Asymptotic behavior of the Gerber–Shiu discounted penalty function in the Erlang(2) risk process with subexponential claims*

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Abstract. We investigate the asymptotic behavior of the Gerber–Shiu discounted penalty function

$$\phi(u) = E\big(\mathrm{e}^{-\delta T}\mathbf{1}_{\{T<\infty\}} \;\big|\; U(0) = u\big),$$

where T denotes the time to ruin in the Erlang(2) risk process. We obtain an asymptotic expression for the discounted penalty function when claim sizes are subexponentially distributed.

Keywords: Gerber–Shiu discounted penalty function, subexponential claim sizes, defective renewal equation.

1 Introduction

In this paper we consider the insurer's surplus process $\{U(t),\, t \geqslant 0\}$ which is defined by the equality

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Y_i, \tag{1}$$

where $u \geq 0$ is the insurer's initial surplus, $c > 0$ is the rate of premium income per unit time, and $\{N(t)\}_{t\geq 0}$ is the renewal counting process for the number of claims up to time t. As usual,

$$N(t) = \sum_{i=1}^{\infty} \mathbf{1}_{\{\theta_1+\theta_2+\cdots+\theta_i \leqslant t\}}, \tag{2}$$

where $\theta_1, \theta_2, \ldots$ is a sequence of independent and identically distributed random variables, which represent the inter-arrival times, with θ_1 being the time until the first claim. In

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addition, in this paper we suppose that θ_1 has the Erlang(2) density function with scale parameter $\lambda > 0$:

$$k(y) = \lambda^2 y \mathrm{e}^{-\lambda y}, \quad y \geqslant 0.$$

Individual claims $Y_1, Y_2, \ldots$ are non-negative, independent and identically distributed random variables with distribution function $H(y) = \mathbb{P}(Y_1 \leqslant y)$ and finite mean $\mathbb{E}Y_1 = a$. In addition, we suppose that the claim sequence is independent of the renewal process $N(t)$. Also we assume that the safety loading condition holds, i.e.

$$\varrho := \frac{c\mathbb{E}\theta_1}{\mathbb{E}Y_1} - 1 = \frac{2c}{\lambda a} - 1 > 0. \tag{3}$$

Suppose that

$$T = \inf\{t > 0\colon U(t) < 0 \mid U(0) = u\}$$

is the time to ruin. Hence $|U(T)|$ is the deficit at ruin and $U(T-)$ is the surplus before ruin. In [1], Gerber and Shiu considered a function associated with a given penalty function ω and the joint distribution of $(T, U(T-), |U(T)|)$. The authors named this function by an expected discounted penalty function and defined it by the following equality

$$\phi_\omega(u) = \mathbb{E}\big(\mathrm{e}^{-\delta T}\omega\big(U(T-), |U(T)|\big)\mathbf{1}_{\{T<\infty\}} \bigm| U(0) = u\big), \tag{4}$$

where $\mathbf{1}$ is an indicator function; $\omega(x, y)$, $0 \leqslant x, y < \infty$ is some non-negative function, which can be interpreted as the penalty at the time to ruin; and $\delta \geq 0$ is a force of interest. It is important to mention that, since the concept of expected discounted penalty function has been introduced, many authors started to investigate the Gerber–Shiu discounted penalty function. A number of significant results about this function was obtained, but many problems haven't been studied yet and the investigation of the Gerber–Shiu discounted penalty function is still actual and important.

Cheng and Tang [2] investigated the discounted penalty function in Erlang(2) risk process. They demonstrated that if $\delta = 0$ and

$$\int_0^\infty \int_0^\infty \omega(x, y) h(x + y)\,\mathrm{d}x\,\mathrm{d}y < \infty, \tag{5}$$

then the function $\phi_\omega(u)$ satisfies some defective renewal equation. Here $h(y)$ is the continuous density function of individual claim sizes. In the case where $\delta \geqslant 0$, using a similar approach as in Cheng and Tang, Sun showed (see [3]) that under condition (5) the function $\phi_\omega(u)$ satisfies the following defective renewal equation

$$\phi_\omega(u) = \frac{1}{1+\beta}\int_0^u \phi_\omega(u-y)\,\mathrm{d}G(y) + \frac{1}{1+\beta}B(u), \tag{6}$$

where

$$B(u) = \frac{\lambda^2}{c^2}(1+\beta)\int_u^\infty \mathrm{e}^{-\rho_2(s-u)}\int_s^\infty \mathrm{e}^{-\rho_1(x_1-s)}\int_{x_1}^\infty \omega(x_1, x_2 - x_1)\,\mathrm{d}H(x_2)\,\mathrm{d}x_1\,\mathrm{d}s, \tag{7}$$

$$\beta = \begin{cases} \dfrac{2\lambda\delta + \delta^2}{c^2\rho_1\rho_2 - 2\lambda\delta - \delta^2} & \text{if } \delta > 0, \\ \dfrac{\lambda(2c - \lambda a)}{a\lambda^2 - 2\lambda c + c^2\rho_2} & \text{if } \delta = 0, \end{cases}$$

and $\rho_1 = \rho_1(\delta)$, $\rho_2 = \rho_2(\delta)$ are two non-negative roots of Lundberg's fundamental equation (see, for example, [4])

$$\lambda^2 \int_0^\infty \mathrm{e}^{-sx} \, \mathrm{d}H(x) = (cs - \lambda - \delta)^2 \tag{8}$$

provided $0 \leqslant \rho_1 < (\lambda + \delta)/c < \rho_2$, $\rho_1(0) = 0$.

Furthermore, in (6),

$$G(y) = \frac{1}{D} \int_0^y g(x) \, \mathrm{d}x,$$

where

$$g(x) = \frac{\lambda^2}{c^2} \int_x^\infty \mathrm{e}^{-\rho_2(v-x)} \int_{[v,\infty)} \mathrm{e}^{-\rho_1(z-v)} \, \mathrm{d}H(z) \, \mathrm{d}v,$$

and

$$D = \int_0^\infty g(x) \, \mathrm{d}x = \frac{c^2\rho_1\rho_2 - 2\lambda\delta - \delta^2}{c^2\rho_1\rho_2} = \frac{1}{1+\beta}. \tag{9}$$

In this paper we investigate the case where $\omega(x, y) = 1$ for all x, y, namely the penalty at the moment T is accepted to be unit. In this case, for $\delta \geqslant 0$ let

$$\phi(u) = \mathbb{E}\bigl(\mathrm{e}^{-\delta T} \mathbf{1}_{\{T<\infty\}} \bigm| U(0) = u\bigr).$$

We obtain that the partial discounted penalty function $\phi(u)$ satisfies a defective renewal equation without the technical condition (5) which was required by Sun [3] and Cheng and Tang [2]. Below, we formulate this assertion.

Theorem 1. *Consider the Erlang(2) risk model with the safety loading condition* (3)*. If $\delta > 0$, then the penalty function $\phi(u)$ satisfies the renewal equation*

$$\phi(u) = \frac{1}{1+\beta} \int_{[0,u]} \phi(u-y) \, \mathrm{d}G(y) + \frac{1}{1+\beta}\overline{G}(u) \tag{10}$$

with distribution function $G(u) = 1 - \overline{G}(u)$ for which $G(0) = 0$, and

$$\overline{G}(u) = \frac{\lambda^2}{c^2}(1+\beta) \int_u^\infty \mathrm{e}^{-\rho_2(y-u)} \int_y^\infty \mathrm{e}^{-\rho_1(x-y)}\overline{H}(x) \, \mathrm{d}x \, \mathrm{d}y, \quad u \geqslant 0. \tag{11}$$

Here ρ_1, ρ_2 ($\rho_1 < (\lambda+\delta)/c < \rho_2$) are two positive roots of equation (8)*, and the positive constant β is defined in* (9)*.*

The defective renewal equation for the function $\phi_\omega(u)$ defined in (4) has been investigated in many works. The papers of Gerber and Shiu [5], Willmot [6] and Landriault and Willmot [7] should be mentioned where the general case of inter-arrival times $\theta_1, \theta_2, \dots$ was considered. Unfortunately, to obtain the renewal equation for $\phi_\omega(u)$, the authors suppose that the joint distribution of $(T, U(T-), |U(T)|)$ has a density. This questionable assumption greatly facilitates the derivation of the renewal equation. In Section 2, we present the proof of equation (10) which does not require the existence of such a joint density. For this reason, our proof turns out to be much more complicated.

Before formulating the second main result, we need to define the class of subexponential distribution functions.

A distribution function F with support $[0, \infty)$ *is called subexponential if*

$$\lim_{x\to\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} = 2,$$

where $F * F$ *denotes the convolution of F with itself. As usual, the class of all subexponential distribution functions is denoted by* $\mathcal{S}$.

As it was already mentioned, the Gerber–Shiu discounted penalty function was widely investigated. Much less attention has been paid to the asymptotic behavior of this function. We mention only few works. Šiaulys and Asanavičiūtė [8] obtained an asymptotic formula for the Gerber–Shiu discounted penalty function in the classical Poisson risk model with subexponential claim sizes. They showed that the asymptotic formula

$$\phi(u) \sim \frac{\mu}{\delta}\, \overline{H}(u), \quad u \to \infty,$$

holds. Here $H \in \mathcal{S}$, $\delta > 0$, and μ is the intensity of the Poisson process.

Cheng and Tang [2] derived some asymptotic formulas for the moments of the surplus prior to ruin and deficit at ruin in the renewal risk model with convolution-equivalent claim sizes and Erlang(2) inter-arrival times. Tang and Wei [9] studied the renewal risk model with absolutely continuous claim sizes whose density function belongs to some class of heavy-tailed distributions and satisfies some additional conditions. They obtained asymptotic formulas for the Gerber–Shiu discounted penalty function which involve the ladder heights and related quantities in the main terms.

Our purpose is to find the simple asymptotic formula of the Gerber–Shiu discounted penalty function like in [8] for the Erlang(2) risk process with claims distributed according to the classical subexponential law. The second main result of this article is the following theorem.

Theorem 2. *Consider the Erlang(2) model with the safety loading condition* (3) *with the claim distribution function* $H \in \mathcal{S}$. *Then, for* $\delta > 0$,

$$\frac{\phi(u)}{\overline{H}(u)} \sim \frac{\lambda^2}{2\lambda\delta + \delta^2}, \quad u \to \infty.$$

If, in addition,

$$\frac{1}{a}\int_0^u \overline{H}(y)\,\mathrm{d}y \in \mathcal{S},$$

then, for $\delta = 0$*,*

$$\phi(u) \sim \frac{1}{\varrho a}\int_u^\infty \overline{H}(y)\,\mathrm{d}y.$$

The second part of Theorem ($\delta = 0$) is well known (see, for example, [10]). The proof in the case $\delta > 0$ is presented in Section 3. We can check that the main formula of Theorem 2 coincides with the asymptotic formula (3.15) of Tang and Wei [9], which was proved for absolutely continuous claim distributions.

2 Proof of Theorem 1

In this section we show that the Gerber–Shiu discounted penalty function $\phi(u)$ satisfies the defective renewal equation (10) as stated in Theorem 1.

Proof of Theorem 1. Let δ be any fixed positive real number. Let the surplus process $U(t)$ be defined by equality (1) with renewal counting process $N(t)$ defined by equality (2). Denote $T_0 = 0$ and $T_m = \theta_1 + \theta_2 + \cdots + \theta_m$ for $m \geq 1$. It is evident that ruin can occur only at moments T_m, $m \geq 1$. For these moments, $N(T_m) = m$ and

$$U(T_m) = u + cT_m - \sum_{n=1}^{N(T_m)} Y_n = u - S_m,$$

where

$$S_0 = 0, \quad S_m = \sum_{n=1}^{m} (Y_n - c\theta_n), \quad m = 1, 2, \ldots.$$

We have that for all nonnegative u

$$\mathbb{P}(T = T_1) = \mathbb{P}(S_1 > u),$$

and for all $m = 2, 3, \ldots$

$$\mathbb{P}(T = T_m) = \mathbb{P}(S_m > u,\, S_{m-1} \leqslant u,\, S_{m-2} \leqslant u,\, \ldots,\, S_1 \leqslant u).$$

Therefore, a discounted penalty function

$$\begin{aligned}\phi(u) &= \mathbb{E}\big(\mathrm{e}^{-\delta T}\mathbf{1}_{\{T<\infty\}}\big) = \mathbb{E}\Bigg(\mathrm{e}^{-\delta T}\sum_{m=1}^{\infty}\mathbf{1}_{\{T=T_m\}}\Bigg)\\ &= \sum_{m=1}^{\infty}\mathbb{E}\big(\mathrm{e}^{-\delta T_m}\mathbf{1}_{\{S_1\leqslant u,\, S_2\leqslant u,\, \ldots,\, S_{m-1}\leqslant u,\, S_m>u\}}\big)\end{aligned}$$

for nonnegative u. Decomposing the last sum we obtain

$$\phi(u) = \mathbb{E}\mathrm{e}^{-\delta T_1}\mathbf{1}_{\{S_1>u\}} + \sum_{m=2}^{\infty} \mathbb{E}\mathrm{e}^{-\delta\theta_1}\mathrm{e}^{-\delta(T_m-\theta_1)}$$
$$\times \mathbf{1}_{\{S_1\leqslant u,\, S_2-S_1\leqslant u-S_1,\, \dots,\, S_{m-1}-S_1\leqslant u-S_1,\, S_m-S_1>u-S_1\}}.$$

Therefore,

$$\begin{aligned}
\phi(u) &= \mathbb{E}\mathrm{e}^{-\delta T_1}\mathbf{1}_{\{S_1>u\}} + \sum_{m=2}^{\infty} \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}\mathrm{e}^{-\delta t}\mathrm{e}^{-\delta(T_m-\theta_1)} \\
&\quad \times \mathbf{1}_{\{y-ct\leqslant u,\, S_2-S_1\leqslant u-y+ct,\, \dots,\, S_{m-1}-S_1\leqslant u-y+ct,\, S_m-S_1>u-y+ct\}}\,\mathrm{d}H(y)\lambda^2 t\mathrm{e}^{-\lambda t}\,\mathrm{d}t \\
&= \mathbb{E}\,\mathrm{e}^{-\delta T_1}\mathbf{1}_{\{S_1>u\}} + \int_0^{\infty} \mathrm{e}^{-\delta t} \int_{[0,u+ct]} \sum_{m=2}^{\infty} \mathbb{E}\mathrm{e}^{-\delta(T_m-\theta_1)} \\
&\quad \times \mathbf{1}_{\{S_2-S_1\leqslant u-y+ct,\, \dots,\, S_{m-1}-S_1\leqslant u-y+ct,\, S_m-S_1>u-y+ct\}}\,\mathrm{d}H(y)\lambda^2 t\mathrm{e}^{-\lambda t}\,\mathrm{d}t \\
&= \mathbb{E}\,\mathrm{e}^{-\delta T_1}\mathbf{1}_{\{S_1>u\}} + \int_0^{\infty} \mathrm{e}^{-\delta t} \int_{[0,u+ct]} \sum_{l=1}^{\infty} \mathbb{E}\mathrm{e}^{-\delta T_l} \\
&\quad \times \mathbf{1}_{\{S_1\leqslant u-y+ct,\, S_2\leqslant u-y+ct,\, \dots,\, S_{l-1}\leqslant u-y+ct,\, S_l>u-y+ct\}}\,\mathrm{d}H(y)\lambda^2 t\mathrm{e}^{-\lambda t}\,\mathrm{d}t \\
&= \int_0^{\infty} \mathrm{e}^{-\delta t} \int_{y-ct>u} \mathrm{d}H(y)\lambda^2 t\,\mathrm{e}^{-\lambda t}\,\mathrm{d}t + \int_0^{\infty} \mathrm{e}^{-\delta t} \int_{[0,u+ct]} \phi(u+ct-y)\,\mathrm{d}H(y)\lambda^2 t\mathrm{e}^{-\lambda t}\,\mathrm{d}t \\
&= \lambda^2 \int_0^{\infty} t\,\mathrm{e}^{-(\delta+\lambda)t} \biggl(\int_{[0,\,u+ct]} \phi(u+ct-y)\,\mathrm{d}H(y) + \int_{(u+ct,\,\infty)} \mathrm{d}H(y) \biggr)\,\mathrm{d}t.
\end{aligned}$$

Substituting the new variable $z = u + ct$ into obtained equality we get

$$\phi(u) = \frac{\lambda^2}{c} \int_u^{\infty} \frac{z-u}{c}\,\mathrm{e}^{-\frac{(\delta+\lambda)(z-u)}{c}} \biggl(\int_{[0,z]} \phi(z-y)\,\mathrm{d}H(y) + \int_{(z,\infty)} \mathrm{d}H(y) \biggr)\,\mathrm{d}z. \tag{12}$$

Since the function

$$\int_{[0,z]} \phi(z-y)\,\mathrm{d}H(y) + \int_{(z,\infty)} \mathrm{d}H(y) = 1 - \int_{[0,z]} \bigl(1-\phi(z-y)\bigr)\,\mathrm{d}H(y)$$

is decreasing and bounded by unity, the product

$$\frac{z-u}{c}\mathrm{e}^{-\frac{(\delta+\lambda)(z-u)}{c}} \biggl(\int_{[0,z]} \phi(z-y)\,\mathrm{d}H(y) + \int_{(z,\infty)} \mathrm{d}H(y) \biggr)$$

is continuous in z for all positive z with the possible exception some finite or countable subset of $\mathbb{R}$. In addition, this product of functions is integrable over the interval $[0,\infty)$. Hence, it follows from (12) that $\phi(u)$ is absolutely continuous decreasing function, and for almost all nonnegative u

$$\phi' = \frac{\lambda^2}{c}\int_u^\infty \left[\frac{z-u}{c}\mathrm{e}^{-\frac{(\delta+\lambda)(z-u)}{c}}\left(\int_{[0,z]}\phi(z-y)\,\mathrm{d}H(y) + \int_{(z,\infty)}\mathrm{d}H(y)\right)\right]_u' \mathrm{d}z.$$

Simplifying the last equation and using (12), we get that for almost all nonnegative u

$$\phi'(u) = \frac{\delta+\lambda}{c}\phi(u) - \frac{\lambda^2}{c^2}\int_u^\infty \mathrm{e}^{-\frac{(\delta+\lambda)(z-u)}{c}}\left(\int_{[0,z]}\phi(z-y)\,\mathrm{d}H(y) + \int_{(z,\infty)}\mathrm{d}H(y)\right)\mathrm{d}z.$$

Similarly to the above considerations from the last expression we obtain that $\phi'(u)$ is absolutely continuous and for almost all nonnegative u the second derivative of $\phi(u)$ satisfies

$$\begin{aligned}\phi''(u) &= \frac{2(\lambda+\delta)}{c}\,\phi'(u) - \frac{(\lambda+\delta)^2}{c^2}\,\phi(u)\\ &\quad+\frac{\lambda^2}{c^2}\int_{[0,u]}\phi(u-y)\,\mathrm{d}H(y) + \frac{\lambda^2}{c^2}\overline{H}(u). \end{aligned}\tag{13}$$

Now we use the Laplace transform for reconstruction of equation (13). It is well known that the Laplace transform of some function exists in the case if this function has exponential order. Since $0 \leqslant \phi(u) \leqslant 1$ for all $u \geqslant 0$, we have that

$$\left|\phi'(u)\right| \leqslant \frac{\delta+\lambda}{c}\phi(u) + \frac{\lambda^2}{c^2}\int_u^\infty \mathrm{e}^{-\frac{(\delta+\lambda)(z-u)}{c}}\,\mathrm{d}z \leqslant \frac{\delta+\lambda}{c} + \frac{\lambda^2}{c(\delta+\lambda)} := c_1,$$

and

$$\begin{aligned}\left|\phi''(u)\right| &\leqslant \frac{2(\lambda+\delta)}{c}\left|\phi'(u)\right| + \frac{(\lambda+\delta)^2}{c^2}\phi(u) + \frac{\lambda^2}{c^2}\int_{[0,u]}\phi(u-y)\,\mathrm{d}H(y) + \frac{\lambda^2}{c^2}\overline{H}(u)\\ &\leqslant \frac{2(\lambda+\delta)}{c}c_1 + \frac{(\lambda+\delta)^2}{c^2} + \frac{\lambda^2}{c^2}\end{aligned}$$

for almost all $u \geqslant 0$. Thus the functions in equation (13) are bounded, and the Laplace transforms of these functions exist. Taking the Laplace transform on the both sides of equation (13) we obtain that for all complex s with $\Re e(s) > 0$

$$\begin{aligned}&s^2\widehat{\phi}(s) - s\phi(0) - \phi'(0)\\ &\quad= \frac{2(\lambda+\delta)}{c}\big(s\widehat{\phi}(s) - \phi(0)\big) - \frac{(\lambda+\delta)^2}{c^2}\widehat{\phi}(s) + \frac{\lambda^2}{c^2}\widehat{\phi}(s)\widetilde{H}(s) + \frac{\lambda^2}{c^2}\widehat{\overline{H}}(s),\end{aligned}\tag{14}$$

where

$$\widehat{\phi}(s) = \int_0^\infty \mathrm{e}^{-su}\phi(u)\,\mathrm{d}u, \quad \widetilde{H}(s) = \int_{[0,\infty)} \mathrm{e}^{-su}\,\mathrm{d}H(u), \quad \widehat{\overline{H}}(s) = \int_0^\infty \mathrm{e}^{-su}\overline{H}(u)\,\mathrm{d}u$$

because

$$\widehat{\phi''}(s) = s^2\widehat{\phi}(s) - s\phi(0) - \phi'(0), \quad \widehat{\phi'}(s) = s\widehat{\phi}(s) - \phi(0),$$

and by Fubini's theorem

$$\int_0^\infty \mathrm{e}^{-su}\Bigg(\int_{[0,u]} \phi(u-y)\,\mathrm{d}H(y)\Bigg)\,\mathrm{d}u = \int_{[0,\infty)} \Bigg(\int_y^\infty \mathrm{e}^{-su}\phi(u-y)\,\mathrm{d}u\Bigg)\,\mathrm{d}H(y)$$

$$= \int_{[0,\infty)} \Bigg(\int_0^\infty \mathrm{e}^{-s(x+y)}\phi(x)\,\mathrm{d}x\Bigg)\,\mathrm{d}H(y) = \widehat{\phi}(s)\widetilde{H}(s).$$

After some simplifications of (14) we get the following expression for the Laplace transform $\widehat{\phi}(s)$

$$\widehat{\phi}(s) = \frac{c^2 s\phi(0) - 2c(\lambda+\delta)\phi(0) + c^2\phi'(0) + \lambda^2\widehat{\overline{H}}(s)}{(sc-\lambda-\delta)^2 - \lambda^2\widetilde{H}(s)}. \tag{15}$$

For a latter expression we use the same transformations as in Sun's paper (see [3, Thm. 2.2]). Let T_ρ be an operator defined in Dickson and Hipp [4]. For a given integrable function f and any real ρ

$$T_\rho f(x) = \int_x^\infty \mathrm{e}^{-\rho(z-x)} f(z)\,\mathrm{d}z, \quad x \geqslant 0.$$

Simple calculations show that

$$\widehat{T_\rho f}(s) = \frac{\widehat{f}(s) - \widehat{f}(\rho)}{\rho - s} \tag{16}$$

for all $\rho > 0$, $\Re e(s) > 0$, $\rho \neq s$. Non negative real roots of equation (8) ρ_1 and ρ_2 are zeroes for the denominator of expression (15). Hence, they are also zeroes for the numerator of this expression. In particular,

$$c^2\rho_1\phi(0) - 2c(\lambda+\delta)\phi(0) + c^2\phi'(0) + \lambda^2\widehat{\overline{H}}(\rho_1) = 0.$$

Hence, from property (16) and expression (15) we get

$$\widehat{\phi}(s) = \frac{c^2 s\phi(0) + \lambda^2\widehat{\overline{H}}(s) - c^2\rho_1\phi(0) - \lambda^2\widehat{\overline{H}}(\rho_1)}{(sc-\lambda-\delta)^2 - \lambda^2\widetilde{H}(s)}$$

$$= \frac{(s-\rho_1)\big(c^2\phi(0) - \lambda^2\frac{\widehat{\overline{H}}(s) - \widehat{\overline{H}}(\rho_1)}{\rho_1 - s}\big)}{(sc-\lambda-\delta)^2 - \lambda^2\widetilde{H}(s)} = \frac{(s-\rho_1)\big(c^2\phi(0) - \lambda^2\,\widehat{T_{\rho_1}\overline{H}}(s)\big)}{(sc-\lambda-\delta)^2 - \lambda^2\widetilde{H}(s)} \tag{17}$$

for all complex s with positive real part. It was mentioned that ρ_2 is another zero for denominator of the last expression. Hence ρ_2 $(\rho_2 > \rho_1)$ is a zero for numerator. Therefore,

$$c^2\phi(0) = \lambda^2\widehat{T_{\rho_1}\overline{H}}(\rho_2),$$

and, using the property (16) again, we obtain from (17) that for above s

$$\begin{aligned}\widehat{\phi}(s) &= \frac{\lambda^2(s-\rho_1)(s-\rho_2)\frac{\widehat{T_{\rho_1}\overline{H}}(\rho_2)-\widehat{T_{\rho_1}\overline{H}}(s)}{s-\rho_2}}{L(s)}\\ &= \frac{\lambda^2(s-\rho_1)(s-\rho_2)\widehat{T_{\rho_2}T_{\rho_1}\overline{H}}(s)}{L(s)}, \end{aligned}\tag{18}$$

where $L(s)$ denotes the denominator of the fraction in (17). Now we consider this denominator in more details. Evidently,

$$\begin{aligned}L(s) &= (sc-\lambda-\delta)^2 - \lambda^2\widetilde{H}(s) - (\rho_1 c-\lambda-\delta)^2 + \lambda^2\widetilde{H}(\rho_1)\\ &= (s-\rho_1)\left(c^2(s+\rho_1) - 2(\lambda+\delta)c + \lambda^2\frac{\widetilde{H}(\rho_1)-\widetilde{H}(s)}{s-\rho_1}\right). \end{aligned}\tag{19}$$

Let τ_ρ be a new operator defined by equality

$$\tau_\rho F(x) = \int\limits_{[x,\infty)} \mathrm{e}^{-\rho(z-x)}\,\mathrm{d}F(z), \quad x \geqslant 0,$$

where F is a distribution function of a nonnegative random variable and ρ is real positive number. Similarly to relation (16), we obtain that the Laplace transform of function $\tau_\rho F$ has the following property

$$\widehat{\tau_\rho F}(s) = \frac{\widetilde{F}(s)-\widetilde{F}(\rho)}{\rho - s},\tag{20}$$

where $\widetilde{F}(s) = \int_{[0,\infty)} \mathrm{e}^{-su}\,\mathrm{d}F(u)$ denotes the Laplace–Stieltjes transform of the distribution function F. According to this relation, it follows from (19) that

$$L(s) = (s-\rho_1)\bigl(c^2(s+\rho_1) - 2(\lambda+\delta)c + \lambda^2\,\widehat{\tau_{\rho_1}H}(s)\bigr)$$

Since ρ_2 is also a root of $L(s) = 0$, we have

$$c^2(\rho_2+\rho_1) - 2(\lambda+\delta)c + \lambda^2\widehat{\tau_{\rho_1}H}(\rho_2) = 0.$$

Hence, using the property (16) we obtain:

$$\begin{aligned}L(s) &= (s-\rho_1)\bigl(c^2(s-\rho_2) + \lambda^2\widehat{\tau_{\rho_1}H}(s) - \lambda^2\widehat{\tau_{\rho_1}H}(\rho_2)\bigr)\\ &= (s-\rho_1)(s-\rho_2)\bigl(c^2 - \lambda^2\widehat{T_{\rho_2}\tau_{\rho_1}H}(s)\bigr).\end{aligned}$$

Substituting this expression into (18) we get that

$$\widehat{\phi}(s) = \frac{\lambda^2 \widehat{T_{\rho_2} T_{\rho_1} \overline{H}}(s)}{c^2 - \lambda^2 \widehat{T_{\rho_2} \tau_{\rho_1} H}(s)} \tag{21}$$

for complex s, $\Re e(s) > 0$. For non negative u, let

$$\eta(u) := T_{\rho_2} T_{\rho_1} \overline{H}(u) = \int_u^\infty \mathrm{e}^{-\rho_2(x-u)} \left(\int_x^\infty \mathrm{e}^{-\rho_1(y-x)} \overline{H}(y) \, \mathrm{d}y \right) \mathrm{d}x,$$

$$\gamma(u) := T_{\rho_2} \tau_{\rho_1} H(u) = \int_u^\infty \mathrm{e}^{-\rho_2(x-u)} \left(\int_{[x,\infty)} \mathrm{e}^{-\rho_1(y-x)} \, \mathrm{d}H(y) \right) \mathrm{d}x.$$

It follows from (21) that

$$\widehat{\phi}(s) = \frac{\lambda^2}{c^2} \big(\widehat{\phi}(s) \widehat{\gamma}(s) + \widehat{\eta}(s) \big)$$

for any complex s as above. Inverting the Laplace transform, we get the following form of the renewal equation for $\phi(u)$

$$\phi(u) = \frac{\lambda^2}{c^2} \int_0^u \phi(u-y) \gamma(y) \, \mathrm{d}y + \frac{\lambda^2}{c^2} \eta(u).$$

The desired renewal equation (10) follows immediately from the last equality taking

$$\overline{G}(u) = \frac{\lambda^2}{c^2} (1+\beta) \eta(u),$$

because (for details, see [11] or [12])

$$\begin{aligned}
(1+\beta) \int_u^\infty \frac{\lambda^2}{c^2} \gamma(x) \, \mathrm{d}x &= \frac{\lambda^2 (1+\beta)}{c^2} \int_u^\infty \int_x^\infty \mathrm{e}^{-\rho_2(y-x)} \int_{[y,\infty)} \mathrm{e}^{-\rho_1(z-y)} \, \mathrm{d}H(z) \, \mathrm{d}y \, \mathrm{d}x \\
&= \frac{\lambda^2 (1+\beta)}{c^2} \int_u^\infty \int_{(x,\infty)} \mathrm{e}^{\rho_2 x - \rho_1 z} \int_x^z \mathrm{e}^{-(\rho_2 - \rho_1) y} \, \mathrm{d}y \, \mathrm{d}H(z) \, \mathrm{d}x \\
&= \frac{\lambda^2 (1+\beta)}{c^2 (\rho_2 - \rho_1)} \int_u^\infty \int_{(x,\infty)} \big(\mathrm{e}^{-\rho_1(z-x)} - \mathrm{e}^{-\rho_2(z-x)} \big) \, \mathrm{d}H(z) \, \mathrm{d}x \\
&= \frac{\lambda^2 (1+\beta)}{c^2 (\rho_2 - \rho_1)} \int_u^\infty \big(\mathrm{e}^{-\rho_1(x-u)} - \mathrm{e}^{-\rho_2(x-u)} \big) \overline{H}(x) \, \mathrm{d}x
\end{aligned}$$

and

$$
\begin{aligned}
\overline{G}(u) &= \frac{\lambda^2(1+\beta)}{c^2}\int_u^\infty \mathrm{e}^{-\rho_2(y-u)}\int_y^\infty \mathrm{e}^{-\rho_1(x-y)}\overline{H}(x)\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{\lambda^2(1+\beta)}{c^2}\int_u^\infty \overline{H}(x)\mathrm{e}^{-\rho_1 x+\rho_2 u}\int_u^x \mathrm{e}^{-(\rho_2-\rho_1)y}\,\mathrm{d}y\,\mathrm{d}x\\
&= \frac{\lambda^2(1+\beta)}{c^2(\rho_2-\rho_1)}\int_u^\infty \bigl(\mathrm{e}^{-\rho_1(x-u)}-\mathrm{e}^{-\rho_2(x-u)}\bigr)\overline{H}(x)\,\mathrm{d}x.
\end{aligned}
$$

Finally, observe that the properties (16) and (20) imply identity (9). Namely,

$$
\begin{aligned}
\frac{\lambda^2}{c^2}\int_0^\infty \gamma(x)\,\mathrm{d}x &= \frac{\lambda^2}{c^2}\widehat{T_{\rho_2}\tau_{\rho_1}H}(0) = \frac{\lambda^2}{c^2}\,\frac{\widehat{\tau_{\rho_1}H}(0)-\widehat{\tau_{\rho_1}H}(\rho_2)}{\rho_2}\\
&= \frac{\frac{\lambda^2-\lambda^2\widetilde{H}(\rho_1)}{\rho_1}+\frac{\lambda^2\widetilde{H}(\rho_2)-\lambda^2\widetilde{H}(\rho_1)}{\rho_2-\rho_1}}{c^2\,\rho_2}\\
&= \frac{\frac{\lambda^2-(c\rho_1-\lambda-\delta)^2}{\rho_1}+\frac{(c\rho_2-\lambda-\delta)^2-(c\rho_1-\lambda-\delta)^2}{\rho_2-\rho_1}}{c^2\rho_2}\\
&= \frac{c^2\rho_1\rho_2-2\lambda\delta-\delta^2}{c^2\rho_1\rho_2}.
\end{aligned}
$$

Theorem 1 is proved. □

3 Proof of Theorem 2

In this section, we present the proof of Theorem 2. For this we need the following two auxiliary lemmas. The first lemma describes the standard form of a solution of equation (6). The simple proof of lemma can be found, for example, in [12] (see Theorem 2.1).

Lemma 1. *Assume that a function ψ satisfies the defective renewal equation*

$$
\psi(v) = \frac{1}{1+b}\int_{[0,v]}\psi(v-x)\,\mathrm{d}V(x) + \frac{1}{1+b}W(v), \quad v \geqslant 0,
$$

where $b > 0$, $V(x) = 1-\overline{V}(x)$ is a distribution function with $V(0) = 0$, and $W(v)$ is continuous for all $v \geqslant 0$.

The solution of this equation can be expressed in the form

$$
\psi(v) = \frac{1}{b}\int_{[0,v]} W(v-x)\,\mathrm{d}K(x) = \frac{1}{b}\int_{(0,v]} W(v-x)\,\mathrm{d}K(x) + \frac{1}{1+b}W(v).
$$

Here $K(x) = 1 - \overline{K}(x)$ *is the associated compound geometric distribution function, i.e.*

$$\overline{K}(x) = \sum_{n=1}^{\infty} \frac{b}{1+b}\left(\frac{1}{1+b}\right)^n \overline{V^{*n}}(x), \quad x \geqslant 0,$$

and $\overline{V^{*n}}$ *denotes the tail of the n-fold convolution of* V *with itself.*

The following lemma is needed in order to compare the tail of a convolution of distribution functions with a tail of some basic subexponential distribution. The proof of the lemma below can be found in [13] (see Corollary 3.19).

Lemma 2. *Let* F *be some distribution function from* $\mathcal{S}$*, and let* $G_1, G_2, \ldots, G_n$ *be distribution functions such that*

$$\lim_{x\to\infty} \frac{\overline{G}_i(x)}{\overline{F}(x)} = c_i, \quad i = 1, 2, \ldots n,$$

with some nonnegative constants $c_1, c_2, \ldots, c_n$*. Then*

$$\frac{\overline{G_1 * G_2 * \cdots * G_n}(x)}{\overline{F}(x)} \to c_1 + \cdots + c_n \quad as \quad x \to \infty.$$

If, in addition $c_1 + c_2 + \cdots + c_n > 0$*, then* $G_1 * G_2 * \cdots * G_n \in \mathcal{S}$*.*

Proof of Theorem 2. In the case $\delta > 0$, according to Theorem 1, we have that function $\phi(u)$ satisfies the defective renewal equation (10) with the distribution tail $\overline{G}(u)$ defined by (11). Hence, it follows from Lemma 1 that, for any nonnegative u,

$$\phi(u) = \frac{1}{\beta} \int\limits_{[0,u]} \overline{G}(u-x)\,\mathrm{d}L(x) \tag{22}$$

with

$$\overline{L}(x) = 1 - L(x) = \sum_{n=1}^{\infty} \frac{\beta}{(1+\beta)^{n+1}} \overline{G^{*n}}(x).$$

In order to obtain the asymptotic formula for $\phi(u)$ consider first the function $\overline{G}(u)$. Let for nonnegative y

$$r(y) := \int\limits_y^{\infty} \mathrm{e}^{-\rho_1(x-y)} \overline{H}(x)\,\mathrm{d}x.$$

Then, according to (11),

$$\overline{G}(u) = \frac{\lambda^2}{c^2}(1+\beta) \int\limits_u^{\infty} \mathrm{e}^{-\rho_2(y-u)} r(y)\,\mathrm{d}y.$$

We have that

$$r(y) = \frac{1}{\rho_1}\biggl(\overline{H}(y) + \int\limits_{[0,\infty)} \mathrm{e}^{-\rho_1 z}\overline{H}\,(\mathrm{d}z + y)\biggr), \tag{23}$$

and for every positive M

$$\begin{aligned}
&\frac{1}{\overline{H}(y)}\biggl|\int\limits_{[0,\infty)} \mathrm{e}^{-\rho_1 z}\overline{H}\,(\mathrm{d}z + y)\biggr| \\
&\quad = \frac{1}{\overline{H}(y)}\int\limits_{[0,M]} \mathrm{e}^{-\rho_1 z}\bigl(-\overline{H}\,(\mathrm{d}z + y)\bigr) + \frac{1}{\overline{H}(y)}\int\limits_{(M,\infty)} \mathrm{e}^{-\rho_1 z}\bigl(-\overline{H}\,(\mathrm{d}z + y)\bigr) \\
&\quad \leqslant 1 - \frac{\overline{H}(y+M)}{\overline{H}(y)} + \mathrm{e}^{-\rho_1 M}.
\end{aligned}$$

The basic properties of subexponential distribution functions (see, for example, Lemma 1.3.5 in [14]) imply that for every fixed M

$$\lim_{y\to\infty}\biggl(1 - \frac{\overline{H}(y+M)}{\overline{H}(y)}\biggr) = 0.$$

Therefore, according to the obtained estimates,

$$\lim_{y\to\infty}\frac{1}{\overline{H}(y)}\int\limits_{[0,\infty)} \mathrm{e}^{-\rho_1 z}\overline{H}\,(\mathrm{d}z + y) = 0. \tag{24}$$

This and (23) imply

$$r(y) = \frac{1}{\rho_1}\overline{H}(y)\bigl(1 + o(1)\bigr), \quad y \to \infty.$$

Hence for $u \to \infty$ we get

$$\begin{aligned}
\overline{G}(u) &= \frac{\lambda^2}{c^2\rho_1}(1+\beta)\bigl(1+o(1)\bigr)\int\limits_u^\infty \mathrm{e}^{-\rho_2(y-u)}\overline{H}(y)\,\mathrm{d}y \\
&= \frac{\lambda^2}{c^2\rho_1\rho_2}(1+\beta)\bigl(1+o(1)\bigr)\biggl(\overline{H}(u) + \int\limits_{[0,\infty)} \mathrm{e}^{-\rho_2 v}\overline{H}\,(\mathrm{d}v + u)\biggr) \\
&= \frac{\lambda^2}{c^2\rho_1\rho_2}(1+\beta)\overline{H}(u)\bigl(1+o(1)\bigr)
\end{aligned} \tag{25}$$

because, similarly to (24),

$$\lim_{u\to\infty}\frac{1}{\overline{H}(u)}\int\limits_{[0,\infty)} \mathrm{e}^{-\rho_2 v}\overline{H}\,(\mathrm{d}v + u) = 0.$$

Distribution function H belongs to the class $\mathcal{S}$, so, according to Lemma 2, we have that for every fixed n

$$\lim_{u\to\infty}\frac{\overline{G^{*n}}(u)}{\overline{H}(u)}=\frac{\lambda^2}{c^2\rho_1\rho_2}(1+\beta)n,$$

and distribution function G also belongs to the class $\mathcal{S}$. According to the basic properties of subexponential distribution functions (see, for example, Lemma 1.3.5 in [14]), there exists a finite constant K_β such that for all $n\geqslant 2$

$$\sup_{u\geqslant 0}\frac{\overline{G^{*n}}(u)}{\overline{G}(u)}\leqslant K_\beta\left(1+\frac{\beta}{2}\right)^n.$$

Therefore, for every nonnegative u

$$\frac{\overline{G^{*n}}(u)}{(1+\beta)^n\overline{H}(u)}=\frac{\overline{G^{*n}}(u)}{(1+\beta)^n\overline{G}(u)}\frac{\overline{G}(u)}{\overline{H}(u)}\leqslant\frac{K_\beta\lambda^2}{c^2\rho_1\rho_2}(1+\beta)\frac{(1+\frac{\beta}{2})^n}{(1+\beta)^n}.$$

By the dominated convergence theorem we obtain

$$\begin{aligned}\lim_{u\to\infty}\frac{\overline{L}(u)}{\overline{H}(u)}&=\frac{\beta}{(1+\beta)}\lim_{u\to\infty}\sum_{n=1}^{\infty}\frac{1}{(1+\beta)^n}\frac{\overline{G}^{*n}(u)}{\overline{H}(u)}\\&=\frac{\beta}{(1+\beta)}\sum_{n=1}^{\infty}\frac{1}{(1+\beta)^n}\lim_{u\to\infty}\frac{\overline{G}^{*n}(u)}{\overline{H}(u)}\\&=\frac{\beta\lambda^2}{c^2\rho_1\rho_2}\sum_{n=1}^{\infty}\frac{n}{(1+\beta)^n}=\frac{\lambda^2}{c^2\rho_1\rho_2}\frac{(1+\beta)}{\beta}\end{aligned}$$

and we get that

$$\overline{L}(u)\sim\frac{\lambda^2}{c^2\rho_1\rho_2}\frac{(1+\beta)}{\beta}\overline{H}(u),\quad u\to\infty.$$

If u is nonnegative, then

$$\overline{G*L}(u)=\int_{[0,u]}\overline{G}(u-y)\,\mathrm{d}L(y)+\overline{L}(u).$$

According to Lemma 2 and (25)

$$\int_{[0,u]}\overline{G}(u-y)\,\mathrm{d}L(y)\sim\frac{\lambda^2}{c^2\rho_1\rho_2}(1+\beta)\overline{H}(u),$$

Hence, from (22) we get that for $u\to\infty$

$$\phi(u)\sim\frac{\lambda^2}{c^2\rho_1\rho_2}\left(\frac{1}{\beta}+1\right)\overline{H}(u).$$

Theorem 2 is proved. □

4 Examples

In this section we present three simple examples which illustrate the use of Theorem 2 for the evaluation of the Gerber–Shiu discounted penalty function. All obtained formulas are applicable only for a large values of initial capital u.

Example 1. Consider the Pareto claim distribution function

$$H(x) = \left(1 - \frac{1}{(1+x)^3}\right)\mathbf{1}_{\{x \geqslant 0\}},$$

the Erlang(2) scale parameter $\lambda = 2$ and the security loading coefficient $\varrho > 0$. It is well known that distribution function H is subexponential (see, for instance, [14]). In addition, $a = \mathbb{E}Y_1 = 1/2$, and $\int_0^u \overline{H}(y)\,\mathrm{d}y/a$ is also subexponential. Hence, according to Theorem 2 we have that in the described model

$$\phi(u) \sim \begin{cases} \dfrac{4}{\delta^2 + 4\delta}\dfrac{1}{(1+u)^3} & \text{if } \delta > 0, \\ \dfrac{1}{\varrho(1+u)^2} & \text{if } \delta = 0. \end{cases}$$

Example 2. Consider the Weibull distribution function

$$H(x) = \left(1 - \mathrm{e}^{-\sqrt{x}}\right)\mathbf{1}_{\{x \geqslant 0\}},$$

for claims, the inter arrival times scale parameter λ, the force of interest δ and the security loading coefficient $\varrho > 0$. Distribution function H is subexponential (see, for instance, [14]), $a = \mathbb{E}Y_1 = 2$, and

$$\frac{1}{a}\int_0^x \overline{H}(y)\,\mathrm{d}y = \left(1 - (\sqrt{x}+1)\mathrm{e}^{-\sqrt{x}}\right)\mathbf{1}_{\{x \geqslant 0\}} \in \mathcal{S}.$$

In the considered case Theorem 2 implies that

$$\phi(u) \sim \begin{cases} \dfrac{\lambda^2}{\delta^2 + 2\lambda\delta}\mathrm{e}^{-\sqrt{u}} & \text{if } \delta > 0, \\ \dfrac{\sqrt{u}+1}{2\varrho}\mathrm{e}^{-\sqrt{u}} & \text{if } \delta = 0. \end{cases}$$

Example 3. Suppose now that claims of Erlang(2) renewal risk model are distributed according to the discrete law with distribution function

$$H(x) = \frac{1}{A}\sum_{k=1}^{[x]} \mathrm{e}^{-\log^2(k)}\frac{2\log k}{k},$$

where

$$A = \sum_{k=1}^{\infty} \frac{2\log k}{k}\mathrm{e}^{-\log^2 k} \approx 0.851222.$$

Let, in addition, $\lambda = 2$, $\delta = 0.03$ and ϱ is positive. It is evident that $\overline{H}(x) \sim \mathrm{e}^{-\log^2(1+x)}/A$ for large x. Distribution function

$$\left(1 - \mathrm{e}^{-\log^2(1+x)}\right)\mathbf{1}_{\{x \geqslant 0\}}$$

belongs to the class $\mathcal{S}$ (see, for instance, [15, p. 87]. Hence $H \in \mathcal{S}$ according to the basic properties of subexponential distribution (for details see [13, Corol. 3.13], and

$$\phi(u) \sim 38.86786\mathrm{e}^{-\log^2(1+u)},$$

as $u \to \infty$, according to Theorem 2.

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