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On fractional Langevin equation involving two fractional orders in different intervals

Hamid Baghani[a,1], **Juan J. Nieto**[b]

[a]Department of Mathematics,
Faculty of Mathematics, Statistics and Computer Science,
University of Sistan and Baluchestan,
Zahedan, Iran
h.baghani@gmail.com; h.baghani@math.usb.ac.ir

[b]Department of Mathematical Analysis, Institute of Mathematics,
University of Santiago de Compostela,
Santiago de Compostela, 15782 Spain
juanjose.nieto.roig@usc.es

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Abstract. In this paper, we study a nonlinear Langevin equation involving two fractional orders $\alpha \in (0,1]$ and $\beta \in (1,2]$ with initial conditions. By means of an interesting fixed point theorem, we establish sufficient conditions for the existence and uniqueness of solutions for the fractional equations. Some illustrative numerical examples are also discussed.

Keywords: Caputo fractional derivative, initial boundary conditions, existence and uniqueness, fractional Langevin equation.

1 Introduction

Fractional differential equations have proven strong connection to many physical and engineering applications such as viscoelasticity, ground water flows, boundary layer theory, granular flows, dynamics of cold atoms in optical lattices, plasma turbulence, dynamics of polymeric materials. On the other hand, the fractional differential equations involving Riemann–Liouville and Caputo operators of fractional orders have gained intensive interest in the last years. It has been shown that these types of equations have also numerous applications in diverse fields and thus have evolved into multidisciplinary subjects. An excellent account in the study of fractional differential equations can be found in [10, 11, 14, 16, 21, 26, 30, 33]. For more details and recent examples, see [3, 5–7, 9, 20] and the references therein. Some new and recent aspects on fractional calculus can be seen in [1, 2, 15, 22, 25, 28, 29, 31].

[1]Corresponding author.

The Langevin differential equation was first proposed by Paul Langevin in 1908. He gave a mathematical description of Brownian motion

$$x'' + \lambda x' = f,$$

where x represents the position of a unit mass. The force acting on that particle is written as a sum

$$-\lambda x' + f$$

of a viscous force proportional to the particle's velocity $-\lambda D^{\beta} x$, $\beta = 1$ and a nonlinear noisy term f. The parameter λ represents a viscosity related to the velocity or the fractional derivative. Indeed, the equation has been widely used in different disciplines to describe the dynamical processes of various fluctuating environments [8]. However, it has been realized that the ordinary Langevin differential equation comes up with insufficient illustrations for some systems existing in disordered or fractal mediums. Such occurrence was observed in anomalous transport phenomena [17]. To overcome this deficiency, Kubo introduced the generalized Langevin differential equation in which a fractional memory kernel was incorporated to interpret the fractal and memory properties [19].

The nonlinear Langevin equation with two fractional orders was introduced and studied in [32]. Most of the nonlinear fractional equations cannot be analytically solved, so, firstly, the existence and uniqueness of solutions should be verified. Recently, the existence of solutions of the initial and boundary value problems for nonlinear fractional equations are extensively studied. For example, see Kosmatov [18], Li et al. [24], Deng et al. [9], Gorka et al. [12], Lakshmikantham et al. [23], Zhou et al. [34] and Guo et al. [13].

The authors in [11] have investigated the existence and uniqueness of solutions for nonlinear Langevin equation involving two fractional orders with anti-periodic boundary conditions based on coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces.

Motivated by the work, we study, with a new and different method, the existence and uniqueness of solutions for an initial value problem of Langevin equation involving two fractional orders in different intervals as follows:

$$\begin{aligned} &D^{\beta}\big(D^{\alpha} + \lambda\big)x(t) = f\big(t, x(t)\big), \quad 0 \leqslant t \leqslant 1,\ 0 < \alpha \leqslant 1,\ 1 < \beta \leqslant 2, \\ &x(0) = x(1) = 0, \qquad \mathcal{D}^{2\alpha}x(1) + \lambda \mathcal{D}^{\alpha}x(1) = 0, \end{aligned} \tag{1}$$

where D^{α} is the Caputo fractional derivative of order α, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and λ is a real number. Furthermore, $\mathcal{D}^{2\alpha}$ is the sequential fractional derivative presented by Miller and Ross [27] as follows:

$$\begin{aligned} &\mathcal{D}^{\alpha}u = D^{\alpha}u, \\ &\mathcal{D}^{k\alpha}u = \mathcal{D}^{\alpha}\mathcal{D}^{(k-1)\alpha}u, \quad k = 2, 3, \dots . \end{aligned}$$

2 Preliminaries

In this section, we recall some notations and preliminaries for fractional calculus and a new fixed point theorem which are needed prior to addressing our results.

Definition 1. (See [9,18].) The Riemann–Liouville integral of order $\alpha > 0$ for a function $x : (0, \infty) \to \mathbb{R}$ is defined as

$$I^\alpha x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant 1, \tag{2}$$

where Γ is the Gamma function, provided the right-hand side integral exists and is finite.

Definition 2. (See [9,18].) The Caputo fractional derivative of order $\alpha > 0$ for a continuous function $x : (0, \infty) \to \mathbb{R}$ is given by

$$D^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant 1, \tag{3}$$

where n is the smallest integer greater than or equal to α, provided the right-hand side integral exists and is finite.

We note that the Caputo derivative reduces to the conventional nth derivative of a given function as $\alpha \to n$, and the initial conditions keep the same form as that of the classical Langevin equation.

The relationship between (2) and (3) and other properties of these notions are stated in the next theorems. For their proofs, we refer to [18,30].

Theorem 1. *Let $\alpha, \beta \geqslant 0$. If x is continuous, then $I^\alpha I^\beta x = I^\beta I^\alpha x = I^{\alpha+\beta} x$.*

Theorem 2. *Let $\alpha \geqslant 0$. If x is continuous, then $D^\alpha I^\alpha x = x$.*

Theorem 3. (*See* [16,30].) *Let n be a positive integer and $\alpha \in (n-1, n)$. Then the general solution of the fractional differential equation $D^\alpha x = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$.

Now, we present an interesting fixed point theorem (Amini–Emami's fixed point theorem) which plays main role in our discussion.

Definition 3. (See [4].) Let $(X, \preccurlyeq)$ be a partially ordered set and $f : X \to X$ be a self mapping. Then f is called increasing (decreasing) if $fx \preccurlyeq fy$ ($fy \preccurlyeq fx$) whenever $x \preccurlyeq y$. Also, we say that elements $x, y \in X$ are comparable either $x \preccurlyeq y$ or $y \preccurlyeq x$. Moreover, $\{x_n\}$ is called an increasing sequence (a decreasing sequence) if $x_n \preccurlyeq x_{n+1}$ for all $n \in \mathbb{N}$ ($x_{n+1} \preccurlyeq x_n$ for all $n \in \mathbb{N}$).

Theorem 4. (*See* [4].) *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be an* *increasing mapping such that there exists an element $x_0 \in X$ with $x_0 \preccurlyeq f(x_0)$. Suppose that there exists $0 \leqslant \alpha < 1$ such that*

$$d\big(f(x), f(y)\big) \leqslant \alpha\, d(x, y)$$

for all comparable $x, y \in X$. Assume that either f is continuous or X is such that

if an incrasing sequence $\{x_n\} \to x$ in X, then $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$.

Beside, if

for each $x, y \in X$, there exists $z \in X$ which is comparable to x and y.

Then f has a unique fixed point x^*.

3 Existence and uniqueness

In this section, we intend to give an existence and uniqueness result for problem (1). First, we establish the equivalence of this problem and a mixed Fredholm–Volterra integral equation. Then we break our discussion up into two subsections. Two cases of interest are discussed separately in Section 3.1 for the case $\lambda \geqslant 0$ and in Section 3.2 for the case $\lambda < 0$. The reason for discussing them separately is that the definition of coupled lower and upper solution is different in these two cases. Also, in the end of this section, we conduct a comparison between our results and the main results of [11].

Lemma 1. *x is a solution of problem* (1) *if and only if it is a solution of the nonlinear integral equation*

$$\begin{aligned}
x(t) = {} & \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha+\beta-1} f\big(\tau, x(\tau)\big)\,\mathrm{d}\tau - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau)\,\mathrm{d}\tau \\
& + t^\alpha \Bigg[-\int_0^1 \frac{(1-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\big(\tau, x(\tau)\big)\,\mathrm{d}\tau + \lambda \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} x(\tau)\,\mathrm{d}\tau \\
& + \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+2)} \int_0^1 \frac{(1-\tau)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} f\big(\tau, x(\tau)\big)\,\mathrm{d}\tau \Bigg] \\
& - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \Bigg[\Gamma(2-\alpha) \int_0^1 \frac{(1-\tau)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} f\big(\tau, x(\tau)\big)\,\mathrm{d}\tau \Bigg]
\end{aligned} \tag{4}$$

for all $t \in [0, 1]$.

Proof. Let x be a solution of the problem (1). Then from Theorem 2 we obtain

$$D^\beta \big[\big(D^\alpha + \lambda\big) x(t) - I^\beta f\big(\cdot, x(\cdot)\big)(t) \big] = 0.$$

Now by applying Theorem 3 we deduce

$$\big(D^\alpha + \lambda\big)x(t) - I^\beta f\big(\cdot, x(\cdot)\big)(t) = c_1 + c_2 t$$

or, equivalently,

$$D^\alpha\left(x(t) + \lambda I^\alpha x(\cdot)(t) - I^{\alpha+\beta} f\big(\cdot, x(\cdot)\big)(t) - c_1\frac{t^\alpha}{\Gamma(\alpha+1)} - c_2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right) = 0.$$

Applying Theorem 3 again, the general form of problem (1) can be written as

$$x(t) = I^{\alpha+\beta} f\big(\cdot, x(\cdot)\big)(t) - \lambda I^\alpha x(\cdot)(t) + c_0 + c_1\frac{t^\alpha}{\Gamma(\alpha+1)} + c_2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}. \tag{5}$$

Now, by (5) we obtain

$$D^\alpha x(t) = I^\beta f\big(\cdot, x(\cdot)\big)(t) - \lambda x(t) + c_1 + c_2 t,$$

and

$$\mathcal{D}^{2\alpha} x(t) = I^{\beta-\alpha} f\big(\cdot, x(\cdot)\big)(t) - \lambda D^\alpha x(t) + c_2\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}.$$

Hence, by using the boundary conditions for problem (1) we have

$$\begin{aligned}
c_2 &= -\Gamma(2-\alpha)\int_0^1 \frac{(1-\tau)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} f\big(\tau, x(\tau)\big)\,\mathrm{d}\tau,\\
c_1 &= -\Gamma(\alpha+1)\int_0^1 \frac{(1-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\big(\tau, x(\tau)\big)\,\mathrm{d}\tau\\
&\quad + \lambda\Gamma(\alpha+1)\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} x(\tau)\,\mathrm{d}\tau\\
&\quad + \frac{\Gamma(\alpha+1)\Gamma(2-\alpha)}{\Gamma(\alpha+2)}\int_0^1 \frac{(1-\tau)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} f\big(\tau, x(\tau)\big)\,\mathrm{d}\tau,\\
c_0 &= 0.
\end{aligned}$$

Substituting the values of c_0, c_1 and c_2 in (5), we obtain solution (4). On the other hand, it is easy to prove that, if x is a solution of the integral equation (4), then x is also a solution of problem (1). $\square$

3.1 Consideration in the case $\lambda \geqslant 0$

Definition 4. An element $(x_0, y_0) \in C[0,1] \times C[0,1]$ is called a coupled lower and upper solution of (1) if, for all $t \in [0,1]$,

$$\begin{aligned}
x_0(t) \leqslant{}& \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha+\beta-1} f\bigl(\tau, x_0(\tau)\bigr)\,\mathrm{d}\tau - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y_0(\tau)\,\mathrm{d}\tau \\
&+ t^\alpha \Biggl[-\int_0^1 \frac{(1-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\bigl(\tau, y_0(\tau)\bigr)\,\mathrm{d}\tau + \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} x_0(\tau)\,\mathrm{d}\tau \\
&+ \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+2)} \int_0^1 \frac{(1-\tau)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} f\bigl(\tau, x_0(\tau)\bigr)\,\mathrm{d}\tau \Biggr] \\
&- \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \Biggl[\Gamma(2-\alpha) \int_0^1 \frac{(1-\tau)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} f\bigl(\tau, y_0(\tau)\bigr)\,\mathrm{d}\tau \Biggr]
\end{aligned}$$

and

$$\begin{aligned}
y_0(t) \geqslant{}& \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha+\beta-1} f\bigl(\tau, y_0(\tau)\bigr)\,\mathrm{d}\tau - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x_0(\tau)\,\mathrm{d}\tau \\
&+ t^\alpha \Biggl[-\int_0^1 \frac{(1-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\bigl(\tau, x_0(\tau)\bigr)\,\mathrm{d}\tau + \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} y_0(\tau)\,\mathrm{d}\tau \\
&+ \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+2)} \int_0^1 \frac{(1-\tau)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} f\bigl(\tau, y_0(\tau)\bigr)\,\mathrm{d}\tau \Biggr] \\
&- \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \Biggl[\Gamma(2-\alpha) \int_0^1 \frac{(1-\tau)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} f\bigl(\tau, x_0(\tau)\bigr)\,\mathrm{d}\tau \Biggr].
\end{aligned}$$

To prove the main results, we need the following assumptions:

(H1) $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot, x(\cdot)) \in C[0,1]$ for each $x \in C[0,1]$.

(H2) There exists $L > 0$ such that $0 \leqslant f(t,x) - f(t,y) \leqslant L(x-y)$ for all $x, y \in \mathbb{R}$ with $x \geqslant y$.

Theorem 5. *With assumptions* (H1)–(H2)*, if problem* (1) *has a coupled lower and upper solution and* $\Lambda < 1$*, where*

$$\Lambda = \frac{2L}{\Gamma(\alpha+\beta+1)} + \frac{2\lambda}{\Gamma(\alpha+1)} + \frac{2L\Gamma(2-\alpha)}{\Gamma(\alpha+2)\Gamma(\beta-\alpha+1)},$$

then it has a unique solution in $C[0,1]$.

Proof. It is easy to see that $X := C[0,1]$ is a partially ordered set with the following order relation in X:

$$x \preccurlyeq y \quad x, y \in X \quad \Longleftrightarrow \quad x(t) \leqslant y(t) \quad \forall t \in [0,1].$$

Also, (X, d) is a complete metric space with metric $d(x,y) = \max_{t\in[0,1]} |x(t) - y(t)|$. Obviously, if $\{x_n\}$ is an increasing sequence in X that converges to $x \in X$ and $\{y_n\}$ is a decreasing sequence in X that converges to $y \in X$, then $x_n \preccurlyeq x$ and $y \preccurlyeq y_n$ for all n. Also, for any $x, y \in X$, the functions $\max\{x,y\}$ and $\min\{x,y\}$ are the upper and lower bounds of x, y, respectively.

Also, $X \times X$ is a partially ordered set if we define the following order relation in $X \times X$:

$$(x,y) \hat{\preccurlyeq} (u,v) \quad \Longleftrightarrow \quad x \preccurlyeq u, v \preccurlyeq y.$$

Furthermore, for every $(x,y), (u,v) \in X \times X$, there exists a $(\max\{x,u\}, \min\{y,v\}) \in X \times X$ that is comparable to (x,y) and (u,v). Moreover, $(X \times X, \hat{d})$ is a complete metric space, where $\hat{d}((x,y),(u,v)) = d(x,u) + d(y,v)$. Also, if $\{(x_n, y_n)\}$ is an increasing sequence in $X \times X$ that converges to (x,y) then $(x_n, y_n) \hat{\preccurlyeq} (x,y)$ for each n.

Now we define $g_1, g_2 : X \to X$ and $\mathcal{S} : X \times X \to X \times X$ as follows:

$$\begin{aligned}
g_1(x)(t) &= \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha+\beta-1} f\big(\tau, x(\tau)\big)\,\mathrm{d}\tau + t^\alpha \Bigg[\frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} x(\tau)\,\mathrm{d}\tau \\
&\quad + \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+2)\Gamma(\beta-\alpha)} \int_0^1 (1-\tau)^{\beta-\alpha-1} f\big(\tau, x(\tau)\big)\,\mathrm{d}\tau\Bigg], \quad t \in [0,1], \\
g_2(y)(t) &= \frac{-\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau)\,\mathrm{d}\tau - \frac{t^\alpha}{\Gamma(\alpha+\beta)} \int_0^1 (1-\tau)^{\alpha+\beta-1} f\big(\tau, y(\tau)\big)\,\mathrm{d}\tau \\
&\quad - t^{\alpha+1} \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+2)\Gamma(\beta-\alpha)} \int_0^1 (1-\tau)^{\beta-\alpha-1} f\big(\tau, y(\tau)\big)\,\mathrm{d}\tau, \quad t \in [0,1],
\end{aligned}$$

and

$$\mathcal{S}(x,y) = \big(g_1(x) + g_2(y),\ g_1(y) + g_2(x)\big).$$

It is easy to see that $(x_0, y_0) \hat{\preccurlyeq} \mathcal{S}(x_0, y_0)$, g_1 is an increasing mapping and g_2 is a decreasing mapping. Hence, $\mathcal{S}$ is an increasing mapping in $X \times X$.

Now, for $(x,y), (u,v) \in X \times X$ with $(x,y) \hat{\preccurlyeq} (u,v)$ and $t \in [0,1]$, we have

$$\begin{aligned}
&\big|\mathcal{S}(x,y)(t) - \mathcal{S}(u,v)(t)\big| \\
&\quad = \big|\big(g_1x(t) + g_2y(t),\ g_1y(t) + g_2x(t)\big) - \big(g_1u(t) + g_2v(t),\ g_1v(t) + g_2u(t)\big)\big| \\
&\quad \leqslant \big|g_1x(t) - g_1u(t)\big| + \big|g_2y(t) - g_2v(t)\big| + \big|g_1y(t) - g_1v(t)\big| + \big|g_2x(t) - g_2u(t)\big|.
\end{aligned}$$

On the other hand

$$
\begin{aligned}
&\bigl|g_1x(t) - g_1u(t)\bigr| \\
&\quad\leqslant \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha+\beta-1}\bigl|f\bigl(\tau, x(\tau)\bigr) - f\bigl(\tau, u(\tau)\bigr)\bigr|\,\mathrm{d}\tau \\
&\qquad + t^\alpha\Biggl[\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\bigl|x(\tau) - u(\tau)\bigr|\,\mathrm{d}\tau \\
&\qquad + \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+2)\Gamma(\beta-\alpha)}\int_0^1 (1-\tau)^{\beta-\alpha-1}\bigl|f\bigl(\tau, x(\tau)\bigr) - f\bigl(\tau, u(\tau)\bigr)\bigr|\,\mathrm{d}\tau\Biggr] \\
&\quad\leqslant \biggl(\frac{L}{\Gamma(\alpha+\beta+1)} + \frac{\lambda}{\Gamma(\alpha+1)} + \frac{L\Gamma(2-\alpha)}{\Gamma(\alpha+2)\Gamma(\beta-\alpha+1)}\biggr)d(x,u),
\end{aligned}
$$

$$
\begin{aligned}
&\bigl|g_1y(t) - g_1v(t)\bigr| \\
&\quad\leqslant \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha+\beta-1}\bigl|f\bigl(\tau, y(\tau)\bigr) - f\bigl(\tau, v(\tau)\bigr)\bigr|\,\mathrm{d}\tau \\
&\qquad + t^\alpha\Biggl[\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\bigl|y(\tau) - v(\tau)\bigr|\,\mathrm{d}\tau \\
&\qquad + \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+2)\Gamma(\beta-\alpha)}\int_0^1 (1-\tau)^{\beta-\alpha-1}\bigl|f\bigl(\tau, y(\tau)\bigr) - f\bigl(\tau, v(\tau)\bigr)\bigr|\,\mathrm{d}\tau\Biggr] \\
&\quad\leqslant \biggl(\frac{L}{\Gamma(\alpha+\beta+1)} + \frac{\lambda}{\Gamma(\alpha+1)} + \frac{L\Gamma(2-\alpha)}{\Gamma(\alpha+2)\Gamma(\beta-\alpha+1)}\biggr)d(y,v),
\end{aligned}
$$

$$
\begin{aligned}
&\bigl|g_2x(t) - g_2u(t)\bigr| \\
&\quad\leqslant \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\bigl|x(\tau) - u(\tau)\bigr|\,\mathrm{d}\tau \\
&\qquad + \frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-\tau)^{\alpha+\beta-1}\bigl|f\bigl(\tau, x(\tau)\bigr) - f\bigl(\tau, u(\tau)\bigr)\bigr|\,\mathrm{d}\tau \\
&\qquad + \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+2)\Gamma(\beta-\alpha)}\int_0^1 (1-\tau)^{\beta-\alpha-1}\bigl|f\bigl(\tau, x(\tau)\bigr) - f\bigl(\tau, u(\tau)\bigr)\bigr|\,\mathrm{d}\tau \\
&\quad\leqslant \biggl(\frac{L}{\Gamma(\alpha+\beta+1)} + \frac{\lambda}{\Gamma(\alpha+1)} + \frac{L\Gamma(2-\alpha)}{\Gamma(\alpha+2)\Gamma(\beta-\alpha+1)}\biggr)d(x,u)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|g_2 y(t) - g_2 v(t)\right| \\
&\quad\leqslant \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|y(\tau) - v(\tau)\right| \mathrm{d}\tau \\
&\qquad + \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-\tau)^{\alpha+\beta-1} \left|f\bigl(\tau, y(\tau)\bigr) - f\bigl(\tau, v(\tau)\bigr)\right| \mathrm{d}\tau \\
&\qquad + \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+2)\Gamma(\beta-\alpha)} \int_0^1 (1-\tau)^{\beta-\alpha-1} \left|f\bigl(\tau, y(\tau)\bigr) - f\bigl(\tau, v(\tau)\bigr)\right| \mathrm{d}\tau \\
&\quad\leqslant \left(\frac{L}{\Gamma(\alpha+\beta+1)} + \frac{\lambda}{\Gamma(\alpha+1)} + \frac{L\Gamma(2-\alpha)}{\Gamma(\alpha+2)\Gamma(\beta-\alpha+1)}\right) d(y, v)
\end{aligned}
$$

for all $(x, y), (u, v) \in X \times X$ with $(x, y) \hat{\preccurlyeq} (u, v)$ and $t \in [0, 1]$. Then we conclude that

$$
\begin{aligned}
&\left|\mathcal{S}(x, y)(t) - \mathcal{S}(u, v)(t)\right| \\
&\quad\leqslant \left(\frac{2L}{\Gamma(\alpha+\beta+1)} + \frac{2\lambda}{\Gamma(\alpha+1)} + \frac{2L\Gamma(2-\alpha)}{\Gamma(\alpha+2)\Gamma(\beta-\alpha+1)}\right) d(x, u) \\
&\qquad + \left(\frac{2L}{\Gamma(\alpha+\beta+1)} + \frac{2\lambda}{\Gamma(\alpha+1)} + \frac{2L\Gamma(2-\alpha)}{\Gamma(\alpha+2)\Gamma(\beta-\alpha+1)}\right) d(y, v) \\
&\quad= \Lambda \hat{d}\bigl((x, y), (u, v)\bigr)
\end{aligned}
$$

for all $(x, y), (u, v) \in X \times X$ with $(x, y) \hat{\preccurlyeq} (u, v)$ and $t \in [0, 1]$. Hence, for each (x, y), $(u, v) \in X \times X$ with $(x, y) \hat{\preccurlyeq} (u, v)$, we obtain

$$
\hat{d}\bigl(\mathcal{S}(x, y), \mathcal{S}(u, v)\bigr) \leqslant \Lambda \hat{d}\bigl((x, y), (u, v)\bigr)
$$

Thus, according to Theorem 4, there is a unique element $(x^*, y^*) \in X \times X$ such that $(x^*, y^*) = \mathcal{S}(x^*, y^*)$. On the other hand, since (y^*, x^*) is another fixed point of $\mathcal{S}$, then $y^* = x^*$. Hence, $x^* = g_1(x^*) + g_2(x^*)$, i.e., x^* is a unique solution of (1). □

3.2 Consideration in the case $\lambda < 0$

For consideration the existence and uniqueness result in this case, we can use the same technique as in previous subsection. The main difference is the structure of the function $\mathcal{S}$ and consequently the definition of coupled lower and upper solution of problem (1). We could carry out a similar argument to prove the existence and uniqueness result. For this reason, we omit the full details of the processes.

Definition 5. An element $(x_0, y_0) \in C[0,1] \times C[0,1]$ is called a coupled lower and upper solution of (1) if

$$\begin{aligned} x_0(t) \leqslant{}& \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha+\beta-1} f\bigl(\tau, x_0(\tau)\bigr)\,\mathrm{d}\tau - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x_0(\tau)\,\mathrm{d}\tau \\ &+ t^\alpha \Biggl[-\int_0^1 \frac{(1-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\bigl(\tau, y_0(\tau)\bigr)\,\mathrm{d}\tau + \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} y_0(\tau)\,\mathrm{d}\tau \\ &+ \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+2)} \int_0^1 \frac{(1-\tau)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} f\bigl(\tau, x_0(\tau)\bigr)\,\mathrm{d}\tau \Biggr] \\ &- \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \Biggl[\Gamma(2-\alpha) \int_0^1 \frac{(1-\tau)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} f\bigl(\tau, y_0(\tau)\bigr)\,\mathrm{d}\tau \Biggr] \end{aligned}$$

and

$$\begin{aligned} y_0(t) \geqslant{}& \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha+\beta-1} f\bigl(\tau, y_0(\tau)\bigr)\,\mathrm{d}\tau - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y_0(\tau)\,\mathrm{d}\tau \\ &+ t^\alpha \Biggl[-\int_0^1 \frac{(1-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\bigl(\tau, x_0(\tau)\bigr)\,\mathrm{d}\tau + \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} x_0(\tau)\,\mathrm{d}\tau \\ &+ \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+2)} \int_0^1 \frac{(1-\tau)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} f\bigl(\tau, y_0(\tau)\bigr)\,\mathrm{d}\tau \Biggr] \\ &- \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \Biggl[\Gamma(2-\alpha) \int_0^1 \frac{(1-\tau)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} f\bigl(\tau, x_0(\tau)\bigr)\,\mathrm{d}\tau \Biggr] \end{aligned}$$

for all $t \in [0,1]$.

To prove the main results, we need the following assumptions:

(H3) $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot, x(\cdot)) \in C[0,1]$ for each $x \in C[0,1]$.

(H4) There exists $L > 0$ such that $0 \leqslant f(t,x) - f(t,y) \leqslant L(x-y)$ for all $x, y \in \mathbb{R}$ with $x \geqslant y$.

Theorem 6. *With assumptions* (H3)–(H4)*, if problem* (1) *has a coupled lower and upper solution and* $\Lambda^* < 1$*, where*

$$\Lambda^* = \frac{2L}{\Gamma(\alpha+\beta+1)} + \frac{2|\lambda|}{\Gamma(\alpha+1)} + \frac{2L\Gamma(2-\alpha)}{\Gamma(\alpha+2)\Gamma(\beta-\alpha+1)},$$

then it has a unique solution in $C[0,1]$.

Proof. It is enough to define $g_3, g_4 : X \to X$ and $\mathcal{H} : X \times X \to X \times X$ as follows:

$$\begin{aligned}
&g_3(x)(t) \\
&\quad= \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha+\beta-1} f\big(\tau, x(\tau)\big)\,\mathrm{d}\tau - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau)\,\mathrm{d}\tau \\
&\qquad+ \frac{\Gamma(2-\alpha)t^\alpha}{\Gamma(\alpha+2)} \int_0^1 \frac{(1-\tau)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} f\big(\tau, x(\tau)\big)\,\mathrm{d}\tau, \\
&g_4(y)(t) \\
&\quad= t^\alpha \left[\frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} y(\tau)\,\mathrm{d}\tau - \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-\tau)^{\alpha+\beta-1} f\big(\tau, y(\tau)\big)\,\mathrm{d}\tau \right] \\
&\qquad- t^{\alpha+1} \frac{\Gamma(2-\alpha)}{\Gamma(\beta-\alpha)\Gamma(\alpha+2)} \int_0^1 (1-\tau)^{\beta-\alpha-1} f\big(\tau, y(\tau)\big)\,\mathrm{d}\tau
\end{aligned}$$

for all $t \in [0,1]$, and

$$\mathcal{H}(x,y) = \big(g_3(x) + g_4(y),\ g_3(y) + g_4(x)\big).$$

The rest of proof is the same argument as in the proof of Theorem 5. □

Remark 1. Now, we conduct a comparison between our results and the main results of [11]. The authors in [11] have investigated the existence and uniqueness of solutions for nonlinear Langevin equation involving two fractional orders with anti-periodic boundary conditions. Their results are based on coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces. In the present paper and following the arguments of our main results (Theorems 5 and 6), we improve the results of [11] under the weaker condition $(\Lambda_1 + \Lambda_2)/2 < 1$ instead $\max\{\Lambda_1, \Lambda_2\} < 1$, where

$$\begin{aligned}
\Lambda_1 &= \frac{2L}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda|}{\Gamma(\alpha+1)} + \frac{L}{2\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&\quad+ \frac{3+\alpha}{4} \frac{\Gamma(2-\alpha)}{\Gamma(2+\alpha)} \frac{2L}{\Gamma(\beta-\alpha+1)}, \\
\Lambda_2 &= \frac{2|\lambda|}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha+\beta+1)} + \frac{L}{\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&\quad+ \frac{\Gamma(2-\alpha)}{\Gamma(2+\alpha)} \frac{2L}{\Gamma(\beta-\alpha+1)}.
\end{aligned}$$

It indicates that our methods are more general and applicable to different situations and cases.

4 Examples

To illustrate our results, let us consider the following simple examples.

Example 1. Consider the boundary value problem

$$\begin{aligned} &D^{7/4}\left(D^{3/4}+\frac{1}{6}\right)x(t)=\frac{1}{12}\bigl(\cos(t)+\tan^{-1}x(t)\bigr), \quad 0\leqslant t\leqslant 1,\\ &x(0)=x(1)=0, \qquad \mathcal{D}^{2\cdot(3/4)}x(1)+\frac{1}{6}\mathcal{D}^{3/4}x(1)=0. \end{aligned} \tag{6}$$

Here $f(t,x)=(1/12)(\cos(t)+\tan^{-1}x)$, $\alpha=3/4$, $\beta=7/4$ and $\lambda=1/6$.

Clearly, $0\leqslant f(t,x)-f(t,y)\leqslant L(x-y)$, $x\geqslant y$ with $L=1/12$. Moreover,

$$\Lambda=0.5067647970<1.$$

Also, a relatively simple calculus, with the help of Maple, shows that $(x_0(t),y_0(t))=(-t,t)$ is a coupled lower and upper solution of problem (6). Thus, by Theorem 5 the boundary value problem (6) has a unique solution on $[0,1]$.

Example 2. Consider the boundary value problem

$$\begin{aligned} &D^{8/7}\left(D^{3/4}-\frac{1}{4}\right)x(t)=\frac{1+x(t)}{6(1+t)^4}, \quad 0\leqslant t\leqslant 1,\\ &x(0)=x(1)=0, \qquad \mathcal{D}^{2\cdot(3/4)}x(1)-\frac{1}{4}\mathcal{D}^{3/4}x(1)=0. \end{aligned} \tag{7}$$

Here $f(t,x)=(1+x)/(6(1+t)^4)$, $\alpha=3/4$, $\beta=8/7$ and $\lambda=-1/4$.

Clearly, $0\leqslant f(t,x)-f(t,y)\leqslant L(x-y)$, $x\geqslant y$ with $L=1/6$. Moreover,

$$\Lambda^*=0.9392198383<1.$$

Also, a relatively simple calculus, with the help of Maple, shows that $(x_0(t),y_0(t))=(-1,1)$ is a coupled lower and upper solution of problem (7). Thus, by Theorem 6 the boundary value problem (7) has a unique solution on $[0,1]$.

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