A Joint Limit Theorem for Laplace Transforms of the Riemann Zeta-Function

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Abstract. In the paper, a joint limit theorem in the sense of weak convergence of probability measures on the complex plane for Laplace transforms of the Riemann zeta-function is obtained.

Keywords: Laplace transform, limit theorem, probability measure, Riemann zeta-function.

1 Introduction

Let $\mathbb{Z}, \mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ stand for the sets of all integers, positive integers, real and complex numbers, respectively, and let, as usual, $\zeta(s), s = \sigma + it$, denote the Riemann zeta-function. For $k \in \mathbb{N}$, define

$$L_k(s) = \int_0^\infty |\zeta(1/2 + ix)|^{2k} e^{-sx} dx.$$

As [1]

$$\zeta(1/2 + it) \ll_\varepsilon t^{\frac{32}{205}+\varepsilon}, \quad t \geq t_0 > 0,$$

with every $\varepsilon > 0$, the integral for $L_k(s)$ converges absolutely and uniformly on compact subsets of the half – plane $D = \{s \in \mathbb{C}\colon \sigma > 0\}$, and defines there an analytic function.

The function $L_k(s)$ is applied, see, for example, [2], [4–6], for investigations of the mean value

$$\int_0^T |\zeta(1/2 + it)|^{2k} dt, T \to \infty.$$

In [7] we obtained the first probabilistic results for the function $L_1(s)$. Let meas$\{A\}$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and let, for $T > 0$,

$$\nu_T(\ldots) = \frac{1}{T}\text{meas}\{t \in [0,T]\colon \ldots\},$$

where in place of dots a condition satisfied by t is to be written. Denote $\mathcal{B}(S)$ the class of Borel sets of the space S, and define the probability measure

$$P_{T,\sigma} = \nu_T\big(L_1(\sigma+it) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}).$$

Then in [7] the following characterization of the asymptotic behavior of the function $L_1(s)$ has been obtained.

Theorem A. *Let $\sigma > 0$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure P_σ such that the measure $P_{T,\sigma}$ converges weakly to P_σ as $T \to \infty$.*

Let $\mathbb{C}^r = \underbrace{\mathbb{C} \times \ldots \times \mathbb{C}}_{r}$, $\underline{\sigma} = (\sigma_1, \ldots, \sigma_r)$, and

$$\underline{L}(\underline{\sigma}+it) = \big(L_1(\sigma_1+it), \ldots, L_r(\sigma_r+it)\big).$$

The aim of this paper is to obtain a limit theorem for the probability measure

$$P_{T,\underline{\sigma}}(A) = \nu_T\big(\underline{L}(\underline{\sigma}+it) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}^r).$$

Theorem 1. *Suppose that $\min_{1\le j\le r}\sigma_j > 0$. Then on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ there exists a probability measure $P_{\underline{\sigma}}$ such that the measure $P_{T,\underline{\sigma}}$ converges weakly to $P_{\underline{\sigma}}$ as $T \to \infty$.*

Obviously, Theorem A is a corollary of Theorem 1 with $r = 1$.

It is well known that almost periodic functions have limit distributions in the sense of Theorem 1. For example, the majority of functions defined by Dirichlet series have the above property. Almost periodic functions are approximated in some metric by trigonometric polynomials, however, for Laplace transforms this is not known. So, we can not apply the almost periodicity property for Laplace transforms.

2 Case of a finite interval

Let $a > 0$ be a fixed finite number, and

$$L_{k,a}(s) = \int_0^a |\zeta(1/2+ix)|^{2k} e^{-sx}\,dx.$$

In this section, we will consider the limit distribution of the vector

$$\underline{L}_a(\underline{\sigma}+it) = \big(L_{1,a}(\sigma_1+it), \ldots, L_{r,a}(\sigma_r+it)\big).$$

On $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$, define the probability measure $P_{T,a,\underline{\sigma}}$ by

$$P_{T,a,\underline{\sigma}}(A) = \nu_T\big(\underline{L}_a(\underline{\sigma}+it) \in A\big).$$

Theorem 2. *On $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$, there exists a probability measure $P_{a,\underline{\sigma}}$ such that the measure $P_{T,a,\underline{\sigma}}$ converges weakly to $P_{a,\underline{\sigma}}$ as $T \to \infty$.*

Proof. We begin with a limit theorem on one topological group. Let $\gamma = \{s \in \mathbb{C} \colon |s| = 1\}$ denote the unit circle on the complex plane, and

$$\Omega_a = \prod_{u \in [0,a]} \gamma_u,$$

where $\gamma_u = \gamma$ for all $u \in [0,a]$. As γ is a compact, by the Tikhonov theorem, with the product topology and pointwise multiplication, Ω_a is a compact topological Abelian group. Define a probability measure

$$Q_{T,a}(A) = \nu_T\big(\{e^{-itu} \colon u \in [0,a]\} \in A\big), \quad A \in \mathcal{B}(\Omega_a).$$

The dual group of Ω_a is

$$\mathcal{D} \overset{def}{=} \oplus_{u \in [0,a]} \mathbb{Z}_u,$$

where $\mathbb{Z}_u = \mathbb{Z}$ for each $u \in [0,a]$. An element $\underline{k} = \{k_u \colon u \in [0,a]\} \in \mathcal{D}$, where only a finite number of integers k_u are non – zero, acts on Ω_a by the formula

$$\underline{x} \to \underline{x}^{\underline{k}} = \prod_{u \in [0,a]} x_u^{k_u},$$

where $\underline{x} = \{x_u \colon |x_u| = 1, u \in [0,a]\}$. Hence, the Fourier transform $g_{T,a}(\underline{k})$ of the measure $Q_{T,a}$ is of the form

$$\begin{aligned} g_{T,a}(\underline{k}) &= \int\limits_{\Omega_a} \bigg(\prod_{u \in [0,a]} x_u^{k_u}\bigg) dQ_{T,a} = \frac{1}{T}\int\limits_0^T \prod_{u \in [0,a]} e^{-ituk_u} dt \\ &= \frac{1}{T}\int\limits_0^T \exp\bigg\{-it \sum_{u \in [0,a]} uk_u\bigg\} dt, \end{aligned}$$

where, as above, only a finite number of integers k_u are non-zero. Thus,

$$g_{T,a}(\underline{k}) = \begin{cases} 1 & \text{if} \quad \sum\limits_{u \in [0,a]} uk_u = 0, \\ \frac{\exp\{-iT \sum\limits_{u \in [0,a]} uk_u\} - 1}{-iT \sum\limits_{u \in [0,a]} uk_u} & \text{if} \quad \sum\limits_{u \in [0,a]} uk_u \neq 0. \end{cases}$$

Now let, for $\{y_x \colon x \in [0,a]\} \in \Omega_a$,

$$\widehat{y}_x = \begin{cases} y_x \quad \text{if } y_x \text{ is integrable over } [0,a], \\ \text{an arbitrary integrable over } [0,a] \text{ circle function, otherwise.} \end{cases}$$

Define a function $h_a\colon \Omega_a \to \mathbb{C}^r$ by the formula

$$\begin{aligned} &h_a\bigl(\{y_x\colon x\in[0,a]\}\bigr)\\ &\quad=\biggl(\int_0^a |\zeta(1/2+ix)|^2 e^{-\sigma_1 x}\widehat{y}_x dx,\dots,\int_0^a |\zeta(1/2+ix)|^{2r} e^{-\sigma_r x}\widehat{y}_x dx\biggr). \end{aligned}$$

Then by the Lebesgue theorem on bounded convergence the function h_a is continuous, moreover,

$$\begin{aligned} &h_a\bigl(\{e^{-itx}\colon x\in[0,a]\}\bigr)\\ &\quad=\biggl(\int_0^a |\zeta(1/2+ix)|^2 e^{-(\sigma_1+it)x}dx,\dots,\int_0^a |\zeta(1/2+ix)|^{2r} e^{-(\sigma_r+it)x}dx\biggr)\\ &\quad=\underline{L}_{r,a}(\underline{\sigma}+it). \end{aligned}$$

Hence $P_{T,a,\underline{\sigma}} = Q_{T,a}h_a^{-1}$. This, the weak convergence of the measure $Q_{T,a}$ and Theorem 5.1 of [3] show that the measure $P_{T,a,\underline{\sigma}}$ converges weakly to the measure $Q_a h_a^{-1}$ as $T\to\infty$. The theorem is proved.

3 Proof of Theorem 1

First we observe that the uniform convergence on compact subsets of D of the integral for $L_k(s)$ implies, for $\sigma > 1/2$ and each $k\in\mathbb{N}$, the relation

$$\lim_{a\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T |L_k(\sigma+it)-L_{k,a}(\sigma+it)|dt = 0. \tag{1}$$

Let $\underline{z}_1=(z_{11},\dots,z_{1r})$, $\underline{z}_2=(z_{21},\dots,z_{2r})\in\mathbb{C}$. Define a metric ρ in $\mathbb{C}^r$ by

$$\rho(\underline{z}_1,\underline{z}_2)=\biggl(\sum_{j=1}^r |z_{1j}-z_{2j}|^2\biggr)^{1/2},$$

and, for $\underline{z}\in\mathbb{C}^r$, let $|\underline{z}|=\rho(\underline{z},\underline{0})$. Then, clearly, this metric induces the topology of $\mathbb{C}^r$. Since

$$\rho(\underline{z}_1,\underline{z}_2)\le\sum_{j=1}^r |z_{1j}-z_{2j}|,$$

it follows from (1) that, for $\min\limits_{1\le j\le r} \sigma_j > 1/2$,

$$\begin{aligned}\lim_{a\to\infty} \limsup_{T\to\infty} \frac{1}{T}\int\limits_0^T \rho\big(\underline{L}(\underline{\sigma}+it), L_a(\underline{\sigma}+it)\big)dt \\ \le \lim_{a\to\infty} \limsup_{T\to\infty} \sum_{k=1}^{r} \frac{1}{T}\int\limits_0^T |L_k(\sigma_k+it)-L_{k,a}(\sigma_k+it)|dt = 0.\end{aligned} \tag{2}$$

For the further proof, we recall some definitions and results from the probability theory.

Let (S,ρ) be a separable metric space with a metric ρ, and let $Y_n, X_{1n}, X_{2n}, \dots$ be the S – valued random elements defined on a certain probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$. Denote by $\xrightarrow{\mathcal{D}}$ the convergence in distribution.

Lemma 1. *Suppose that* $X_{kn} \xrightarrow[n\to\infty]{\mathcal{D}} X_k$ *for each* $k \in \mathbb{N}$*, and that* $X_k \xrightarrow[k\to\infty]{\mathcal{D}} X$*. If, for every* $\varepsilon > 0$*,*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}\{\rho(X_{kn}, Y_n) \ge \varepsilon\} = 0,$$

then $Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X$.

The lemma is Theorem 4.2 of [3] where its proof is given.

Let $\mathcal{P} = \{P\}$ be a family of probability measures on $(S, \mathcal{B}(S))$. The family $\mathcal{P}$ is called tight if, for arbitrary $\varepsilon > 0$, there exists a compact set $K \subset S$ such that

$$P(K) > 1 - \varepsilon$$

for all P from $\mathcal{P}$. The family $\mathcal{P}$ is relatively compact if every sequence of elements of $\mathcal{P}$ contains a weakly convergent subsequence.

The next lemma (the Prokhorov theorem) relates the relative compactness with the tightness of the family $\mathcal{P}$.

Lemma 2. *If the family* $\mathcal{P}$ *is tight, then it is relatively compact.*

The lemma is Theorem 6.1 from [3].

Proof of Theorem 1. By Theorem 2, the probability measure $P_{T,a,\underline{\sigma}}$ converges weakly to the measure $P_{a,\underline{\sigma}}$ as $T \to \infty$. On a certain probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$, define an uniformly distributed on [0, 1] random variable θ, and put

$$\underline{X}_{T,a} = \underline{X}_{T,a}(\underline{\sigma}) = \underline{L}_{r,a}(\underline{\sigma} + iT\theta).$$

Then, denoting by $\underline{X}_a = \underline{X}_a(\underline{\sigma})$ a $\mathbb{C}^r$ – valued random element with the distribution $P_{a,\underline{\sigma}}$, we can rewrite the assertion of Theorem 2 in the form

$$\underline{X}_{T,a} \xrightarrow[T\to\infty]{\mathcal{D}} \underline{X}_a. \tag{3}$$

Now let M be an arbitrary positive number. Then in view of Chebyshev's inequality

$$\begin{aligned}
\limsup_{T\to\infty} P_{T,a,\underline{\sigma}}\big(\{\underline{z}\in\mathbb{C}^r\colon |\underline{z}|>M\}\big) &= \limsup_{T\to\infty} \nu_T\big(|\underline{L}_a(\underline{\sigma}+it)|>M\big)\\
&\le \limsup_{T\to\infty}\frac{1}{MT}\int_0^T |\underline{L}_a(\underline{\sigma}+it)|dt\\
&\le \frac{1}{M}\sup_{a\ge 0}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\bigg(\sum_{k=1}^r |L_{k,a}(\sigma_k+it)|^2\bigg)^{1/2}dt \qquad (4)\\
&\le \frac{1}{M}\sup_{a\ge 0}\limsup_{T\to\infty}\bigg(\sum_{k=1}^r \frac{1}{T}\int_0^T |L_{k,a}(\sigma_k+it)|^2dt\bigg)^{1/2}.
\end{aligned}$$

Let $\sigma>0$. Then

$$\begin{aligned}
|L_k(\sigma+it)|^2 &= L_k(\sigma+it)L_k(\sigma-it)\\
&= \int_0^\infty |\zeta(1/2+ix)|^{2k}e^{-(\sigma+it)x}dx\int_0^\infty |\zeta(1/2+iy)|^{2k}e^{-(\sigma-it)y}dy.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^T |L_k(\sigma+it)|^2dt &= \int_0^T\bigg(\int_0^\infty |\zeta(1/2+ix)|^{4k}e^{-2\sigma x}dx\bigg)dt\\
&\quad+\int_0^T\bigg(\int_0^\infty\int_{\substack{0\\x\ne y}}^\infty |\zeta(1/2+ix)|^{2k}|\zeta(1/2+iy)|^{2k}e^{-\sigma x-\sigma y}e^{-it(y-x)}dxdy\bigg)dt\\
&= T\int_0^T |\zeta(1/2+ix)|^{4k}e^{-2\sigma x}dx\\
&\quad+\int_0^\infty\int_{\substack{0\\x\ne y}}^\infty |\zeta(1/2+ix)|^{2k}|\zeta(1/2+iy)|^{2k}e^{-\sigma x-\sigma y}\frac{e^{-iT(y-x)}-1}{i(x-y)}dxdy.
\end{aligned}$$

Since, for $\sigma>0$, the integral converges absolutely, hence we find that, for $\sigma>0$,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |L_k(\sigma+it)|^2dt = \int_0^\infty |\zeta(1/2+ix)|^{4k}e^{-2\sigma x}dx \qquad (5)$$

for all $k\in\mathbb{N}$.

Using (2), (4) and (5), we obtain that

$$\limsup_{T\to\infty} P_{T,a,\underline{\sigma}}\big(\{\underline{z}\in\mathbb{C}^r\colon |\underline{z}|>M\}\big)\le \frac{R}{M},$$

where

$$R=\Bigg(\sum_{k=1}^{r}\int_0^\infty |\zeta(1/2+ix)|^{4k}e^{-2\sigma_k x}dx\Bigg)^{1/2}$$
$$+\sup_{a\ge 0}\limsup_{T\to\infty}\Bigg(\sum_{k=1}^{r}\frac{1}{T}\int_0^T |L_k(\sigma_k+it)-L_{k,a}(\sigma_k+it)|^2dt\Bigg)^{1/2}<\infty.$$

Hence, taking $M=R\varepsilon^{-1}$, we find that

$$\limsup_{T\to\infty} P_{T,a,\underline{\sigma}}\big(\{\underline{z}\in\mathbb{C}^r\colon |\underline{z}|>M\}\big)\le \varepsilon.$$

Thus, in view of (3)

$$P_{a,\underline{\sigma}}\big(\{\underline{z}\in\mathbb{C}^r\colon |\underline{z}|>M\}\big)\le \varepsilon. \tag{6}$$

Let $K_\varepsilon=\{\underline{z}\in\mathbb{C}^r\colon |\underline{z}|\le M\}$. Then the set K_ε is compact, and by (6), for all $a>0$,

$$P_{a,\underline{\sigma}}(K_\varepsilon)\ge 1-\varepsilon.$$

This shows that the family $\{P_{n,\underline{\sigma}}\}$ is tight. Hence, by the Prokhorov theorem (Lemma 2) it is relatively compact. Therefore, there exists a subsequence $\{P_{a_1,\underline{\sigma}}\}\subset\{P_{a,\underline{\sigma}}\}$ such that $P_{a_1,\underline{\sigma}}$ converges weakly to some probability measure $P_{\underline{\sigma}}$ on $(\mathbb{C},\mathcal{B}(\mathbb{C}))$ as $a_1\to\infty$. Then

$$\underline{X}_{a_1}\ \xrightarrow[a_1\to\infty]{\mathcal{D}}\ P_{\underline{\sigma}}. \tag{7}$$

Now define

$$\underline{X}_T=X_T(\underline{\sigma})=\underline{L}(\underline{\sigma}+iT\theta).$$

Then, taking into account (2), we find

$$\begin{aligned}&\lim_{a\to\infty}\limsup_{T\to\infty}\mathbb{P}\big(\rho\big(\underline{X}_{T,a}(\underline{\sigma}),X_T(\underline{\sigma})\big)\ge\varepsilon\big)\\ &\quad=\lim_{a\to\infty}\limsup_{T\to\infty}\nu_T\big(\rho\big(\underline{L}_a(\underline{\sigma}+it),\underline{L}(\underline{\sigma}+it)\big)\ge\varepsilon\big)\\ &\quad\le\frac{1}{\varepsilon T}\int_0^T\rho\big(\underline{L}_a(\underline{\sigma}+it),\underline{L}(\underline{\sigma}+it)\big)dt=0.\end{aligned}$$

This, (3), (7) yield

$$\underline{X}_T\ \xrightarrow[T\to\infty]{\mathcal{D}}\ \mathcal{P},$$

and the theorem is proved. □

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