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Practical stability for fractional impulsive control systems with noninstantaneous impulses on networks*

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Abstract. This paper investigates practical stability for a class of fractional-order impulsive control coupled systems with noninstantaneous impulses on networks. Using graph theory and Lyapunov method, new criteria for practical stability, uniform practical stability as well as practical asymptotic stability are established. In this paper, we extend graph theory to fractional-order system via piecewise Lyapunov-like functions in each vertex system to construct global Lyapunov-like functions. Our results are generalization of some known results of practical stability in the literature and provide a new method of impulsive control law for impulsive control systems with noninstantaneous impulses. Examples are given to illustrate the effectiveness of our results.

Keywords: practical stability, fractional order, noninstantaneous impulses, networks, graph theory.

1 Introduction

Impulsive control has wide applications in real world. Some useful impulsive control approaches have been presented in many fields such as in financial models, epidemic models, neural networks and so on [6, 7, 17, 19, 21, 25]. As is known to us, impulsive control is a discontinuous control. In some situation, it can perform better than continuous case for special control purpose. There has been great interest in this area as witnessed by scholars new contributions. Compared with instantaneous impulses, the action of non-instantaneous impulses still starts at an arbitrary fixed point but it remains active on a finite time interval. While, there are few works about impulsive control concerning noninstantaneous impulses.

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Coupled systems of differential equations on networks (CSDENs) have been widely applied in various fields of biology, engineering, social science and physical science such as in modeling the spread of infectious diseases in heterogeneous populations, neural networks, ecosystems and so on [4, 5, 12, 24, 28]. Especially, the stability analysis of CSDENs is one of the most essential topics in practice. Li and Shuai [16] proposed a new method by combining graph theory with Lyapunov methods to investigate global stability problem for CSDENs. Since then, Suo [23] applied results from graph theory to construct global Lyapunov functions and then established a new asymptotic stability and exponential stability principles. However, many results about stability of coupled system on networks utilize differential equations of integer order [22, 26]. Until now, there are few relevant researches about stability analysis for coupled systems of fractional-order differential equations on networks (CSFDENs). Li and Jiang [14] investigated CSFDENs, they obtained a global Mittag-Leffler stability principles by Lyapunov method and graph theory. Recently, Li [15] studied stability of fractional-order impulsive coupled nonautonomous (FOIC) systems on networks using graph theory and Lyapunov method to get stability for a kind of FOIC systems. Some remarkable achievements have been made in [8, 13–15, 18, 22, 26] during the past few years.

Practical stability analysis is one of the most important types for stability theory. In 2016, Stamova [20] derived the practical stability criteria of fractional-order impulsive control systems by using fractional comparison principle, scalar and vector Lyapunov-like functions. In 2017, Agarwal [2] investigated practical stability of nonlinear fractional differential equations with noninstantaneous impulses and presented a new definition of the derivative of a Lyapunov-like function; see literatures [2, 3, 9, 11, 20] for more details.

The purpose of this paper is to study the practical stability for a class of impulsive CSFDENs with noninstantaneous impulses. Generally speaking, we investigate systems on networks by studying each individual vertex dynamics to determine the collective dynamics and explore the noninstantaneous impulses effect on systems. We establish new practical stability criteria for the systems. Some sufficient conditions are given to meet the practical stability, uniform practical stability and practical asymptotic stability of this coupled systems on networks.

Our results generalize relevant results in [2]. We provide a new method of impulsive control law for impulsive CSFDENs with noninstantaneous impulses by using graph theory and Lyapunov method. The systems in [2] can be considered as a special case for $i = 1$. It is the first time to consider fractional-order coupled systems with noninstantaneous impulses via graph theory. We also illustrate that the topology property of systems have a close connection with the practical stability of the systems.

Compared with the existing method for studying impulsive CSFDENs, we develop a systematic approach to construct a Lyapunov-like function by using the Lyapunov-like function of each vertex system, which avoids the difficulty of finding it directly of the whole system. Especially for systems with noninstantaneous systems, it is a creative work. In this paper, we are interested in whether the dynamical behaviors can be effected by network encoded in the directed graph. Therefore, to better solve this problem, we construct piecewise continuous Lyapunov-like functions V_i in each vertex system, then construct a global Lyapunov-like function V for coupled systems as $V(x) = c_i V_i(x)$,

$c_i \geqslant 0$. Besides, this method constructs a relation between the practical stability criteria and topology property of the network, which can help analyzing the practical stability of fractional-order complex networks.

The rest of our paper are organized as follows. In Section 2, we introduce some necessary notions, definitions and lemmas. Practical stability criteria about fractional-order coupled systems on networks are given in Section 3. In Section 4, examples are given to show the applicability of our results.

2 Preliminaries

In this section, we recall some basic and essential definitions of fractional calculus as well as concepts and lemmas of graph theories for better obtaining our main results.

The following knowledge of graph theories can be found in [16].

A nonempty directed graph $G = [V, E]$ is defined with a vertex set $V = \{1, 2, \dots, n\}$ and an edge set E, each element of E denotes an arc (i, j) leading from the initial vertex i to terminal vertex j. Two diagraph $G = [V, E]$ and $G' = [V', E']$ are given. If $V' \subseteq V$, $E' \subseteq E$, then G' is called a subgraph of G. A subgraph G' of G is a spanning subgraph if G' contains all vertices from G.

A digraph is weighted if a positive weight a_{ij} is assigned to each arc. Denote $a_{ij} > 0$ if and only if there exists arc from vertex i to j in G, otherwise, $a_{ij} = 0$. The weight $W(G)$ of G denotes the product of the weights on all its arcs. A directed path P is a subgraph of G with vertices $\{i_1, i_2, \dots, i_n\}$ and a set of arcs $\{i_k, i_{k+1}, k = 1, 2, \dots, n-1\}$. If $i_n = i_1$, then P is a directed cycle.

Assume that G is a weighted diagraph with n vertices. A is a matrix $(a_{ij})_{n\times n}$, whose element equals the weight of each arc (i, j). Denote weighted diagraph with weight matrix A as (G, A). (G, A) is said to be balanced if $W(C) = W(-C)$, C covers all directed cycle in G, $-C$ means the reverse of C constructed by reversing direction of all arcs in C. A connected subgraph is a tree if it has no cycle. We call i the root of a tree if i is not a terminal vertex of any arc and each of the remaining vertices is a terminal arc of one arc. A subgraph Q is a unicyclic graph when it is a disjoint union of root trees, whose roots form a directed cycle. Q and Q' are unicyclic graphs with the cycles C_Q and $-C_Q$, respectively. When (G, A) is balanced, $W(Q) = W(-Q)$. The Laplacian matrix of (G, A) is defined as $L = (c_{ij})_{n\times n}$, where $c_{ij} = -a_{ij}$ for $i \neq j$, $c_{ij} = \sum_{k\neq i} a_{ik}$ for $i = j$. The constant $\lambda_{\max}(A)$ denotes the maximum eigenvalue of matrix A.

Lemma 1. (*See* [16].) *Assume $n \geqslant 2$. Let c_i denote the cofactor of the ith diagonal element of L. Then the following equation holds*:

$$\sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(x_i, x_j) = \sum_{Q\in\mathbb{Q}} W(Q) \sum_{(s,r)\in E(C_Q)} F_{rs}(x_r, x_s),$$

where $F_{ij}(x_i, x_j)$ are arbitrary functions, $1 \leqslant i, j \leqslant n$, a_{ij} are the elements of L, $\mathbb{Q}$ is the set of all spanning unicyclic graphs of (G, A), $W(Q)$ is the weight of Q, C_Q denotes the directed cycle of Q.

If (G, A) is balanced, then

$$\sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(x_i, x_j) = \frac{1}{2} \sum_{Q \in \mathbb{Q}} W(Q) \sum_{(j,i) \in E(C_Q)} \big[F_{ij}(t, x_i, x_j) + F_{ji}(t, x_j, x_i)\big],$$

and if (G, A) is strongly connected, then $c_i > 0$ for $i = 1, \dots, n$.

Definition 1. (See [27].) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f : [t_0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s)\, \mathrm{d}s, \quad t > t_0,$$

where $\Gamma(\alpha)$ is the Gamma function, provided the right side is pointwise defined on $[t_0, +\infty)$.

Definition 2. (See [27].) The Caputo fractional derivative of order $\alpha > 0$ of a function $f : [t_0, +\infty) \to \mathbb{R}$ is given by

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s)\, \mathrm{d}s, \quad t > t_0, \tag{1}$$

where n is the smallest integer greater than or equal to α, provided that the right side is pointwise defined on $[t_0, +\infty)$.

In case $0 < \alpha < 1$, we have

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-s)^{-\alpha} f'(s)\, \mathrm{d}s, \quad t \geqslant t_0.$$

Definition 3. (See [2].) We say $m \in C^{\alpha}([t_0, T], R^n)$ if $m(t)$ is differentiable, the Caputo derivative $^{C}D^{\alpha} f(t)$ exists and satisfies (1) for $t \in [t_0, T]$.

Now, we introduce the definition of Grunwald–Letnikov fractional derivative and Grunwald–Letnikov fractional Dini derivative, then we use the relation between Caputo fractional derivative and Grunwald–Letnikov fractional derivative to define Caputo fractional Dini derivative. The details can be found in [10].

Definition 4. (See [10].) The Grunwald–Letnikov fractional derivative of a function x is given by

$$^{GL}D^{\alpha} x(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{r=0}^{[(t-t_0)/h]} (-1)^r q_{C_r} x(t - rh), \quad t \geqslant t_0,$$

and the Grunwald–Letnikov fractional Dini derivative of a function x is defined as

$$^{GL}D_+^\alpha x(t) = \limsup_{h\to 0^+} \frac{1}{h^\alpha} \sum_{r=0}^{[(t-t_0)/h]} (-1)^r q_{C_r} x(t-rh), \quad t \geqslant t_0,$$

where q_{C_r} are the Binomial coefficients, and $[(t-t_0)/h]$ denotes the integer part of $(t-t_0)/h$.

Definition 5. (See [10].) The Caputo fractional Dini derivative of a function x is defined as

$$^{C}D_+^\alpha x(t) = {}^{GL}D_+^\alpha \bigl(x(t)-x_0\bigr),$$

i.e.,

$$^{C}D_+^\alpha x(t) = \limsup_{h\to 0^+} \frac{1}{h^\alpha}\bigl[x(t)-x_0-S(x,h,q,r)\bigr], \quad t \geqslant t_0,$$

where

$$S(x,h,q,r) = \sum_{r=0}^{[(t-t_0)/h]} (-1)^{r+1} q_{C_r}\bigl(x(t-rh)-x_0\bigr).$$

Consider a network represented by a diagraph G with n vertices. A fractional-order impulsive control coupled system with noninstantaneous impulses can be built on G by assigning dynamics on each vertex, then coupling these individual vertex dynamics in G. In this way, for $1 \leqslant i \leqslant n$, the ith vertex dynamics is defined as the following system:

$$\begin{aligned} {}^{C}D^\alpha x_i &= f_i(t,x_i), \quad t \in (s_k, t_{k+1}], \\ x_i(t) &= I_k\bigl(t, x_i(t_k-0)\bigr), \quad t \in (t_k, s_k], \\ x_i\bigl(t_0^+\bigr) &= x_{i0}, \end{aligned} \tag{2}$$

where $0<\alpha<1$, $x_i \in \mathbb{R}^{m_i}$, $f_i : \bigcup_{k=0}^\infty [s_k, t_{k+1}] \times \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$, $I_k : (t_k, s_k] \times \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$. $\{t_k\}_{k=1}^\infty$, $\{s_k\}_{k=0}^\infty$ are two increasing sequences such that $0 = s_0 < t_k \leqslant s_k < t_{k+1}$, $k = 1, 2, \dots$, $\lim_{k\to\infty} t_k = \infty$. $t_0 \in \bigcup_{k=0}^\infty (s_k, t_{k+1}]$ be a given arbitrary point. Without loss of generality, we make an assumption that $t_0 \in [s_0, t_1)$.

The solution $x_i(t) = x_i(t, t_0, x_{i0})$, $t \geqslant t_0$, of the ith vertex system (2) satisfies

$$x_i(t) = \begin{cases} x_{i0} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f_i(s, x_i(s))\,\mathrm{d}s, & t \in [t_0, t_1], \\ I_1(t, x_i(t_1-0)), & t \in (t_1, s_1], \\ I_1(s_1, x_i(t_1-0)) + \frac{1}{\Gamma(\alpha)} \int_{s_1}^t (t-s)^{\alpha-1} f_i(s, x_i(s))\,\mathrm{d}s, & t \in [s_1, t_2], \\ \dots, & \\ I_k(t, x_i(t_k-0)), & t \in (t_k, s_k], \\ I_k(s_k, x_i(t_k-0)) + \frac{1}{\Gamma(\alpha)} \int_{s_k}^t (t-s)^{\alpha-1} f_i(x_i(s))\,\mathrm{d}s, & t \in [s_k, t_{k+1}], \\ \dots . & \end{cases}$$

We can refer to [2] for detailed proof.

Let $J \subset \mathbb{R}_+$ be a given interval, for $1 \leqslant i \leqslant n$, $\Omega_i \subset \mathbb{R}^{m_i}$. We introduce the following class of functions:

$$\begin{aligned}
&PC^{\alpha}(J, \Omega_i) \\
&\quad = \Biggl\{ x_i \in C^{\alpha}\Biggl(J \cap \Biggl(\bigcup_{k=0}^{\infty} (s_k, t_{k+1}] \Biggr),\ \Omega_i \Biggr) \bigcup C\Biggl(J \cup \Biggl(\bigcup_{k=1}^{\infty} (t_k, s_k] \Biggr), \Omega_i \Biggr): \\
&\qquad\quad x_i(t_k) = \lim_{t \to t_k^-} x_i(t) < \infty \text{ and } x_i(t_k + 0) = \lim_{t \to t_k^+} x_i(t) < \infty \text{ for } k\text{: } t_k \in J; \\
&\qquad\quad x_i(s_k) = x_i(s_k - 0) = \lim_{t \to s_k^-} x_i(t) = x(s_k + 0) = \lim_{t \to t_k^+} x_i(t) \text{ for } k\text{: } s_k \in J \Biggr\}.
\end{aligned}$$

Remark 1. From the above description of any solution for system (2) we can conclude that $x_i(t) = x_i(t, t_0, x_{i0})$ ($1 \leqslant i \leqslant n$, $t \geqslant t_0$) of (2) is discontinuous at points $t_k, k = 1, 2, \dots$.

Definition 6. (See [2].) Let $J \subset \mathbb{R}_+$ be a given interval. For $1 \leqslant i \leqslant n$, $\Omega_i \subset \mathbb{R}^{m_i}$, $0 \in \Omega_i$ are given sets. We say that the functions $V_i(t, x_i) : J \times \Omega_i \to \mathbb{R}_+$, $V_i(t, 0) \equiv 0$ belong to the classes $\Theta_i(J, \Omega_i)$ if

(i) The functions $V_i(t, x_i)$ are continuous on $J \setminus \{t_1, t_2, \dots\} \times \Omega_i$ and they are locally Lipschitzian with respect to the second argument;

(ii) For each $t_k \in J$ and $x_i \in \Omega_i$, there exist finite limits

$$\begin{aligned}
V_i(t_k - 0,\ x_i) &= \lim_{t \to t_k^-} V_i(t, x_i) < \infty, \\
V_i(t_k + 0,\ x_i) &= \lim_{t \to t_k^+} V_i(t, x_i) < \infty,
\end{aligned}$$

and the following equalities are valid:

$$V_i(t_k - 0,\ x_i) = V_i(t_k, x_i).$$

For $V_i \in \Theta_i(J, \Omega_i)$, $1 \leqslant i \leqslant n$, we define the generalized Caputo fractional Dini derivative with respect to system (2) as

$$\begin{aligned}
{}^{C}D_{+}^{\alpha} V_i(t, x_i) = \limsup_{h \to 0^+} \frac{1}{h^{\alpha}} \Biggl\{ & V_i(t, x_i) - V_i(t_0, x_{i0}) \\
& - \sum_{r=0}^{[(t-t_0)/h]} (-1)^{r+1} q_{C_r} \bigl(V_i(t - rh,\ x_i - h^{\alpha} f_i(t, x_i) - V_i(t_0, x_{i0}) \bigr) \Biggr\},
\end{aligned}$$

where $t_0 \in J$, and for any $t \in (s_k, t_{k+1}) \cap J$, $k = 0, 1, 2, \dots$, there exists $h_t > 0$ such that $t - h \in (s_k, t_{k+1}) \cap J, x - h^{\alpha} f_i(t, x_i) \in \Omega_i$ for $0 < h \leqslant h_t$.

Together with system (2), we consider the scalar comparison system on graph. The ith vertex dynamics is described as follows:

$$\begin{aligned} {}^{C}D^{\alpha}u_i &= h_i(t, u_i), \quad t \in (s_k, t_{k+1}], \\ u_i(t) &= S_k\bigl(t, u_i(t_k - 0)\bigr), \quad t \in (t_k, s_k], \\ u_i(t_0^+) &= u_{i0}, \end{aligned} \tag{3}$$

where $u_i, u_{i0} \in \mathbb{R}$, $h_i \in \bigcup_{k=0}^{\infty}[s_k, t_{k+1}] \times \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$, $S_k : (t_k, s_k] \times \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$, $k = 1, 2, \dots$, $u = (u_1, u_2, \dots, u_n)$, $u_0 = u(t_0^+) = (u_{10}, u_{20}, \dots, u_{n0})$.

Next, we prove some comparison results for noninstantaneous impulsive Caputo fractional-order system (2) using Definition 4 for fractional Dini derivative. Without loss of generality, we assume $t_0 \in [s_0, t_1)$. We will use results in Lemma 2 of [2] to obtain comparison results for system (2).

Lemma 2. (*See* [2].) *For $1 \leqslant i \leqslant n$, we let*:

(i) *The function $x_i(t) = x_i(t; t_0, x_{i0}) \in PC^{\alpha}([t_0, T], \varOmega_i)$ is the solution of initial value problems (IVPs) for the ith vertex system* (2)*, where $t_0 \in \bigcup_{k=0}^{\infty}[s_k, t_{k+1})$, $\varOmega_i \subset \mathbb{R}^n$, $T > t_0 \geqslant 0$.*

(ii) *For $t_k \in (t_0, T)$, the functions $S_k \in C([t_k, s_k] \times \mathbb{R}, \mathbb{R})$ are such that $S_k(t, z) \leqslant S_k(t, w)$ for $z \leqslant w$, $t \in [t_k, s_k], z, w \in \mathbb{R}$.*

(iii) *The function $h_i \in C([t_0, T] \bigcup_{k=0}^{\infty}[s_k, t_{k+1}] \times \mathbb{R}, \mathbb{R})$ and for a given $u_0 \in \mathbb{R}$, the IVPs for the ith vertex of the scalar system* (3) *has a maximal solution $u_i^*(t) = u_i(t; t_0, u_{i0}) \in PC^{\alpha}([t_0, T], \mathbb{R})$.*

(iv) *The function $V_i \in \varTheta_i([t_0, T], \varOmega_i)$, and the following inequalities hold*:

$$ {}^{C}D_{+}^{\alpha}V_i\bigl(t, x_i(t); t_0, x_{i0}\bigr) \leqslant h_i\bigl(t, V_i\bigl(t, x_i(t)\bigr)\bigr), \quad t \in (t_0, T) \bigcap_{k=0}^{\infty}[s_k, t_{k+1}),$$

$$V_i\bigl(t, I_k(t,\, x_i(t_k - 0))\bigr) \leqslant S_k\bigl(V_i\bigl(t_k - 0,\, x_i(t_k - 0)\bigr)\bigr), \quad t \in [t_0, T] \cap (t_k, s_k].$$

Then the inequality $V_i(t_0, x_0) \leqslant u_0$ implies $V_i(t, x_i(t)) \leqslant u_i^*(t)$ on $[t_0, T]$.

Corollary 1. *For $1 \leqslant i \leqslant n$, if the function $V_i \in \varTheta_i([t_0, T], \varOmega_i)$ satisfies*

$$ {}^{C}D_{+}^{\alpha}V_i(t, x_i) \leqslant 0, \quad t \in (t_0, T) \bigcap_{k=0}^{\infty}[s_k, t_{k+1}),$$

then

$$V_i\bigl(t, x_i(t)\bigr) \leqslant V_i(t_0, x_{i0}) \quad \textit{on } [t_0, T].$$

Lemma 3. *For $1 \leqslant i \leqslant n$, if the function $V_i \in \varTheta_i([t_0, T], \varOmega_i)$ satisfies*

$$ {}^{C}D_{+}^{\alpha}V_i\bigl(t, x_i(t)\bigr) \leqslant MV_i\bigl(t, x_i(t)\bigr), \quad t \in (t_0, T) \bigcap_{k=0}^{\infty}[s_k, t_{k+1}),$$

then

$$V_i\bigl(t, x_i(t)\bigr) \leqslant V_i(t_0, x_{i0})E_{\alpha}\bigl(M(t - t_0)^{\alpha}\bigr) \quad \textit{on } [t_0, T]. \tag{4}$$

Proof. Take $h_i(t, u_i) = Mu_i$, $S_k(u_i(t_k - 0)) = u_i(t_k - 0)$, $u_i(0) = V_i(t_0, x_{i0})$ in system (3). The solutions of IVPs for (3) satisfy

$$u_i(t) = \begin{cases} V_i(t_0, x_{i0}) E_\alpha(M(t - t_0)^\alpha), & t \in [t_0, t_1], \\ V_i(t_0, x_{i0}) E_\alpha(M(t_1 - t_0)^\alpha), & t \in (t_1, s_1], \\ V_i(t_0, x_{i0}) E_\alpha(M(t_1 - t_0)^\alpha) E_\alpha(M(t - t_1)^\alpha), & t \in [s_1, t_2], \\ \dots, \\ \prod_{k=1}^n V_i(t_0, x_{i0}) E_\alpha(M(t_k - t_{k-1})^\alpha), & t \in (t_k, s_k], \\ \prod_{k=1}^n V_i(t_0, x_{i0}) E_\alpha(M(t_k - t_{k-1})^\alpha) E_\alpha(M(t - t_k)^\alpha), & t \in [s_k, t_{k+1}], \\ \dots . \end{cases}$$

Without loss of generality, we assume $t_0 \in [s_0, t_1)$. Let $u_i(0) = V_i(t_0, x_{i0})$. We prove by induction.

Let $t \in [t_0, t_1] \cap [t_0, T]$, $x_i(t; t_0, x_{i0}) \in C^\alpha([t_0, t_1] \cap [t_0, T], \mathbb{R}^n)$ be a solution of (2). Then

$$V_i\bigl(t, x_i(t)\bigr) \leqslant V_i(t_0, x_{i0}) E_\alpha\bigl(M(t - t_0)^\alpha\bigr),$$

the conclusion follows from Corollary 2.3.2 in [1], i.e., inequality (4) holds on $[t_0, t_1] \cap [t_0, T]$.

Let $T > t_1$, $t \in [t_1, s_1] \cap [t_0, T]$. Then we get $u_i(t_1) = V_i(t_0, x_{i0}) E_\alpha(M(t_1 - t_0)^\alpha)$. After that, still by Corollary 2.3.2 in [1] we have

$$\begin{aligned} V_i\bigl(t, x_i(t)\bigr) &\leqslant V_i(t_0, x_{i0}) E_\alpha\bigl(M(t_1 - t_0)^\alpha\bigr) E_\alpha\bigl(M(t - t_1)^\alpha\bigr) \\ &\leqslant V_i(t_0, x_{i0}) E_\alpha\bigl(M(t - t_0)^\alpha\bigr), \end{aligned}$$

where $t \in [t_0, T]$. So inequality (4) holds on $[t_0, s_1] \cap [t_0, T]$. We can continue this process, then induction proves that Lemma 3 is true. □

Remark 2. The results of Lemmas 2, 3 and Corollary 1 are true on the half-line. In other words, conditions in Lemma 2 are satisfied for $T = \infty$. Then the conclusions still hold.

3 Practical stability analysis for fractional-order coupled systems with noninstantaneous impulses on networks

In this section, we investigate the following fractional-order coupled system with noninstantaneous impulses on graph G:

$$\begin{aligned} {}^C D^\alpha x_i &= f_i(t, x_i) + \sum_{j=1}^n g_{ij}(t, x_i, x_j), \quad t \in (s_k, t_{k+1}], \\ x_i(t) &= I_k\bigl(t, x_i(t_k - 0)\bigr), \quad t \in (t_k, s_k], \\ x_i(t_0^+) &= x_{i0} \end{aligned} \tag{5}$$

for $1 \leqslant i \leqslant n$, where $g_{ij} : \bigcup_{k=0}^{\infty}[s_k, t_{k+1}] \times \mathbb{R}^{m_i} \times \mathbb{R}^{m_j} \to \mathbb{R}^{m_i}$ $(k = 1, 2, \dots)$ represents the influence of vertex j on vertex i. If there is no arc from j to i in graph G, $g_{ij} = 0$, $i, j = 1, 2, \dots, n$.

The following assumptions are given:

(H1) The function $f_i \in C(\bigcup_{k=0}^{\infty}[s_k, t_{k+1}] \times \mathbb{R}^{m_i}, \mathbb{R}^{m_i})$, $f_i(t, 0) = 0$, $g_{ij} \in C(\bigcup_{k=0}^{\infty}(s_k, t_{k+1}] \times \mathbb{R}^{m_i} \times \mathbb{R}^{m_j} \to \mathbb{R}^{m_i})$, $g_{ij}(t, 0, 0) = 0$. f_i, g_{ij} satisfy the globally Lipschitz conditions.

(H2) The functions $I_k \in C([t_k, s_k] \times \mathbb{R}^{m_i}, \mathbb{R}^{m_i})$, $I_k(t, 0) = 0$ for $t \in [t_k, s_k]$, I_k, $k = 1, 2, \dots$, satisfies the globally Lipschitz conditions.

If (H1)–(H2) are satisfied, IVPs for (5) has a trivial solution $x = (x_1, x_2 \dots, x_n) = 0$.

For $V_i \in \Theta_i(J, \Omega_i)$, $1 \leqslant i \leqslant n$, we define the generalized Caputo fractional Dini derivative with respect to system (5) as

$$\begin{aligned} {}^{C}D_{+}^{\alpha} V_i(t, x_i; t_0, x_{i0}) = \limsup_{h \to 0^+} \frac{1}{h^\alpha} \Biggl\{ & V_i(t, x_i) - V_i(t_0, x_{i0}) \\ & - \sum_{r=0}^{[(t-t_0)/h]} (-1)^{r+1} q_{C_r} V_i \Biggl(t - rh,\ x_i - h^\alpha f_i(t, x_i) \\ & - h^\alpha \sum_{j=1}^{n} g_{ij}(t, x_i, x_j) - V_i(t_0, x_{i0}) \Biggr) \Biggr\}, \end{aligned}$$

where $t_0 \in J$, and for any $t \in (s_k, t_{k+1}) \cap J$, $k = 0, 1, 2, \dots$, there exists $h_t > 0$ such that $t - h \in (s_k, t_{k+1}) \cap J$, $x - h^\alpha f_i(t, x_i) \in \Omega_i$ for $0 < h \leqslant h_t$.

Let $\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_n \subset \mathbb{R}^m$, $m = m_1 + m_2 + \dots + m_n$, $x_0 = x(t_0^+) = (x_{10}, x_{20} \dots, x_{n0})$.

We define $V : J \times \Omega \to \mathbb{R}_+$ as follows:

$$V(t, x) = \sum_{i=1}^{n} c_i V_i(t, x_i),$$

where c_i is defined in Lemma 1, $i = 1, 2, \dots, n$.

Definition 7. The zero solution of system (5) is said to be

(S1) practically stable with respect to (λ, A), $0 < \lambda < A$, if there exists $t_0 \in \bigcup_{k=0}^{\infty}[s_k, t_{k+1})$ such that for any $x_0 \in \mathbb{R}^m$, the inequality $\|x_0\| < \lambda$ implies $\|x(t; t_0, x_0)\| < A$ for $t \geqslant t_0$;

(S2) uniformly practically stable with respect to (λ, A) if (S1) holds for every $t_0 \in \bigcup_{k=0}^{\infty}[s_k, t_{k+1})$;

(S3) practically asymptotically stable with respect to (λ, A) if (S1) holds and

$$\lim_{t \to \infty} \bigl\|x(t, t_0, x_0)\bigr\| = 0.$$

Remark 3. From Definition 7 we can see that practical stability is neither stronger nor weaker than stability in the sense of Lyapunov. Practical stability is not defined in the neighborhood of the origin, but an arbitrary set. To some extent, the range of this set can better reflect the essence of the study of practical problems. In detail, a system considered may be unstable in the sense of Lyapunov stability, whereas in practical problems, the dynamic behavior of the system can meet the actual demand within a certain range. For example, rocket launchers are considered to have unstable navigation trajectory, while the effect of rocket system under oscillation can be accepted, hence it is practical stability. The key point of the creation for practical stability theory is that practical stability and other means of stability are completely independent concepts.

We use the following sets:

1. $\mathcal{K} = \{a \in C(\mathbb{R}_+, \mathbb{R}_+)$: a is strictly increasing and $a(0) = 0\}$;
2. $S(\lambda) = \{x \in \mathbb{R}^m\colon \|x\| \leqslant \lambda\}$, $\lambda > 0$, $S(A) = \{x \in \mathbb{R}^m\colon \|x\| \leqslant A\}$, $A > 0$.

Theorem 1. *For* $1 \leqslant i \leqslant n$, *let the following conditions be fulfilled*:

(i) *The function* $S_k \in C([t_k, s_k]\times\mathbb{R}, \mathbb{R})$ *is such that* $S_k(t, z) \leqslant S_k(t, w)$ *for* $z \leqslant w$, $t \in [t_k, s_k]$, $z, w \in \mathbb{R}$, $k = 1, 2 \dots .$

(ii) *The function* $h_i \in C(\bigcup_{k=0}^{\infty}[s_k, t_{k+1}] \times \mathbb{R}, \mathbb{R})$, *and for a given* $u_{i0} \in \mathbb{R}$, *the IVPs for the scalar system* (3) *has maximal solutions* $u_i^*(t) = u_i(t; t_0, u_{i0}) \in PC^{\alpha}([t_0, \infty], \mathbb{R})$, $k = 0, 1, 2 \dots .$

(iii) *There exist functions* $V_i(t, x_i) \in \varTheta_i(J, \varOmega_i)$, $F_{ij}(x_i, x_j)$ *and a matrix* $A = (a_{ij})_{n\times n}$ *in which* $a_{ij} \geqslant 0$ *such that*

$$ {}^{C}D_{+}^{\alpha} V_i(t, x_i) \leqslant \sum_{j=1}^{n} a_{ij} F_{ij}(x_i, x_j), t \in \bigcup_{k=0}^{\infty} [s_k, t_{k+1}], \quad t \geqslant t_0. $$

(iv) *Along each directed cycle* C *of the weighted digraph* (G, A), (G, A) *is strongly connected,*

$$ \sum_{(s,r)\in E(C)} F_{r,s}(x_r, x_s) \leqslant 0, \quad t \geqslant t_0,\ x_r \in \varOmega_r,\ x_s \in \varOmega_s. $$

(v) *There exist functions* $V_i \in \varTheta_i(J, \varOmega_i)$ *such that*

$$ \begin{aligned} &V_i\bigl(t,\, I_k\bigl(t, x_i(t_k - 0)\bigr)\bigr) \leqslant V_i(t_k - 0,\, x_i), \\ &V_i\bigl(t, I_k\bigl(t,\, x_i(t_k - 0)\bigr)\bigr) \leqslant S_k\bigl(V_i\bigl(t_k - 0,\, x_i(t_k - 0)\bigr)\bigr), \end{aligned} $$

where $t \in (t_k, s_k]$, $t_k \geqslant t_0$, $k = 1, 2 \dots,$ $x_i \in S_i(\rho)$, $S_i(\rho) = \{x_i \in \mathbb{R}^{m_i}\colon \|x_i\| < \rho\}$.

(vi) *There exist functions* $b_i \in \mathcal{K}$ *such that*

$$ b_i\bigl(\|x_i\|\bigr) \leqslant V_i(t, x_i), \quad x_i \in S_i(\rho),\ i = 1, 2, \dots, n. $$

Then the trivial solution of system (5) *is practically stable.*

Proof. Define a function $V(t,x) = \sum_{i=1}^{n} c_i V_i(t,x_i)$. According to condition (iii), when $t \in \bigcup_{k=0}^{\infty}[s_k, t_{k+1}]$, we get

$$\begin{aligned} {}^{C}D_{+}^{\alpha} V(t,x) &= {}^{C}D_{+}^{\alpha} \sum_{i=1}^{n} c_i V_i(t,x_i) \leqslant \sum_{i=1}^{n} c_i \bigl({}^{C}D_{+}^{\alpha} V_i(t,x_i)\bigr) \\ &\leqslant \sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(x_i,x_j). \end{aligned}$$

Making use of Lemma 1 in weighted digraph (G,A), we obtain

$$\sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(x_i,x_j) = \sum_{Q\in\mathbb{Q}} W(Q) \sum_{(s,r)\in E(C_Q)} F_{rs}(x_r,x_s).$$

Combing condition (iv) with the fact $W(Q) > 0$, we have

$$ {}^{C}D_{+}^{\alpha} V(t,x) \leqslant \sum_{Q\in\mathbb{Q}} W(Q) \sum_{(s,r)\in E(C_Q)} F_{rs}(x_r,x_s) \leqslant 0. \tag{6}$$

Define $\lambda = \min\{\lambda_1,\lambda_2,\dots,\lambda_n\}$, $b(\|x\|) = n\min\{c_1 b_1\|x_1\|, c_2 b_2\|x_2\|,\dots,c_n b_n\|x_n\|\}$. On account of $b_i \in \mathcal{K}$, $i = 1,2,\dots,n$, we can deduce that $b \in \mathcal{K}$.

Two constants (λ, A) are given, and $0 < \lambda < A$. Let $x_0 \in \Omega$. There exists a $t_0 \in \bigcup_{k=0}^{\infty}[s_k, t_{k+1})$ such that for $\|x_0\| < \lambda$,

$$\sup_{\|x_0\|<\lambda} V(t_0^+, x_0) < b(A).$$

Then from conditions in Theorem 1 and conclusions of (6), by Corollary 1, we have

$$V(t,x) \leqslant V\bigl(t_0^+, x_0\bigr) < b(A). \tag{7}$$

In view of condition (vi), we derive

$$V(t,x) = \sum_{i=1}^{n} c_i V_i(t,x_i) \geqslant \sum_{i=1}^{n} c_i b_i\bigl(\|x_i\|\bigr) \geqslant \sum_{i=1}^{n} \frac{1}{n} b\bigl(\|x\|\bigr) = b\bigl(\|x\|\bigr). \tag{8}$$

Combining (7) and (8), we obtain

$$b\bigl(\|x\|\bigr) < b(A)$$

for $t \geqslant t_0$, provided that $x_0 \in S(\lambda)$, which completes the proof. □

Corollary 2. *Assume that (G,A) is balanced such that*

$$\sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(x_i,x_j) = \frac{1}{2} \sum_{Q\in\mathbb{Q}} W(Q) \sum_{(s,r)\in E(C_Q)} \bigl[F_{rs}(x_r,x_s) + F_{sr}(x_s,x_r)\bigr].$$

Condition (iv) *of Theorem* 1 *is replaced by*

$$\text{(iv}'\text{)} \qquad \sum_{(s,r)\in E(C_Q)} \big[F_{rs}(x_r,x_s)+F_{sr}(x_s,x_r)\big] \leqslant 0, \quad t \geqslant t_0,\ x_r \in \Omega_r,\ x_s \in \Omega_s.$$

Then the trivial solution of system (5) *is practically stable.*

Remark 4. In Theorem 1, we assume that (G, A) is strong connected, which means the topology property of coupled system (5) in a close connection with the practical stability of (5). In fact, without the strong connectedness of (G, A), we can only judge the practical stability of vertex system, but we can not judge the practical stability of the whole system. We give an example to illustrate.

Given a weighted graph (G, A) with 3 vertices, where

$$A = (a_{ij})_{3\times 3} = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

The Laplacian matrix of (G, A) is defined as

$$L = (p_{ij})_{3\times 3} = \begin{bmatrix} 4 & -1 & 3 \\ -1 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Through calculation, we get $c_1 = c_2 = 0$, $c_3 = 15$, which means that the practical stability of the third vertex can be checked, but the practical stability of the whole system is unable to be determined. So the strong connectedness can definitely have effect on the practical stability.

Theorem 2. *Assume conditions of Theorem* 1 *hold, and let following condition holds*:

(I) *There exist functions $a_i \in \mathcal{K}$ such that*

$$V_i(t,x_i) \leqslant a_i\big(\|x_i\|\big), \quad x_i \in S_i(\rho),\ i = 1,2,\dots,n.$$

Then the trivial solution of system (5) *is uniformly practically stable with respect to* (λ, A).

Proof. For a function $V(t,x) = \bigcup_{i=1}^{n} c_i V_i(t,x_i)$, where c_i, $i = 1,2,\dots,n$, is defined in Lemma 1. Two constants λ, A ($0 < \lambda < A$) are given such that $n \cdot a(\|x\|) < b(A)$, provided that $x \in S(\lambda)$.

Define $a(\|x\|) = \max\{c_1 a_1 \|x_1\|, c_2 a_2 \|x_2\|, \dots, c_n a_n \|x_n\|\}$. On account of $a_i \in \mathcal{K}$, $i = 1,2,\dots,n$, we can deduce that $a \in \mathcal{K}$.

If $x_0 \in S(\lambda)$, it follows from conditions (v), (vi) and (7) that for $t \geqslant t_0$,

$$\begin{aligned} V(t,x) &\leqslant V\big(t_0^+,x\big) = \sum_{i=1}^{n} c_i V_i\big(t_0^+,x_i\big) \leqslant \sum_{i=1}^{n} c_i a_i\big(\|x_i\|\big) \leqslant \sum_{i=1}^{n} a\big(\|x\|\big) \\ &= n \cdot a\big(\|x\|\big) < b(A). \end{aligned} \tag{9}$$

On the other hand, in view of condition (vi), one has

$$V(t,x)=\sum_{i=1}^{n}c_iV_i(t,x_i)\geqslant b\bigl(\|x\|\bigr). \tag{10}$$

Combining (9) and (10), we obtain for $t\geqslant t_0$,

$$b\bigl(\|x\|\bigr)<b(A)$$

This proves the uniformly practically stable of the trivial solution of system (5). □

Theorem 3. *Assume conditions* (i)–(ii), (iv)–(vi) *in Theorem* 1, *and let following condition hold*:

(iii′) *There exist functions $V_i(t,x_i)\in\varTheta_i(J,\varOmega_i),\mathbb{R}_+)$, $F_{ij}(t,x_i,x_j)$, the matrix $A=(a_{ij})_{n\times n}$ in which $a_{ij}\geqslant 0$ and $d_i>0$ such that for $i=1,2,\dots,n$,*

$$\begin{gathered}{}^{C}D_+^{\alpha}V_i(t,x_i)\leqslant -d_iV_i(t,x_i)+\sum_{j=1}^{n}a_{ij}F_{ij}(x_i,x_j),\\ t\in\bigcup_{k=0}^{\infty}[s_k,t_{k+1}],\ t\geqslant t_0.\end{gathered}$$

Then the trivial solution of system (5) *is practically asymptotically stable.*

Proof. Define $d=\min\{d_1,d_2,\dots,d_n\}$. When $t\in\bigcup_{k=0}^{\infty}[s_k,t_{k+1}]$, in view of condition (iii′), we get

$$\begin{aligned}{}^{C}D_+^{\alpha}V(t,x)&={}^{C}D_+^{\alpha}\sum_{i=1}^{n}c_iV_i(t,x_i)\leqslant\sum_{i=1}^{n}c_i^{C}D_+^{\alpha}V_i(t,x_i)\\ &\leqslant\sum_{i=1}^{n}c_i\Biggl[-d_iV_i(t,x_i)+\sum_{j=1}^{n}a_{ij}F_{ij}(x_i,x_j)\Biggr]\\ &\leqslant-\sum_{i=1}^{n}c_id_iV_i(t,x_i)\leqslant -dV(t,x).\end{aligned}$$

Then by Lemma 3 we can get

$$V(t,x)\leqslant V\bigl(t_0,x(0)\bigr)E_\alpha\bigl(-d(t-t_0)^\alpha\bigr).$$

We get the fact that the trivial solution of system (5) is practically asymptotically stable. The proof is complete. □

Remark 5. Theorems 1–3 provide a technique by graph theory to construct global Lyapunov functions using piecewise continuous Lyapunov functions V_i, $i=1,\dots,n$ in each vertex. This method overcomes the difficulty of directly finding appropriate Lyapunov functions. Furthermore, it is easier to obtain the practical stability of these types of fractional coupled systems with noninstantaneous on networks.

4 Examples

Example 1. We consider the following fractional-order impulsive control coupled system with noninstantaneous impulses on network:

$$
\begin{aligned}
{}^{C}D^{\alpha}x_i &= \mu_i y_i - \sum_{j=1}^{n} \beta_{ij} x_i y_j, \quad t \in (s_k, t_{k+1}],\ k = 0, 1, \dots, \\
{}^{C}D^{\alpha}y_i &= \sum_{j=1}^{n} \beta_{ij} x_i y_j - \mu_i y_i, \quad t \in (s_k, t_{k+1}],\ k = 0, 1, \dots, \\
x_i(t) &= C_k y_i(t_k), \quad t \in (t_k, s_k],\ k = 1, 2, \dots, \\
y_i(t) &= C_k x_i(t_k), \quad t \in (t_k, s_k],\ k = 1, 2, \dots, \\
x_i(t_0^+) &= x_{i0}, \quad y_i(t_0^+) = y_{i0}, \quad i = 1, 2, \dots, n.
\end{aligned}
\tag{11}
$$

Here $0 < \alpha < 1$, x_i, y_i are n-dimensional column vectors. The parameters μ_i are nonnegative constants, $\beta_{ii} \leqslant 0$, $\beta_{ij} = -\beta_{ji}$, and when $i \neq j$, $\beta_{ij} \neq 0$, $x_j y_i = x_i y_j$, C_k are $n \times n$ constant matrices. Let $s_0 = t_0 = 0$, $s_k = 2k$, $t_k = 2k - 1$ for $k = 1, 2, \dots$.

Let G be a graph with n vertices, $\alpha_{ij} = |\beta_{ij}|$, $i, j = 1, 2, \dots, n$, $A = (\alpha_{ij})_{n \times n}$. (G, A) is strongly connected, so $c_i > 0$, $\lambda_{\max} C_k \leqslant 2$, $k = 1, 2, \dots$.

Let $X_i = (x_i, y_i)$ for $i = 1, 2, \dots, n$. We now construct Lyapunov-like functions as $V_i(t, X_i) = (|x_i + y_i| + |x_i - y_i|)/2$. For $t \in \bigcup_{k=0}^{\infty}[s_k, t_{k+1}]$, through calculation, we have

$$
\begin{aligned}
{}^{C}D_{+}^{\alpha}V_i(t, X_i) &= \frac{1}{2}\bigl({}^{C}D_{+}^{\alpha}|x_i + y_i| + {}^{C}D_{+}^{\alpha}|x_i - y_i|\bigr) \\
&= \frac{1}{2}\bigl(\operatorname{sgn}(x_i + y_i) D_{+}^{\alpha}(x_i + y_i) + \operatorname{sgn}(x_i - y_i) D_{+}^{\alpha}(x_i - y_i)\bigr) \\
&= \sum_{j=1}^{n} |\beta_{ij} x_i y_j| = \sum_{j=1}^{n} \alpha_{ij} |x_i y_j| \\
&= \sum_{j=1}^{n} \alpha_{ij} F_{ij}(X_i, X_j),
\end{aligned}
$$

where $F_{ij}(X_i, X_j) = |x_i y_j|$. Therefore, conditions (i)–(iii) in Theorem 1 are satisfied.

Furthermore, for $i \neq j$,

$$
F_{ji}(X_j, X_i) = \operatorname{sgn}(\beta_{ji})|x_j y_i| = -\operatorname{sgn}(\beta_{ij})|x_i y_j| = -F_{ij}(X_i, X_j).
$$

So, along each cycle C of (G, A), we have

$$
\sum_{(i,j) \in E(C_Q)} \bigl[F_{ij}(X_i, X_j) + F_{ji}(X_j, X_i)\bigr] = 0.
$$

Thus, condition (iv′) is satisfied.

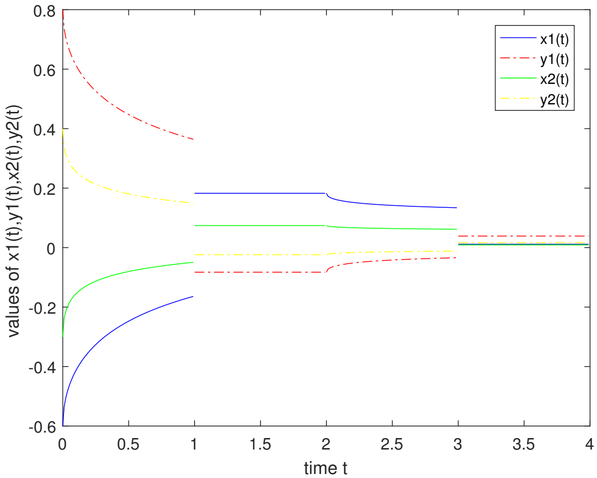

Figure 1. Time-series of two-dimension system with the initial values $(-0.6, 0.8, -0.3, 0.4)^{\mathrm{T}}$, α=0.5.

Also,

$$\begin{aligned} &V_i\bigl(t, I_k\bigl(t,\, x_i(t_k-0)\bigr)\bigr) \\ &\quad = \frac{1}{2}\bigl(\bigl|C_k y_i(t_k) + C_k x_i(t_k)\bigr| + \bigl|C_k y_i(t_k) - C_k x_i(t_k)\bigr|\bigr) \\ &\quad = \frac{1}{2}C_k\bigl(\bigl|x_i(t_k) + y_i(t_k)\bigr| + \bigl|x_i(t_k) - y_i(t_k)\bigr|\bigr) \\ &\quad = \frac{1}{2}C_k V_i(t_k-0,\, x_i) \leqslant \frac{1}{2}\lambda_{\max}(C_k)V_i(t_k-0, x_i) \\ &\quad \leqslant V_i(t_k-0, x_i), \end{aligned}$$

where $t \in (t_k, s_k]$, $t_k \geqslant t_0$, $x_i \in S_i(\rho)$, it follows that condition (v) in Theorem 1 is satisfied.

At last, let $b_i(x) = \|x\|$, $i = 1, 2, \dots, n$. It is easy to verify $b_i \in \mathcal{K}$. We can deduce that condition (vi) in Theorem 1 is satisfied.

According to Corollary 1, taking all the factors into consideration, we can conclude the trivial solution of system (11) is practically stable.

Now we give a numerical simulation to illustrate the effectiveness of our results. Let $\mu_i = 1, i = 1, 2, \dots, n$, $\beta_{ii} = 0$, $\beta_{ij} = 1/(n-1)$ if $i > j$, $\beta_{ij} = -1/(n-1)$ if $i < j$, $C_k = I_n/2$, I_n is $n \times n$ identity matrix. When $i \neq j$, $\beta_{ij} = -\beta_{ji}$, $\lambda_{\max} C_k = 1/2$. According to Example 1, the above system is practically stable. Numerical simulation can be seen in Fig. 1.

Corollary 3. *Let* $a_i(x) = 4\|x\|$, $i = 1, 2, \dots, n$, *in Example* 1. *According to Theorem* 2, *we can conclude that the trivial solution of system* (11) *is uniformly practically stable.*

Example 2. Consider the following fractional-order impulsive control coupled system with noninstantaneous impulses on network:

$$\begin{aligned}
{}^{C}D^{\alpha}x_i &= -a_i x_i - \frac{1}{2}\sum_{j=1}^{n}\beta_{ij}(x_i - y_j), \quad t \in (s_k, t_{k+1}],\ k = 0, 1, \dots,\\
{}^{C}D^{\alpha}y_i &= -a_i y_i - \frac{1}{2}\sum_{j=1}^{n}\beta_{ij}(x_i - y_j), \quad t \in (s_k, t_{k+1}],\ k = 0, 1, \dots,\\
x_i(t) &= C_k y_i(t_k), \quad t \in (t_k, s_k],\ k = 1, 2, \dots,\\
y_i(t) &= C_k x_i(t_k), \quad t \in (t_k, s_k],\ k = 1, 2, \dots,\\
x_i(t_0^+) &= x_{i0}, y_i(t_0^+) = y_{i0}, \quad i = 1, 2, \dots, n,
\end{aligned} \tag{12}$$

$0 < \alpha < 1$, x_i, y_i are n-dimensional column vectors. The parameters a_i are nonnegative constants, $\beta_{ii} \leqslant 0$, $\beta_{ij} = -\beta_{ji}$, and when $i \neq j$, $\beta_{ij} \neq 0$, $|x_i - y_j| = |x_j - y_i|$, C_k are $n \times n$ constant matrix. Let $s_0 = t_0 = 0$, $s_k = 2k$, $t_k = 2k - 1$ for $k = 1, 2, \dots$. Let G be a graph with n vertices, $\alpha_{ij} = |\beta_{ij}|$, $i, j = 1, 2, \dots, n$, $A = (\alpha_{ij})_{n\times n}$. (G, A) is strongly connected, so $c_i > 0$, $\lambda_{\max} C_k \leqslant 2$, $k = 1, 2, \dots$.

Let $X_i = (x_i, y_i)$ for $i = 1, 2, \dots, n$. We now construct Lyapunov-like functions as $V_i(t, X_i) = |x_i + y_i| + |x_i - y_i|$. For $t \in \bigcup_{k=0}^{\infty}[s_k, t_{k+1}]$, through calculation, we have

$$\begin{aligned}
{}^{C}D_{+}^{\alpha}V_i(t, X_i) &= {}^{C}D_{+}^{\alpha}|x_i + y_i| + {}^{C}D_{+}^{\alpha}|x_i - y_i|\\
&\leqslant -a_i\bigl(|x_i + y_i| + |x_i - y_i|\bigr) + \sum_{j=1}^{n}\alpha_{ij}|x_i - y_j|\\
&= -a_i V_i(t, X_i) + \sum_{j=1}^{n}\alpha_{ij}F_{ij}(X_i, X_j),
\end{aligned}$$

where $F_{ij}(X_i, X_j) = |x_i - y_j|$. Therefore, condition (iii$'$) in Theorem 3 is satisfied.

Furthermore, we can easily get that condition (iv$'$) is satisfied.

In the same way, we can get

$$V_i\bigl(t, I_k\bigl(t, x_i(t_k - 0)\bigr)\bigr) \leqslant V_i(t_k - 0, x_i),$$

where $t \in (t_k, s_k]$, $t_k \geqslant t_0$, $x_i \in S_i(\rho)$, it follows that condition (v) in Theorem 1 is satisfied.

Then let $b_i(x) = \|x\|$, $i = 1, 2, \dots, n$. We can deduce that condition (vi) is satisfied.

According to Theorem 3, we can conclude that the trivial solution of system (12) is practically asymptotically stable.

We give the numerical simulation of to verify the effectiveness of the obtained results. Let $\beta_{ii} = 0$, $\beta_{ij} = 1/(n-1)$ if $i > j$, $\beta_{ij} = -1/(n-1)$ if $i < j$, $C_k = I_n/2$, I_n is $n \times n$ identity matrix. When $i \neq j$, $\beta_{ij} = -\beta_{ji}$, $\lambda_{\max} C_k = 1/2$. According to Example 2, the above system is practically asymptotically stable, which can be seen in Fig. 2.

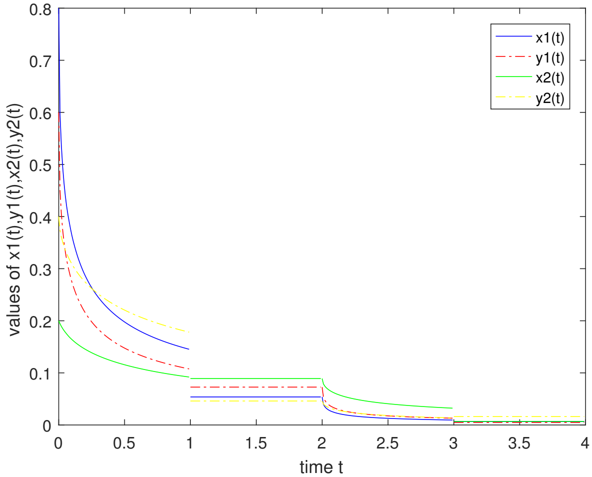

Figure 2. Time-series of two-dimension system with the initial values $(0.8, 0.6, 0.2, 0.4)^{\mathrm{T}}$, where $a_1 = 3, a_2 = 1$, α=0.5.

5 Conclusions

In this paper, we investigate a class of fractional impulsive control systems with noninstantaneous impulses on networks. We give sufficient conditions to obtain the practical stability, uniform practical stability and practical asymptotic stability of this coupled systems on networks for the first time. Meantime, we provide an appropriate way to construct global Lyapunov-like functions in view of noninstantaneous impulses. Then, using Lyapunov method and graph theory, the practical stability principles are obtained, which have a close relation to the topology property of the networks. Our results generalize relevant results in [2] to networks and provide an impulsive control law for impulsive control systems with noninstantaneous impulses.

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