_{n-1}, Ta_n). \end{aligned} \tag{7}$$

Since S is a $(\mathcal{Z}_d, T)$-contraction and by (7) and (ζ_1) we have

$$\begin{aligned} 0 &\leqslant \zeta\bigl(d(Sa_{n-1}, Sa_n), d(Ta_{n-1}, Ta_n)\bigr) \\ &< d(Ta_{n-1}, Ta_n) - (d(Sa_{n-1}, Sa_n) \\ &= d(u_{n-2}, u_{n-1}) - d(u_{n-1}, u_n) \end{aligned}$$

for every positive integer n, then

$$d(u_{n-1}, u_n) < d(u_{n-2}, u_{n-1}).$$

This implies that the sequence $d(u_{n-1}, u_n)$ is decreasing, and so there is a $d \geqslant 0$ such that $d(u_{n-1}, u_n) \to d$. Suppose that $d > 0$, using property (ζ_2) of simulation function, with $p_n = d(Sa_{n-1}, Sa_n)$ and $q_n = d(Ta_{n-1}, Ta_n)$, we have

$$0 \leqslant \limsup \zeta\bigl(d(Sa_{n-1}, Sa_n), d(Ta_{n-1}, Ta_n)\bigr) < 0,$$

which is a contradiction, and hence $d = 0$.

The next step is to show that the sequence $\{u_n\}$ is a Cauchy. Assume that $\{u_n\}$ is not Cauchy. Then by Lemma 2.1 of [13], there exists an $\epsilon > 0$ and two subsequences $\{u_{n_k}\}$ and $\{u_{m_k}\}$ of $\{u_n\}$ such that $n_k > m_k \geqslant k$, $d(u_{n_k}, u_{m_k}) \geqslant \epsilon$ for all $k \in \mathbb{N}$, and

$$\lim_{k\to+\infty} d(u_{n_k}, u_{m_k}) = \lim_{k\to+\infty} d(u_{n_k+1}, u_{m_k+1}) = \epsilon. \tag{8}$$

Then we can assume that $d(u_{n_k+1}, u_{m_k+1}) > 0$ for all $k \in \mathbb{N}$. Since pair (A, B) has the P-property, therefore, by (6) we have for all $k \in \mathbb{N}$,

$$\begin{aligned} d(u_{n_k}, u_{m_k}) &= d(Ta_{n_k+1}, Ta_{m_k+1}),\\ d(u_{n_k+1}, u_{m_k+1}) &= d(Sa_{n_k+1}, Sa_{m_k+1}). \end{aligned} \tag{9}$$

By (8) and (9) we have

$$\lim_{k\to+\infty} d(Sa_{n_k+1}, Sa_{m_k+1}) = \lim_{k\to+\infty} d(Ta_{n_k+1}, Ta_{m_k+1}) = \epsilon.$$

Let $p_k = d(Sa_{n_k+1}, Sa_{m_k+1})$ and $q_k = d(Ta_{n_k+1}, Ta_{m_k+1})$.

Since $d(Sa_{n_k+1}, Sa_{m_k+1}) > 0$ and $d(Ta_{n_k+1}, Ta_{m_k+1}) > 0$ for all $k \in \mathbb{N}$ and $p_k < q_k$ by Remark 1(ii), then by using property (ζ_2) of simulation function we obtain

$$0 \leqslant \limsup_{k\to+\infty} \zeta\bigl(d(Sa_{n_k+1}, Sa_{m_k+1}), d(Ta_{n_k+1}, Ta_{m_k+1})\bigr) < 0,$$

which is a contradiction. We conclude that the sequence $\{u_n\}$ is Cauchy. Since (X, d) is a complete metric space and A_0 is a closed subset of X, there exists u in A_0 such that $\lim_{n\to+\infty} u_n = u$. Because of the continuity of the mappings S and T,

$$Tu = \lim_{n\to+\infty} Tu_n = \lim_{n\to+\infty} Su_{n-1} = Su. \tag{10}$$

Therefore, u is a coincidence point of S and T. Then by Lemmas 1 and 2, S and T have a unique common best proximity point. $\square$

Now, in the following theorem, we replace the continuity condition of f and g with another condition to get the same result.

Theorem 3. *Let A and B be nonempty subsets of a complete metric space (X, d), and let A_0 be nonempty. Let also the nonself mappings $S, T : A \to B$ satisfy the following conditions*:

(i) *S and T commute proximally*;
(ii) *pair (A, B) has the P-property*;
(iii) *S is a $(\mathcal{Z}_d, T)$-contraction*;
(iv) *$SA_0 \subseteq TA_0$ and $SA_0 \subseteq B_0$ (or $TA_0 \subseteq B_0$)*;
(v) *SA_0 (or TA_0) is closed.*

Then S and T have a unique common best proximity point.

Proof. We take the same sequence $\{a_n\}$ as in the proof of Theorem 2 and get that $d(Ta_{n-1}, Ta_n)$ and $d(Sa_{n-1}, Sa_n)$ both have positive values for every $n \in \mathbb{N}$. Since S is a $(\mathcal{Z}_d, T)$-contraction and by (ζ_1) we have

$$\begin{aligned} 0 &\leqslant \zeta\bigl(d(Sa_{n-1}, Sa_n), d(Ta_{n-1}, Ta_n)\bigr) \\ &< d(Ta_{n-1}, Ta_n) - d(Sa_{n-1}, Sa_n) \\ &= d(Ta_{n-1}, Ta_n) - d(Ta_n, Ta_{n+1}) \end{aligned}$$

for every positive integer n, then

$$d(Ta_n, Ta_{n+1}) < d(Ta_{n-1}, Ta_n).$$

This implies that the sequence $d(Ta_n, Ta_{n+1})$ is decreasing, and so there is a $d \geqslant 0$ such that $d(Ta_n, Ta_{n+1}) \to d$. Suppose that $d > 0$, using property (ζ_2) of simulation function, with $p_n = d(Sa_{n-1}, Sa_n)$ and $q_n = d(Ta_{n-1}, Ta_n)$, we have

$$0 \leqslant \limsup \zeta\bigl(d(Sa_{n-1}, Sa_n), d(Ta_{n-1}, Ta_n)\bigr) < 0,$$

which is a contradiction and hence $d = 0$.

The next step is to show that the sequence $\{Ta_n\}$ is a Cauchy. Assume that $\{Ta_n\}$ is not Cauchy. Then by Lemma 2.1 of [13] there exists an $\epsilon > 0$ and two subsequences $\{Ta_{n_k}\}$ and $\{Ta_{m_k}\}$ of $\{Ta_n\}$ such that $n_k > m_k \geqslant k$ and $d(Ta_{n_k}, Ta_{m_k}) \geqslant \epsilon$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to+\infty} d(Ta_{n_k}, Ta_{m_k}) = \lim_{k\to+\infty} d(Ta_{n_k+1}, Ta_{m_k+1}) = \epsilon.$$

Then we can assume that $d(Ta_{n_k+1}, Ta_{m_k+1}) > 0$ for all $k \in \mathbb{N}$. Therefore,

$$\lim_{k\to+\infty} d(Ta_{n_k}, Ta_{m_k}) = \lim_{k\to+\infty} d(Sa_{n_k}, Sa_{m_k}) = \epsilon.$$

Let $p_k = d(Sa_{n_k}, Sa_{m_k})$ and $q_k = d(Ta_{n_k}, Ta_{m_k})$, then by using the simulation function property (ζ_2) we obtain

$$0 \leqslant \limsup_{k\to+\infty} \zeta\bigl(d(Sa_{n_k}, Sa_{m_k}), d(Ta_{n_k}, Ta_{m_k})\bigr) < 0,$$

which is a contradiction. We conclude that the sequence $\{Ta_n\}$ is Cauchy.

Since $Ta_n = Sa_{n-1} \in S(A_0) \subseteq T(A_0)$ for all $n \in \mathbb{N}$, therefore, $\{Ta_n\}$ is included in $S(A_0) \subseteq T(A_0)$. Since $(T(A_0), d)$ (or $(S(A_0), d)$) is a closed subset of X and (X, d) is a complete metric space, then there exists $v \in TA_0$ such that $Ta_n \to v$, that is,

$$\lim_{n\to+\infty} d(Ta_n, v) = 0. \tag{11}$$

Since $Sa_{n-1} = Ta_n$ for all $n \in \mathbb{N}$, we also have that

$$\lim_{n\to+\infty} d(Sa_n, v) = 0. \tag{12}$$

Let $u \in A_0$ be any point such that $Tu = v$.

Now, we show that u is a coincidence point of S and T. If not, that is, $v = Tu \neq Su$, then we have $d(Su, Tu) = \gamma > 0$. Then using (11), there exists $m_0 \in \mathbb{N}$ such that $d(Ta_n, Tu) < \gamma$ for all $n \geqslant m_0$. Therefore,

$$d(Ta_n, Tu) < \gamma = d(Su, Tu) \quad \text{for all } n \geqslant m_0.$$

Then $Ta_n \neq Su$ for all $n \geqslant m_0$, and therefore,

$$d(Sa_n, Su) = d(Ta_{n+1}, Su) > 0 \quad \text{for all } n \geqslant m_0. \tag{13}$$

If there exists $m_1 \in \mathbb{N}$ such that

$$Ta_n = Tu \quad \text{for all } n \geqslant m_1,$$

then $Ta_n = Ta_{n+1}$ for all $n \geqslant m_1$, which contradicts the positiveness of $d(Ta_n, Ta_{n+1})$ for all $n \in \mathbb{N}$. Therefore, there exists $\{a_{\beta(n)}\} \subseteq \{a_n\}$ such that

$$Ta_{\beta(n)} \neq Tu \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

Let $m_2 \in \mathbb{N}$ such that $\beta(m_2) \geqslant m_0$, then by (13) and (14) we have

$$d(Sa_{\beta(n)}, Su) > 0 \quad \text{and} \quad d(Ta_{\beta(n)}, Tu) > 0 \quad \text{for all } n \geqslant m_2.$$

Then by using property (ζ_2) of simulation function we obtain

$$\begin{aligned} 0 &\leqslant \zeta\big(d(Sa_{\beta(n)}, Su), d(Ta_{\beta(n)}, Tu)\big) \\ &< d(Ta_{\beta(n)}, Tu) - d(Sa_{\beta(n)}, Su) \quad \text{for all } n \geqslant n_2. \end{aligned}$$

Letting $n \to +\infty$, by (11) and (12) we obtain

$$0 \leqslant \zeta\big(d(Sa_{\beta(n)}, Su), d(Ta_{\beta(n)}, Tu)\big) < 0,$$

which is a contradiction. Therefore, u is a coincidence point of S and T.

Finally, using Lemmas 1 and 2, it is proved that u is unique common best proximity point of S and T. □

4 Consequences

In this section, we present some results where Theorems 2 and 3 can be applied. In other words, we show that the simulation functions can be used for different types of contraction conditions in an only method.

Corollary 1 [Banach type]. *Let A and B be nonempty subsets of a complete metric space (X, d), and let A_0 be nonempty. Moreover, the nonself mappings $S, T : A \to B$ satisfy the following conditions*:

(i) *S and T commute proximally*;
(ii) *pair (A, B) has the P-property*;

(iii) *there exists* $\alpha \in [0,1)$ *such that* $d(Sx,Sy) \leqslant \alpha d(Tx,Ty)$ *for all* $x,y \in X$ *such that* $Tx \neq Ty$;
(iv) $SA_0 \subseteq TA_0$ *and* $SA_0 \subseteq B_0$ (*or* $TA_0 \subseteq B_0$);
(v) S *and* T *are continuous, and* A_0 *is closed or at least one of the sets of* SA_0 *and* TA_0 *are closed.*

Then S *and* T *have a unique common best proximity point.*

Proof. The proof follows from Theorem 2 (or 3) by choosing the simulation function as $\zeta(p,q) = \alpha q - p$ for all $p,q \in [0,+\infty)$. □

Corollary 2. *Let* A *and* B *be nonempty subsets of a complete metric space* (X,d), *and let* A_0 *be nonempty. Moreover,* $\chi,\varphi : [0,+\infty) \to [0,+\infty)$ *be continuous, nondecreasing functions such that* $\chi^{-1}(0) = \varphi^{-1}(0) = \{0\}$ *and* $\chi(t) < t \leqslant \varphi(t)$ *for all* $t > 0$. *Let also the nonself mappings* $S,T : A \to B$ *satisfy the following conditions*:

(i) S *and* T *commute proximally*;
(ii) *pair* (A,B) *has the P-property*;
(iii) $\varphi(d(Sx,Sy)) \leqslant \chi(d(Tx,Ty))$ *for all* $x,y \in X$ *such that* $Tx \neq Ty$;
(iv) $SA_0 \subseteq TA_0$ *and* $SA_0 \subseteq B_0$ (*or* $TA_0 \subseteq B_0$);
(v) S *and* T *are continuous, and* A_0 *is closed or at least one of the sets of* SA_0 *and* TA_0 *are closed.*

Then S *and* T *have a unique common best proximity point.*

Proof. The proof follows from Theorem 2 (or 3) by choosing the simulation function as $\zeta(p,q) = \chi(q) - \varphi(p)$ for all $p,q \in [0,+\infty)$. □

Corollary 3. *Let* A *and* B *be nonempty subsets of a complete metric space* (X,d), *and let* A_0 *be nonempty. Moreover,* $\psi : [0,+\infty) \to [0,+\infty)$ *is a lower semicontinuous function such that* $\psi^{-1}(0) = \{0\}$. *Let also the nonself mappings* $S,T : A \to B$ *satisfy the following conditions*:

(i) S *and* T *commute proximally*;
(ii) *pair* (A,B) *has the P-property*;
(iii) $d(Sx,Sy) \leqslant d(Tx,Ty) - \psi(d(Tx,Ty))$ *for all* $x,y \in X$ *such that* $Tx \neq Ty$;
(iv) $SA_0 \subseteq TA_0$ *and* $SA_0 \subseteq B_0$ (*or* $TA_0 \subseteq B_0$);
(v) S *and* T *are continuous, and* A_0 *is closed or at least one of the sets of* SA_0 *and* TA_0 *are closed.*

Then S *and* T *have a unique common best proximity point.*

Proof. The proof follows from Theorem 2 (or 3) by choosing the simulation function as $\zeta(p,q) = q - \psi(q) - p$ for all $p,q \in [0,+\infty)$. □

Corollary 4 [Rhoades type]. *Let* A *and* B *be nonempty subsets of a complete metric space* (X,d), *and let* A_0 *be nonempty. Moreover,* $\psi : [0,+\infty) \to [0,+\infty)$ *is a continuous* *function such that* $\psi^{-1}(0) = \{0\}$. *Let also the nonself mappings* $S, T : A \to B$ *satisfy the following conditions*:

(i) S *and* T *commute proximally*;
(ii) *pair* (A, B) *has the P-property*;
(iii) $d(Sx, Sy) \leqslant d(Tx, Ty) - \psi(d(Tx, Ty))$ *for all* $x, y \in X$ *such that* $Tx \neq Ty$;
(iv) $SA_0 \subseteq TA_0$ *and* $SA_0 \subseteq B_0$ (*or* $TA_0 \subseteq B_0$)*;*
(v) S *and* T *are continuous, and* A_0 *is closed or at least one of the sets of* SA_0 *and* TA_0 *are closed.*

Then S *and* T *have a unique common best proximity point.*

Proof. It is a special instance of the above consequence. □

Corollary 5. *Let* A *and* B *be nonempty subsets of a complete metric space* (X, d)*, and let* A_0 *be nonempty. Moreover,* $\eta : [0, +\infty) \to [0, 1)$ *is a function, which satisfies* $\limsup_{t \to r^+} \eta(t) < 1$ *for all* $r > 0$. *Let also the nonself mappings* $S, T : A \to B$ *satisfy the following conditions*:

(i) S *and* T *commute proximally*;
(ii) *pair* (A, B) *has the P-property*;
(iii) $d(Sx, Sy) \leqslant d(Tx, Ty)\eta(d(Tx, Ty))$ *for all* $x, y \in X$ *such that* $Tx \neq Ty$;
(iv) $SA_0 \subseteq TA_0$ *and* $SA_0 \subseteq B_0$ (*or* $TA_0 \subseteq B_0$);
(v) S *and* T *are continuous, and* A_0 *is closed or at least one of the sets of* SA_0 *and* TA_0 *are closed.*

Then S *and* T *have a unique common best proximity point.*

Proof. The proof follows from Theorem 2 (or 3) by choosing the simulation function as $\zeta(p, q) = q\eta(q) - p$ for all $p, q \in [0, +\infty)$. □

Corollary 6. *Let* A *and* B *be nonempty subsets of a complete metric space* (X, d)*, and let* A_0 *be nonempty. Moreover,* $\phi : [0, +\infty) \to [0, +\infty)$ *is a function such that* $\int_0^\epsilon \phi(u)\,\mathrm{d}u$ *exists and* $\int_0^\epsilon \phi(u)\,\mathrm{d}u > \epsilon$ *for all* $\epsilon > 0$. *Let also the nonself mappings* $S, T : A \to B$ *satisfy the following conditions*:

(i) S *and* T *commute proximally*;
(ii) *pair* (A, B) *has the P-property*;
(iii) $\int_0^{d(Sx,Sy)} \phi(u)\,\mathrm{d}u \leqslant d(Tx, Ty)$ *for all* $x, y \in X$ *such that* $Tx \neq Ty$;
(iv) $SA_0 \subseteq TA_0$ *and* $SA_0 \subseteq B_0$ (*or* $TA_0 \subseteq B_0$);
(v) S *and* T *are continuous, and* A_0 *is closed or at least one of the sets of* SA_0 *and* TA_0 *are closed.*

Then S *and* T *have a unique common best proximity point.*

Proof. The proof follows from Theorem 2 (or 3) by choosing the simulation function as $\zeta(p, q) = q - \int_0^p \phi(u)\,\mathrm{d}u$ for all $p, q \in [0, +\infty)$. □

Corollary 7. *Let A and B be nonempty subsets of a complete metric space (X, d), and let A_0 be nonempty. Moreover, $h, k : [0, +\infty) \times [0, +\infty) \to (0, +\infty)$ be two continuous functions with respect to each variable such that $h(p, q) > k(p, q)$ for all $p, q > 0$. Let also the nonself mappings $S, T : A \to B$ satisfy the following conditions*:

(i) *S and T commute proximally*;

(ii) *pair (A, B) has the P-property*;

(iii) *for all $x, y \in X$ such that $Tx \neq Ty$,*

$$\frac{h(d(Sx, Sy), d(Tx, Ty))}{k(d(Sx, Sy), d(Tx, Ty))} d(Sx, Sy) \leqslant d(Tx, Ty);$$

(iv) *$SA_0 \subseteq TA_0$ and $SA_0 \subseteq B_0$ (or $TA_0 \subseteq B_0$)*;

(v) *S and T are continuous, and A_0 is closed or at least one of the sets of SA_0 and TA_0 are closed.*

Then S and T have a unique common best proximity point.

Proof. The proof follows from Theorem 2 (or 3) by choosing the simulation function as $\zeta(p, q) = q - h(p, q)/k(p, q)p$ for all $p, q \in [0, +\infty)$. □

5 Examples

Example 2. Suppose $X = \mathbb{R}$ is equipped with Euclidean metric. Let

$$A := \{(0, a)\colon\ 0 < a \leqslant 1\},$$
$$B := \{(1, a)\colon\ 0 < a \leqslant 1\}.$$

It is easy to see $d(A, B) = 1$, $A_0 = A$ and $B_0 = B$. We define $S, T : A \to B$ by

$$S(0, a) = (1, 1), \qquad T(0, a) = (1, a).$$

Assume that

$$d(u, Sx) = d(v, Tx) = d(A, B) = 1,$$

we conclude from the above equation $u = (0, 1)$, $v = x$. Then $Sv = Tu = (1, 1)$, and therefore, S and T commute proximally. Furthermore,

$$\zeta_\lambda\bigl(d(Sx, Sy), d(Tx, Ty)\bigr) = \lambda|y - x| \geqslant 0.$$

Then S is a $(\mathcal{Z}_d, T)$-contraction with respect to ζ_λ. Clearly, S and T are continuous, $S(A_0) \subseteq T(A_0)$, $S(A_0) \subseteq B_0$, and (A, B) has P-property. Finally, by Theorem 2 we can conclude that $(0, 1)$ is the unique common best proximity point of S and T.

In the next example, we suppose that $\zeta(p, q) : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ with $\zeta(p, q) = q - (p + 4)/(p + 2)p$. Clearly, ζ is a simulation function.

Example 3. Consider $X = \{0,1,2,3,\dots\}$, $A = \{0,1,3,5,\dots\}$, and $B = \{0,2,4,6,\dots\}$. Let $d : X \times X \to [0,+\infty)$ be a metric on X defined by

$$d(x,y) = \begin{cases} x+y & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Clearly, $A_0 = \{0\}$, $B_0 = \{0\}$, $d(A,B) = 0$, and the pair (A,B) has the P-property. Suppose that $S,T : A \to B$ are defined by

$$Sx = \begin{cases} x-3 & \text{if } x \in \{5,7,9,\dots\}, \\ 0 & \text{if } x = 0,1,3, \end{cases} \qquad Tx = \begin{cases} x-1 & \text{if } x \in \{3,5,7,\dots\}, \\ 0 & \text{if } x = 0,1. \end{cases}$$

If $d(u,Sx) = d(v,Tx) = d(A,B) = 0$, then $u = Sx$ and $v = Tx$, and therefore, according to the definitions of S and T, we have $u = v = 0$. Then $Sv = Tu$, or S and T commute proximally.

Now, in the following cases, we show that S is a $(\mathcal{Z}_d, T)$-contraction.

Case 1. If $x = 0,1$ then we have the following subcases.

(i) If $y = 3$, then $Sx = Sy = Tx = 0$ and $Ty = 2$. Then (1) is satisfied.

(ii) If $y \in \{5,7,9,\dots\}$, then $Sx = Tx = 0$, $Sy = y-3$, and $Ty = y-1$. Therefore,

$$\zeta\bigl(d(Sx,Sy), d(Tx,Ty)\bigr) = \zeta(y-3,\ y-1) = y-1-\frac{y+1}{y-1}(y-3) = \frac{4}{y-1} \geqslant 0.$$

Case 2. If $x = 3$ then, we have the following subcases.

(i) If $y = 0,1$, then $Sx = Sy = Ty = 0$ and $Tx = 2$. Then (1) is satisfied.

(ii) If $y \in \{5,7,9,\dots\}$, then $Sx = 0$, $Sy = y-3$, $Tx = 2$, and $Ty = y-1$. Therefore,

$$\zeta(d(Sx,Sy), d(Tx,Ty)) = \zeta(y-3,\ y+1) = y+1-\frac{y+1}{y-1}(y-3) = \frac{2y+2}{y-1} \geqslant 0.$$

Case 3. If $x \in \{5,7,9,\dots\}$, then we have the following subcases.

(i) If $y = 0,1$, then $Sx = x-3$, $Sy = Ty = 0$, and $Tx = x-1$. Then, in this subcase, similar to subcase 1(ii), (1) is satisfied.

(ii) If $y = 3$, then $Sx = x-3$, $Sy = 0$, $Tx = x-1$, and $Ty = 2$. Then, in this subcase, similar to subcase 2(ii), (1) is satisfied.

(iii) If $y \in \{5,7,9,\dots\}$, then $Sx = x-3$, $Sy = y-3$, $Tx = x-1$, and $Ty = y-1$. Then

$$\begin{aligned}\bigl(d(Sx,Sy), d(Tx,Ty)\bigr) &= \zeta(x+y-6,\ x+y-2) \\ &= x+y-2-\frac{x+y-2}{x+y-4}(x+y-6) \\ &= \frac{2x+2y-4}{x+y-4} \geqslant 0.\end{aligned}$$

Thus (1) is verified.

It is easy to see that the other hypotheses of Theorem 3 are satisfied. Therefore, 0 is a unique common best proximity point of S and T.

Remark 2. Using the Archimedean property for all $\alpha \in (0,1)$, we know that there exists $x \in \{5,7,9,\dots\}$ such that $(1-\alpha)x > \alpha + 3$, then

$$(\alpha-1)x < -(\alpha+3) \quad \text{implies} \quad \alpha(x+1)-(x-3) < 0.$$

If $\zeta(p,q) = \alpha q - p$ and $y = 3$ in the previous example, then we have

$$\zeta\bigl(d(Sx,Sy), d(Tx,Ty)\bigr) = \alpha(x+1)-(x-3) < 0.$$

Therefore, the previous example does not apply to the Banach contraction.

6 Application to nonlinear fractional differential equation

In the last decades, two topics have been densely studied: “fixed point theory” and “fractional differential”. Relatively, fractional calculus and fractional differential are very fresh topics for the researchers, and recently, several notable results in fixed point have been recorded [2–4].

In this section, we provide an application for our results in the fractional equations, and we investigate the existence of a solution for Caputo fractional boundary value problems of order $\beta \in (n-1,\, n]$, where $n \geqslant 2$.

Let $\beta \in \mathbb{R}^+$, and let $M(t)$ be a continuous function. Then we define the Caputo derivative of fractional order β as follows:

$${}^{c}D^{\beta}M = J^{[\beta]-\beta}D^{[\beta]}M,$$

where $[\beta]$ is the smallest integer, which is greater than or equal to β, and J^{β} is the Riemann–Liouville integral operator of order $\beta \geqslant 0$ defined by

$$J^{\beta}M(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}M(s)\,\mathrm{d}s$$

such that $\Gamma(\beta) = \int_0^{+\infty} t^{\beta-1}\mathrm{e}^{-t}\,\mathrm{d}t$, and J^0 is the identity operator.

According to the conditions, we consider the following nonlinear fractional differential equation:

$$\bigl({}^{c}D^{\beta}u\bigr)(t) = h\bigl(t,u(t)\bigr), \quad t \in [0,1],\ n-1 < \beta \leqslant n, \tag{15}$$

with

$$u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0 \quad \text{and} \quad u(1) = \int_0^{\nu} u(s)\,\mathrm{d}s, \tag{16}$$

where $\nu \in [0,1]$ and $h : [0,1]\times\mathbb{R} \to \mathbb{R}$.

Now, we discuss the application of common techniques to the solution of boundary value problem (BVP) (15)–(16).

We define the operator equation $S : C[0,1] \to C[0,1]$ as follows:

$$\begin{aligned} S(x)(t) = {} & \frac{nt^{n-1}}{(n-\nu^n)\Gamma(\beta)} \int_0^{\nu} \int_0^{s} (s-\theta)^{\beta-1} h\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\,\mathrm{d}s \\ & - \frac{nt^{n-1}}{(n-\nu^n)\Gamma(\beta)} \int_0^{1} (1-s)^{\beta-1} h\bigl(s, x(s)\bigr)\,\mathrm{d}s \\ & + \frac{1}{\Gamma(\beta)} \int_0^{t} (t-s)^{\beta-1} h\bigl(s, x(s)\bigr)\,\mathrm{d}s. \end{aligned}$$

Meanwhile, the metric space $(C[0,1], \|\cdot\|)$ is endowed with the metric d defined by

$$d(x,y) = \|x-y\|_\infty = \sup\bigl\{\bigl|x(t)-y(t)\bigr| : t \in [0,1]\bigr\},$$

for all $x, y \in C[0,1]$.

Theorem 4. *If for all $t \in [0,1]$ and for all $x, y \in C[0,1]$, there exists K_1 with*

$$K_1 \leqslant \frac{(n-\nu^n)\Gamma(\beta+2)(d(Sx+Sy)+2)}{(n\nu^{\beta+1}+(\beta+1)(2n-\nu^n))(d(Sx,Sy)+4)} \tag{17}$$

such that $|f(t,x(t)) - f(t,y(t))| \leqslant K_1(|x(t)-y(t)|)$. Then BVP (15)–(16) *has a unique solution in $C[0,1]$.*

Proof. Using the definition of S and the assumptions of the theorem, we have

$$\begin{aligned} & \frac{d(Sx,Sy)+4}{d(Sx,Sy)+2}\bigl|Sx(t)-Sy(t)\bigr| \\ & \quad = \frac{d(Sx,Sy)+4}{d(Sx,Sy)+2}\left| \frac{nt^{n-1}}{(n-\nu^n)\Gamma(\beta)} \int_0^{\nu}\int_0^{s} (s-\theta)^{(\beta-1)}\bigl(h\bigl(\theta,x(\theta)\bigr) - h\bigl(\theta,y(\theta)\bigr)\bigr)\,\mathrm{d}\theta\,\mathrm{d}s \right. \\ & \qquad - \frac{nt^{n-1}}{(n-\nu^n)\Gamma(\beta)} \int_0^{1} (1-s)^{\beta-1}\bigl(h\bigl(s,x(s)\bigr) - h\bigl(s,y(s)\bigr)\bigr)\,\mathrm{d}s \\ & \qquad \left. + \frac{1}{\Gamma(\beta)} \int_0^{t} (t-s)^{\beta-1}\bigl(h\bigl(s,x(s)\bigr) - h\bigl(s,y(s)\bigr)\bigr)\,\mathrm{d}s \right| \\ & \quad \leqslant \frac{(d(Sx,Sy)+4)nt^{n-1}}{(d(Sx,Sy)+2)(n-\nu^n)\Gamma(\beta)} \int_0^{\nu}\int_0^{s} \bigl|(s-\theta)\bigr|^{(\beta-1)}\bigl|h\bigl(\theta,x(\theta)\bigr) - h\bigl(\theta,y(\theta)\bigr)\bigr|\,\mathrm{d}\theta\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+ \frac{(d(Sx,Sy)+4)nt^{n-1}}{(d(Sx,Sy)+2)(n-\nu^n)\Gamma(\beta)} \int_0^1 |1-s|^{\beta-1} \bigl| h\bigl(s,x(s)\bigr) - h\bigl(s,y(s)\bigr) \bigr| \, \mathrm{d}s \\
&+ \frac{d(Sx,Sy)+4}{(d(Sx,Sy)+2)\Gamma(\beta)} \int_0^t |t-s|^{\beta-1} \bigl| h\bigl(s,x(s)\bigr) - h\bigl(s,y(s)\bigr) \bigr| \, \mathrm{d}s \\
\leqslant{} & \frac{(d(Sx,Sy)+4)nK_1t^{n-1}}{(d(Sx,Sy)+2)(n-\nu^n)\Gamma(\beta)} \int_0^\nu \int_0^s |s-\theta|^{(\beta-1)} \bigl| x(\theta) - y(\theta) \bigr| \, \mathrm{d}\theta \, \mathrm{d}s \\
&+ \frac{(d(Sx,Sy)+4)nK_1t^{n-1}}{(d(Sx,Sy)+2)(n-\nu^n)\Gamma(\beta)} \int_0^1 |1-s|^{\beta-1} \bigl| x(s) - y(s) \bigr| \, \mathrm{d}s \\
&+ \frac{(d(Sx,Sy)+4)K_1}{(d(Sx,Sy)+2)\Gamma(\beta)} \int_0^t |t-s|^{\beta-1} \bigl| x(s) - y(s) \bigr| \, \mathrm{d}s \\
\leqslant{} & \frac{(d(Sx,Sy)+4)nK_1t^{n-1}}{(d(Sx,Sy)+2)(n-\nu^n)\Gamma(\beta)} d(x,y) \int_0^\nu \int_0^s |s-\theta|^{(\beta-1)} \, \mathrm{d}\theta \, \mathrm{d}s \\
&+ \frac{(d(Sx,Sy)+4)nK_1t^{n-1}}{(d(Sx,Sy)+2)(n-\nu^n)\Gamma(\beta)} d(x,y) \int_0^1 |1-s|^{\beta-1} \, \mathrm{d}s \\
&+ \frac{(d(Sx,Sy)+4)K_1}{(d(Sx,Sy)+2)\Gamma(\beta)} d(x,y) \int_0^t |t-s|^{\beta-1} \, \mathrm{d}s.
\end{aligned}
$$

Since $t \in [0,1]$, then using a simple calculation and (17), we have

$$
\begin{aligned}
&\frac{d(Sx,Sy)+4}{d(Sx,Sy)+2} \bigl| Sx(t) - Sy(t) \bigr| \\
&\quad \leqslant \frac{(d(Sx,Sy)+4)K_1 d(x,y)}{d(Sx,Sy)+2} \frac{1}{\Gamma(\beta+2)} \left[\frac{n\nu^{\beta+1} + (\beta+1)(2n-\nu^n)}{(n-\nu^n)} \right] \\
&\quad \leqslant d(x,y)
\end{aligned}
$$

and

$$
\frac{d(Sx,Sy)+4}{d(Sx,Sy)+2} d(Sx,Sy) \leqslant d(x,y).
$$

If we consider $\zeta(p,q) = q - (p+4)/(p+2)p$ and $T = I$, then all the conditions of Theorem 3 are satisfied. This means that S has a unique fixed point, that is, BVP (15)–(16) has a unique solution in $C[0,1]$. □

7 Conclusion

In this work, we consider a pair of nonlinear operators satisfying a nonlinear contraction involving a simulation function in a complete metric space. For this pair of operators with and without continuity, we establish common best proximity point results. Moreover, an application of our results is given to prove the existence of a solution for a nonlinear fractional differential equation.

Competing interests. The authors declare that they have no competing interests.

References

1. A. Abkar, M. Gabeleh, A note on some best proximity point theorems proved under P-property, **2013**:189567, 2013, https://doi.org/10.1155/2013/189567.
2. R.S. Adigüzel, Ü. Aksoy, E. Karapınar, I.M. Erhan, On the solution of a boundary value problem associated with a fractional differential equation, *Math. Methods Appl. Sci.*, 2020, https://doi.org/10.1002/mma.6652.
3. R.S. Adigüzel, Ü. Aksoy, E. Karapınar, I.M. Erhan, On the solutions of fractional differential equations via Geraghty type hybrid contractions, *Appl. Comput. Math.*, **20**(2):313–333, 2021.
4. R.S. Adigüzel, Ü. Aksoy, E. Karapınar, I.M. Erhan, Uniqueness of solution for higher-order nonlinear fractional differential equations with multi-point and integral boundary conditions, *RACSAM, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat.*, **115**(3):155, 2021, https://doi.org/10.1007/s13398-021-01095-3.
5. M.A. Al-Thagafi, N. Shahzad, Convergence and existence results for best proximity points, *Nonlinear Anal., Theory Methods Appl.*, **70**(10):3665–3671, 2009, https://doi.org/10.1016/j.na.2008.07.022.
6. M.A. Alghamdi, S. Gulyaz-Ozyurt, E. Karapınar, A note on extended Z-contraction, *Mathematics*, **8**(2):195, 2020, https://doi.org/10.3390/math8020195.
7. M.A. Alghamdi, N. Shahzad, F. Vetro, Best proximity points for some classes of proximal contractions, **2013**, 2013, https://doi.org/10.1155/2013/713252.
8. A. Almeida, E. Karapınar, K. Sadarangani, A note on best proximity point theorems under weak-property, **2014**, 2014, https://doi.org/10.1155/2014/716825.
9. O. Alqahtani, E. Karapınar, A bilateral contraction via simulation function, *Filomat*, **33**(15): 4837–4843, 2019, https://doi.org/10.2298/FIL1915837A.
10. R. Alsubaie, B. Alqahtani, E. Karapınar, A.F. Roldán López de Hierro, Extended simulation function via rational expressions, *Mathematics*, **8**(5):710, 2020, https://doi.org/10.3390/math8050710.
11. H. Argoubi, B. Samet, C. Vetro, Nonlinear contractions involving simulation functions in a metric space with a partial order, *J. Nonlinear Sci. Appl.*, **8**(6):1082–1094, 2015, https://doi.org/10.22436/jnsa.008.06.18.
12. C. Di Bari, T. Suzuki, C. Vetro, Best proximity points for cyclic Meir–Keeler contractions, *Nonlinear Anal., Theory Methods Appl.*, **69**(11):3790–3794, 2008, https://doi.org/10.1016/j.na.2007.10.014.

13. M. Jleli, V. Čojbašić Rajić, B. Samet, C. Vetro, Fixed point theorems on ordered metric spaces and applications to nonlinear elastic beam equations, *J. Fixed Point Theory Appl.*, **12**:175–192, 2012, https://doi.org/10.1007/s11784-012-0081-4.
14. E. Karapınar, Best proximity points of cyclic mappings, *Appl. Math. Lett.*, **25**(11):1761–1766, 2012, https://doi.org/10.1016/j.aml.2012.02.008.
15. E. Karapınar, Fixed points results via simulation functions, *Filomat*, **30**(8):2343–2350, 2016, https://doi.org/10.2298/FIL1608343K.
16. E. Karapınar, R.P. Agarwal, Interpolative Rus-Reich-Ćirić type contractions via simulation functions, *An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat.*, **27**(3):137–152, 2019, https://doi.org/10.2478/auom-2019-0038.
17. E. Karapınar, C.M. Chen, C.T. Lee, Best proximity point theorems for two weak cyclic contractions on metric-like spaces, *Mathematics*, **7**(4):349, 2019, https://doi.org/10.3390/math7040349.
18. E. Karapınar, S. Karpagam, P. Magadevan, B. Zlatanov, On Ω class of mappings in a p-cyclic complete metric space, *Symmetry*, **11**(4):534, 2019, https://doi.org/10.3390/sym11040534.
19. E. Karapınar, F. Khojasteh, An approach to best proximity points results via simulation functions, *J. Fixed Point Theory Appl.*, **19**:1983–1995, 2017, https://doi.org/10.1007/s11784-016-0380-2.
20. F. Khojasteh, S. Shukla, S. Radenović, A new approach to the study of fixed point theory for simulation functions, *Filomat*, **29**(6):1189–1194, 2015, https://doi.org/10.2298/FIL1506189K.
21. W.A. Kirk, S. Reich, P. Veeramani, Proximinal retracts and best proximity pair theorems, *Numer. Funct. Anal. Optim.*, **24**:851–862, 2003, https://doi.org/10.1081/NFA-120026380.
22. A. Kostić, E. Karapınar, V. Rakočević, Best proximity points and fixed points with R-functions in the framework of w-distances, *Bull. Aust. Math. Soc.*, **99**(3):497–507, 2019, https://doi.org/10.1017/S0004972718001193.
23. A. Kostić, V. Rakočević, S. Radenović, Best proximity points involving simulation functions with w_0-distance, *RACSAM, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat.*, **113**(2):715–727, 2019, https://doi.org/10.1007/s13398-018-0512-1.
24. P. Lo′lo′, M. Shabibi, Common best proximity points theorems for H-contractive non-self mappings, *Adv. Theory Nonlinear Anal. Appl.*, **5**(2):173–179, 2021, https://doi.org/10.31197/atnaa.776709.
25. P. Lo′lo′, M. Shams, M. De la Sen, Existence of a solution for a nonlinear integral equation by nonlinear contractions involving simulation function in partially ordered metric space, *Nonlinear Anal. Model. Control*, **28**(3):578–596, 2023, https://doi.org/10.15388/namc.2023.28.32119.
26. P. Lo′lo′, S.M. Vaezpour, J. Esmaily, Common best proximity points theorem for four mappings in metric-type spaces, *Fixed Point Theory Appl.*, **2015**(47):1–7, 2015, https://doi.org/10.1186/s13663-015-0298-1.
27. P. Lo′lo′, S.M. Vaezpour, R. Saadati, Common best proximity points results for new proximal C-contraction mappings, *Fixed Point Theory Appl*, **2016**(1):1–10, 2016, https://doi.org/10.1186/s13663-016-0545-0.

28. P. Lo′lo′, S.M. Vaezpour, R. Saadati, C. Park, Existence of a common solution of an integral equations system by (ψ, α, β)-weakly contractions, *J. Inequal. Appl.*, **2014**(517):1–20, 2014, https://doi.org/10.1186/1029-242X-2014-517.

29. P. Magadevan, S. Karpagam, E. Karapınar, Existence of fixed point and best proximity point of p-cyclic orbital ϕ-contraction map, *Nonlinear Anal. Model. Control*, **27**(1):91–101, 2022, https://doi.org/10.15388/namc.2022.27.25188.

30. H.K. Pathak, N. Shahzad, Convergence and existence results for best C-proximity points, *Georgian Math. J.*, **19**(2):301–316, 2012, https://doi.org/10.1515/gmj-2012-0012.

31. V. Pragadeeswarar, R. Gopi, M. De la Sen, S. Radenović, Proximally compatible mappings and common best proximity points, *Symmetry*, **12**(3):353, 2020, https://doi.org/10.3390/sym12030353.

32. A.F. Roldán-López-de-Hierro, E. Karapınar, C. Roldán-López de Hierro, J. Martínez-Moreno, Coincidence point theorems on metric spaces via simulation functions, *J. Comput. Appl. Math.*, **275**:345–355, 2015, https://doi.org/10.1016/j.cam.2014.07.011.

33. F. Tchier, C. Vetro, F. Vetro, Best approximation and variational inequality problems involving a simulation function, *Fixed Point Theory Appl.*, **2016**(1):1–15, 2016, https://doi.org/10.1186/s13663-016-0512-9.