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On the validity of use of physical equations and principles in the socio-economic field and on the predictability of socio-economic system dynamics*

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Abstract. Usually, the various empirical and semi-empirical equations and mathematical models are used to study the dynamics of socio-economic systems. In this case, there are very often questions related to the validity of application of such equations and models.

In this paper, it is shown that the dynamics of socio-economic systems can be described by mathematical equations analogous to the motion equations that are well known in physics (particularly in classical mechanics). In this connection, it is possible to say that the predictability of the dynamics of a socio-economic system is described by the motion equations (i.e., Newton's equations) to the same degree that these equations predict the dynamics of the physical system. Such models allow us to determine the trends of development in the dynamics of socio-economic systems. The importance of this investigation in using adequate mathematical models is that they allow us to notice those or other adverse trends in the socio-economic systems, therefore, ensuring timely optimal management decisions.

Keywords: motion equations, system identification, making optimal management decisions, mathematical models of socio-economic systems, predator–prey models, identification of tendencies in the socio-economic systems.

1 Introduction

The development of mathematical models is very important for the adequate description of the various processes that occur in socio-economic systems. These models identify the hypothesis about the main factors of the development of socio-economic systems,

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which are expressed particularly in terms of satisfaction by social needs. The importance of such models lies in the possibility of forecasting the final results of the evolution of various socio-economic processes [17, 24]. During the development and implementation of reform, the model allows us to take into account the problems that are most problematic in terms of the development of the social sector, which may be both the direct targets of state reform and the sources of risk. The identification of the main trends that prevail in the dynamics of the socio-economic system and that determine its future evolution is a key element in making certain management decisions. The possibility of predicting the undesirable and uncontrolled aspects of the evolution of the processes before making a decision allows the most optimal choice.

In previous years, for the description of the processes in socio-economic systems, along with traditional statistical analysis and data processing, the methods based on the use of approaches previously developed and used in physical sciences have been actively developed [1, 15, 29, 35]. The mathematical formalism for analysing different aspects of the evolution of the socio-economic system [27, 33, 36], including network methods [23], has been intensively developed. Moreover, in the studies of the dynamics of socio-economic systems, the various phenomenological approaches based on physical principles [12, 16, 22] as well as the mathematical models related to the class of so-called predator–prey models [13, 20, 34, 38, 39], have been actively developed and applied. In the listed papers, the Lotka–Volterra model is investigated. In one paper [13], the Lotka–Volterra model was used to study the evolutionary dynamics of nationalism and migration. In general, the models of population dynamics are used extensively to study various processes of socio-economic dynamics [31, 32].

In our previous studies [3, 4, 5, 6, 7, 8, 9, 10, 11, 25, 26] the mathematical models based on the predator–prey principles and intended for the investigation of the dynamics of socio-economic systems are proposed and investigated. Such models were used for the investigation of the dynamics of the socio-economic systems of Russia [3, 4, 5, 10, 11, 25, 26] and the United States [6, 7, 8, 9] at various stages of their evolution.

The predator–prey model allows us to identify the main trends of the dynamics of the socio-economic system to a certain point in time, which determines its future evolution. The main advantage of such mathematical models and methods of analysis is that they may be used to control the dynamic states of socio-economic systems. By varying model parameters that correspond to the change in impacts at various levels of governance at certain time points, we can apply the model to determine the most acceptable tendencies in the evolution of the socio-economic system in future. For example, it is possible to obtain the estimation of results to which those or other tendencies that have prevailed in the dynamics of the socio-economic system at present will lead in the future. If the expected results in the future are not desirable, then on the basis of the model, it is possible to define and develop such control actions for the socio-economic system that will initiate the appearance of new tendencies that are favourable to long-term system functioning in the desirable mode [10].

However, along with the effectiveness of the predator–prey model in the study of the dynamics of socio-economic systems, there remains the question of the reasonableness of its application in this field. This paper is devoted to the research of this problem.

2 Interrelation between predator–prey models and motion equations

The complex system to which the socio-economic system belongs consists of a number of interactions in accordance with certain laws of its constituent elements (i.e., subsystems). Let us consider a complex system S. This system can be separated into a finite number of parts S_i, $i = 1, \dots, n$, called subsystems. It should be noted that the selection of individual parts S_i is not necessarily performed uniquely. As a rule, the decomposition is caused by the aim and the research direction of specific features or steps of system evolution. Subsystem S_i, on the one hand, is a simpler part of a higher level, but on the other hand, it may be a complex system of several elements of a lower level.

At each time t the element S_i of a complex system S is located in one of the possible states $x_i(t)$. From one state to another it passes under the influence of external factors and internal patterns. Let us denote the "force" of the interaction between the subsystems S_i and S_j through F_{ij}. The dynamics of the behaviour of element S_i of complex system S is manifested in the following: the state of element S_i and the output's effects on other elements S_j and $j \neq i$ of a complex system at each time point are determined by the previous states and by the input effects (including those at the moment of time t) on the element S_i from the other elements of the complex system (Fig. 1).

Usually, the dynamics of the continuous systems with lumped parameters are described by a system of ordinary differential equations of the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}\big(\mathbf{x}^{\mathrm{T}}, \mathbf{a}, t\big). \tag{1}$$

Here $\mathbf{x} = (x_1, \dots, x_n)^{\mathrm{T}}$ is the vector of state variables of the system; t is the current time; $\mathbf{f}(\mathbf{x}^{\mathrm{T}}, \mathbf{a}, t)$ is the vector-function (generally non-linear) that characterizes the internal structure of the system; $\mathbf{a} = (a_1, \dots, a_m)$ is the vector of system parameters that is generally time-dependent.

It should be noted that the equations of system (1) are in fact motion equations if the right-hand side of system (1) is interpreted as a type of force acting on the system S

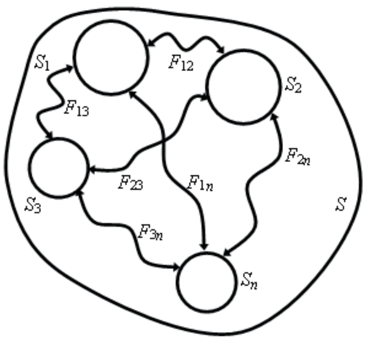

Fig. 1. The scheme of interaction between the subsystems $S_1, S_2, S_3, \dots, S_n$, belonging to the system S. By the curves conditionally the "forces" of interaction between two subsystems are labeled.

elements. In particular, if as the system elements S_i, $i = 1, \dots, n$, consider the point electric charges q_i, $i = 1, \dots, n$, then the force of interaction between two charges q_i and q_j is determined by Coulomb's law

$$F_{ij} = \frac{kq_iq_j}{R_{ij}^2}.$$

Here k is the electric constant; R_{ij} is the distance between two point electric charges q_i and q_j.

Similarly, for the system S consisting of massive point bodies in accordance with Newton's law of universal gravitation, the force of interaction between two bodies with masses m_i and m_j is expressed by the following equation:

$$F_{ij} = \frac{Gm_im_j}{R_{ij}^2}.$$

Here G is the gravitational constant; R_{ij} is the distance between two massive point bodies with masses m_i and m_j.

The state of such systems of point electric charges q_i, $i = 1, \dots, n$, which have the mass m_i or massive point bodies interacting in accordance with the law of universal gravity is determined by the motion equations (Newton's second law)

$$m_i \frac{\mathrm{d}^2\mathbf{r}_i}{\mathrm{d}t^2} = \sum_{j=1,\, j\neq i}^{n} \mathbf{F}_{ij}, \quad i = 1, \dots, n. \tag{2}$$

Here $\mathbf{r}_i$ is the radius-vector that defines the position of the ith particle; $R_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ is the distance between two massive point bodies with masses m_i and m_j. The system of the second-order differential equations (2), taking into account the substitutions of view,

$$\mathbf{Y}_{1i} = \mathbf{r}_i, \quad \mathbf{Y}_{2i} = \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}, \quad i = 1, \dots, n,$$

can be represented as a system of differential equations of the first order

$$\frac{\mathrm{d}\mathbf{Y}_{1i}}{\mathrm{d}t} = \mathbf{Y}_{2i}, \quad m_i \frac{\mathrm{d}\mathbf{Y}_{2i}}{\mathrm{d}t} = \sum_{j=1,\, j\neq i}^{n} \mathbf{F}_{ij}, \quad i = 1, \dots, n. \tag{3}$$

Taking into account that, for the particle with mass m_i, the multiplication

$$m_i \mathbf{Y}_{2i} = m_i \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} = \mathbf{p}_i$$

determines its momentum $\mathbf{p}_i$, motion equations (2) (or equations (3)) also can be written in the form of equations system

$$\frac{\mathrm{d}\mathbf{p}_i}{\mathrm{d}t} = \sum_{j=1,\, j\neq i}^{n} \mathbf{F}_{ij}, \quad i = 1, \dots, n. \tag{4}$$

Obviously, after the redesignation of some variables and parameters, the systems of ordinary differential equations (3) and (4) will take the same form as in equations system (1).

It should be noted that it is not always possible in the system to distinguish the individual elements (i.e., subsystems); the interactions are also described by such "simple" expressions, as considered in the above cases of interactions between point electric charges or point massive bodies. In the study of complex system S when the question about the laws that describe the interactions between its individual subsystems S_i, $i = 1, \dots, n$, is open, some approximation methods, such as the Taylor series expansion of the vector-function $\mathbf{F}_{ij}$ in the right-hand sides of motion equations (2)–(4), are usually applied.

Let the interaction intensity of the subsystems S_i with each other be characterized by set of variables $\mathbf{r}_i$, i.e., $\mathbf{F}_{ij} = \mathbf{F}_{ij}(\mathbf{r}_i, \mathbf{r}_j)$. The Taylor series expansion to the second order of vector-function $\mathbf{F}_{ij}(\mathbf{r}_i, \mathbf{r}_j)$ at a point with coordinates $(\mathbf{r}_i, \mathbf{r}_j) = (\mathbf{r}_{0i}, \mathbf{r}_{0j})$ takes the form of the following equation:

$$\begin{aligned} \mathbf{F}_{ij}(\mathbf{r}_i, \mathbf{r}_j) &= \mathbf{F}_{ij}(\mathbf{r}_{0i}, \mathbf{r}_{0j}) + (\mathbf{r}_i - \mathbf{r}_{0i}) \frac{\partial \mathbf{F}_{ij}(\mathbf{r}_i, \mathbf{r}_j)}{\partial \mathbf{r}_i} \bigg|_{\substack{\mathbf{r}_i = \mathbf{r}_{0i} \\ \mathbf{r}_j = \mathbf{r}_{0j}}} \\ &\quad + (\mathbf{r}_j - \mathbf{r}_{0j}) \frac{\partial \mathbf{F}_{ij}(\mathbf{r}_i, \mathbf{r}_j)}{\partial \mathbf{r}_j} \bigg|_{\substack{\mathbf{r}_i = \mathbf{r}_{0i} \\ \mathbf{r}_j = \mathbf{r}_{0j}}} + \frac{(\mathbf{r}_i - \mathbf{r}_{0i})^2}{2} \frac{\partial^2 \mathbf{F}_{ij}(\mathbf{r}_i, \mathbf{r}_j)}{\partial \mathbf{r}_i^2} \bigg|_{\substack{\mathbf{r}_i = \mathbf{r}_{0i} \\ \mathbf{r}_j = \mathbf{r}_{0j}}} \\ &\quad + (\mathbf{r}_i - \mathbf{r}_{0i})(\mathbf{r}_j - \mathbf{r}_{0j}) \frac{\partial^2 \mathbf{F}_{ij}(\mathbf{r}_i, \mathbf{r}_j)}{\partial \mathbf{r}_i \partial \mathbf{r}_j} \bigg|_{\substack{\mathbf{r}_i = \mathbf{r}_{0i} \\ \mathbf{r}_j = \mathbf{r}_{0j}}} \\ &\quad + \frac{(\mathbf{r}_j - \mathbf{r}_{0j})^2}{2} \frac{\partial^2 \mathbf{F}_{ij}(\mathbf{r}_i, \mathbf{r}_j)}{\partial \mathbf{r}_j^2} \bigg|_{\substack{\mathbf{r}_i = \mathbf{r}_{0i} \\ \mathbf{r}_j = \mathbf{r}_{0j}}} + \cdots . \end{aligned} \tag{5}$$

In equation (5), let us introduce the notation $\mathbf{z}_i = \mathbf{r}_i - \mathbf{r}_{0i}$, $i = 1, \dots, n$. Then, after the introduction of designations A_{ij}, B_{ij}, $\mathbf{C}_{ij}$, $\mathbf{D}_{ij}$ and $\mathbf{E}_{ij}$, equation (5) takes the following form:

$$\mathbf{F}_{ij}(\mathbf{z}_i, \mathbf{z}_j) = \mathbf{F}_{ij}(0,0) + A_{ij}\mathbf{z}_i + B_{ij}\mathbf{z}_j + \mathbf{C}_{ij}\mathbf{z}_i^2 + \mathbf{D}_{ij}(\mathbf{z}_i\mathbf{z}_j) + \mathbf{E}_{ij}\mathbf{z}_j^2 + \cdots . \tag{6}$$

Here

$$A_{ij} = \frac{\partial \mathbf{F}_{ij}(\mathbf{z}_i, \mathbf{z}_j)}{\partial \mathbf{z}_i} \bigg|_{\substack{\mathbf{z}_i = 0 \\ \mathbf{z}_j = 0}}, \qquad B_{ij} = \frac{\partial \mathbf{F}_{ij}(\mathbf{z}_i, \mathbf{z}_j)}{\partial \mathbf{z}_j} \bigg|_{\substack{\mathbf{z}_i = 0 \\ \mathbf{z}_j = 0}},$$

$$\mathbf{C}_{ij} = \frac{\partial^2 \mathbf{F}_{ij}(\mathbf{z}_i, \mathbf{z}_j)}{\partial \mathbf{z}_i^2} \bigg|_{\substack{\mathbf{z}_i = 0 \\ \mathbf{z}_j = 0}}, \qquad \mathbf{D}_{ij} = \frac{\partial^2 \mathbf{F}_{ij}(\mathbf{z}_i, \mathbf{z}_j)}{\partial \mathbf{z}_i \partial \mathbf{z}_j} \bigg|_{\substack{\mathbf{z}_i = 0 \\ \mathbf{z}_j = 0}},$$

$$\mathbf{E}_{ij} = \frac{\partial^2 \mathbf{F}_{ij}(\mathbf{z}_i, \mathbf{z}_j)}{\partial \mathbf{z}_j^2} \bigg|_{\substack{\mathbf{z}_i = 0 \\ \mathbf{z}_j = 0}}.$$

Thus, if we limit the terms of the Taylor series (6) to the second order, then motion equations (3) will take the following form:

$$\begin{aligned}
&\frac{\mathrm{d}\mathbf{Y}_{1i}}{\mathrm{d}t} = \mathbf{Y}_{2i},\\
&m_i\frac{\mathrm{d}\mathbf{Y}_{2i}}{\mathrm{d}t} = \sum_{j=1,\, j\neq i}^{n} \bigl(\mathbf{F}_{ij}(0,0) + A_{ij}\mathbf{z}_i + B_{ij}\mathbf{z}_j + \mathbf{C}_{ij}\mathbf{z}_i^2 + \mathbf{D}_{ij}(\mathbf{z}_i\mathbf{z}_j) + \mathbf{E}_{ij}\mathbf{z}_j^2\bigr),\\
&i = 1,\dots,n.
\end{aligned} \tag{7}$$

Or, in case of using of equations of view (4), we have following equations:

$$\begin{aligned}
&\frac{\mathrm{d}\mathbf{p}_i}{\mathrm{d}t} = \sum_{j=1,\, j\neq i}^{n} \bigl(\mathbf{F}_{ij}(0,0) + A_{ij}\mathbf{z}_i + B_{ij}\mathbf{z}_j + \mathbf{C}_{ij}\mathbf{z}_i^2 + \mathbf{D}_{ij}(\mathbf{z}_i\mathbf{z}_j) + \mathbf{E}_{ij}\mathbf{z}_j^2\bigr),\\
&i = 1,\dots,n.
\end{aligned} \tag{8}$$

In particular, if in the system of equations (7) and (8), $\mathbf{Y}_{2i} = \mathbf{z}_i$ and $\mathbf{C}_{ij} = 0$, $\mathbf{E}_{ij} = 0$, then we obtain the following system of equations:

$$\begin{aligned}
&m_i\frac{\mathrm{d}\mathbf{z}_i}{\mathrm{d}t} = \sum_{j=1,\, j\neq i}^{n} \bigl(\mathbf{F}_{ij}(0,0) + A_{ij}\mathbf{z}_i + B_{ij}\mathbf{z}_j + \mathbf{D}_{ij}(\mathbf{z}_i\mathbf{z}_j)\bigr),\\
&i = 1,\dots,n.
\end{aligned} \tag{9}$$

The mathematical model of form (9) describes the dynamics of a certain class of complex systems. In particular, when $n = 2$, it represents the classic Lotka–Volterra model (the predator–prey model), which describes the population dynamics of predators and prey in the system in which the predators eat prey. In this case, the components of the vector $\mathbf{z} = (z_1, z_2)$ represent the population size of predators (z_1) and prey (z_2) accordingly.

In the case that $n > 2$, equations (9) can also be interpreted as a model that is related to the class of predator–prey. In this case, for example, the ith element is a predator in relation to some elements of a system and the prey in relation to the other elements. It should also be noted that in equations (9), the assumptions $\mathbf{C}_{ij} = 0$ and $\mathbf{E}_{ij} = 0$ are not required in a general case.

It is probable that the approximation of the right-hand sides of motion equations (3) and (4) by the members of second-order Taylor series (5) and (6) is not sufficient for the correct description of the dynamics of complex systems. In this case, it is necessary to add to the right-hand side of motion equations (7) and (8) the members of the Taylor series of the third or higher orders. In this case, motion equations (7) and (8) have to be different from the equations of the classical Lotka–Volterra model in which in the right-hand sides of the model equations, there are members of the Taylor series up to the second order only. Thus, in the case that $\mathbf{Y}_{2i} = \mathbf{z}_i$, the system of differential equations in the vector form describing the dynamics of the socio-economic system can be written in a general form as equations

$$\mathbf{m}_i \frac{\mathrm{d}\mathbf{z}_i}{\mathrm{d}t} = \sum_{j=1,\, j\neq i}^{n} \big(\mathbf{F}_{ij}(0,0) + A_{ij}\mathbf{z}_i + B_{ij}\mathbf{z}_j + \mathbf{C}_{ij}\mathbf{z}_i^2 + \mathbf{D}_{ij}(\mathbf{z}_i\mathbf{z}_j) + \mathbf{E}_{ij}\mathbf{z}_j^2 + \cdots\big), \quad i = 1, \ldots, n. \tag{10}$$

Here the values m_i can be considered as some parameters of the system. It should also be noted that in terms of equations (9) and (10) in the classical model of predator–prey, it is reasonable to consider state variables z_1 and z_2, respectively, not as the numbers of predators and prey but as the rate of the change of their numbers.

3 Identification of the parameters of motion equations

When studying the dynamics of complex systems by means of differential equation systems (7)–(10), a problem arises to determine the unknown parameters $\mathbf{F}_{ij}(0,0)$, A_{ij}, B_{ij}, $\mathbf{C}_{ij}$, $\mathbf{D}_{ij}$ and $\mathbf{E}_{ij}$. If the dynamics of the physical system is analysed and separate elements interact with each other, particularly in accordance with the laws of universal gravity or Coulomb, then the mentioned parameters are the coefficients of expansion in the Taylor series at the points with the coordinates $(\mathbf{r}_i, \mathbf{r}_j) = (\mathbf{r}_{0i}, \mathbf{r}_{0j})$ of the following functions, respectively:

$$F_{ij} = \frac{kq_iq_j}{R_{ij}^2}, \qquad F_{ij} = \frac{Gm_im_j}{R_{ij}^2}.$$

In this case of the known masses m_i of bodies or electric charges q_i of the interacting point bodies, all coefficients of the Taylor series can be easily calculated, and the number of terms of the Taylor series is determined by the required accuracy of the calculations. However, socio-economic systems cannot be shown by formulas that describe the interaction of their individual subsystems (similar to the laws of universal gravity or Coulomb). Therefore, the studying method of dynamics of the socio-economic system is, obviously, used in which its main interacting elements, i.e., moving forces, are determined in the first stage. Such elements, depending on the research aims, can be the following factors: the consolidated budget revenues, the gross domestic product, the expenditure on science funding, the incomes of the population, the size of population, the population's welfare, the capital flight, etc. The problem of choosing the basic interacting elements of socio-economic systems was discussed in papers [3, 4, 5, 6, 7, 8, 9, 10, 11, 25, 26]. Therefore, let us write the system of differential equations in one of forms (7)–(10). For example, for the definition of the unknown coefficients $\mathbf{F}_{ij}(0,0)$, A_{ij}, B_{ij}, $\mathbf{C}_{ij}$, $\mathbf{D}_{ij}$ and $\mathbf{E}_{ij}$, the data from government statistics can be used. These coefficients are determined so that at the specified time interval $[t_0, t_F]$, the discrepancy between the data of the government statistics and the corresponding solutions of differential equations given in one of forms (7)–(10) was minimal for each of the selected main interacting elements. The Levenberg–Marquardt algorithm in the Fletcher modification [19, 21, 28, 30] is one of the most effective methods for solving of this problem.

The method of parametric identification by the Levenberg–Marquardt algorithm in the Fletcher modification consists of minimizing the sum of squared discrepancies. The discrepancies are determined by equation

$$\mathbf{r} = |\mathbf{Z} - \mathbf{Y}|. \tag{11}$$

The matrix

$$\mathbf{Y} = \begin{pmatrix} \mathbf{y}(t_1) \\ \vdots \\ \mathbf{y}(t_l) \end{pmatrix}$$

represents the experimental data obtained in the moments of time $t_i \in [t_0, t_F]$. Depending on the subject area of research, as the such values $\mathbf{y}(t_i)$ data from government statistics, expert estimates, data from physical experiments or the indicators of continued monitoring, etc., can be used. The matrix

$$\mathbf{Z} = \begin{pmatrix} \mathbf{z}(t_1, \mathbf{P}) \\ \vdots \\ \mathbf{z}(t_l, \mathbf{P}) \end{pmatrix}$$

represents the solutions of systems of differential equations, which are given in one of forms (7)–(10). These solutions are taken at the appropriate time moments $t_i \in [t_0, t_F]$. Here $P = P\{\mathbf{F}_{ij}(0,0), A_{ij}, B_{ij}, \mathbf{C}_{ij}, \mathbf{D}_{ij}, \mathbf{E}_{ij}t\}$ is the parameters vector, depending on which of the differential equations systems of forms (7)–(10) have different solutions.

Thus, the goal is to obtain the regression of the vector-function $\mathbf{z}(t, \mathbf{P})$ on vector-function $\mathbf{y}(t)$ at the specified time interval $[t_0, t_F]$. At the same time, the column vector

$$\mathbf{r} = \begin{pmatrix} \mathbf{r}(t_1, \mathbf{P}) \\ \vdots \\ \mathbf{r}(t_l, \mathbf{P}) \end{pmatrix}$$

of discrepancy (11) should be minimal. For the definition of the minimum vector of discrepancy $\mathbf{r}$, the method of least squares is usually used:

$$S = \mathbf{r}^{\mathrm{T}}\mathbf{r} \to \min. \tag{12}$$

As a result of the minimization of value S, we obtain the vector of parameters $\mathbf{P}_{\mathrm{opt}}$ for the system of differential equations given in one of forms (7)–(10), for which the solutions $\mathbf{z}_{\mathrm{opt}} = \mathbf{z}(t, \mathbf{P}_{\mathrm{opt}})$ on the specified time interval $[t_0, t_F]$ will be more accurately described the vector-function $\mathbf{y}(t)$ of the experimental data.

The necessary conditions for a minimum of function (12) take the form

$$\begin{pmatrix} \partial S(\mathbf{P})/\partial P_1 \\ \vdots \\ \partial S(\mathbf{P})/\partial P_m \end{pmatrix} = 2\mathbf{J}^{\mathrm{T}}\mathbf{r} = 2\mathbf{v} = 0. \tag{13}$$

Here P_i is the component of the parameters vector $\mathbf{P}$; m is the number of components of the vector $\mathbf{P}$; $\mathbf{v} = \mathbf{J}^{\mathrm{T}}\mathbf{r}$; $\mathbf{J}$ is the Jacobi matrix, which is determined by the following formula:

$$\mathbf{J} = \begin{pmatrix} \partial\mathbf{r}(t_1, \mathbf{P})/\partial P_1 & \dots & \partial\mathbf{r}(t_1, \mathbf{P})/\partial P_m \\ \vdots & \vdots & \vdots \\ \partial\mathbf{r}(t_l, \mathbf{P})/\partial P_1 & \dots & \partial\mathbf{r}(t_l, \mathbf{P})/\partial P_m \end{pmatrix}.$$

The vector $\mathbf{v}$ must strive to the zero vector at point $\mathbf{P}_{\mathrm{opt}}$, corresponding to the optimal parameter identification of models (7)–(10).

In the general case, the vector of parameters $\mathbf{P}_{\mathrm{opt}}$ cannot be obtained in explicit form from equation (13). It must be calculated in the iterative process by using equation

$$\mathbf{P}^{(k+1)} = \mathbf{P}^{(k)} + \Delta\mathbf{P}^{(k)}. \tag{14}$$

In the Levenberg–Marquardt method [21], the amendments $\Delta\mathbf{P}^{(k)}$ of iterative equation (14) are calculated by the following equation:

$$\big(\mathbf{A}^{(k)} + \lambda^{(k)}\mathbf{D}\big)\Delta\mathbf{P}^{(k)} = -\mathbf{v}^{(k)}. \tag{15}$$

Here $\lambda^{(k)}$ is the scaling parameter; $\mathbf{D}$ is the suitable diagonal matrix of weights, which is often equal to the unit matrix $\mathbf{I}$ or $\mathbf{D} = \mathrm{diag}(\mathbf{A}^{(k)})$; $\mathbf{A}^{(k)} = (\mathbf{J}^{\mathrm{T}})^{(k)}\mathbf{J}^{(k)}$; $\mathbf{J}^{(k)}$ is the Jacobi matrix on the kth step of iteration. The idea of algorithm (15) belongs to Levenberg [28]. Marquardt [30] improved the search strategy and introduced the diagonal matrix of weights $\mathbf{D} = \mathrm{diag}(\mathbf{A}^{(k)})$ instead of unit matrix $\mathbf{I}$. Then Fletcher [19] greatly improved the strategy of Marquardt with an adaptation of parameter λ.

Iterations are repeated for as long as the following inequality (16) will not be correct:

$$\big|S^{(k+1)} - S^{(k)}\big| < \varepsilon. \tag{16}$$

Here $S^{(k)}$ is the sum of the squares of the discrepancies after k steps of iteration.

Previously, the application software realizing the algorithm of the identification (11)–(16) of unknown parameters of the system of differential equations (7)–10, was developed in paper [6]. It is noteworthy that in the case of an arbitrary number of elements of the experimental data $\mathbf{Y}$ (i.e., in the case of an arbitrary length of time interval $[t_0, t_F]$ in which the experimental data are taken), the mathematical model of forms (7)–(10) in which the right-hand sides of the Taylor series are taken into account to only the second order may not provide the desired accuracy in the description of experimental data. In this case, it is necessary to take into account the right-hand sides of motion equations (7)–(10) with members of higher-order Taylor series. However, this increases the intensity of the computation process in the case of the identification of unknown model parameters.

If the number of elements of experimental data $\mathbf{Y}$ are not arbitrary but according to certain criteria, then in some cases, it is possible to achieve at the specified time interval $[t_0, t_F]$ a desirable agreement between vectors $\mathbf{Y}$ and $\mathbf{Z}$ in the case of the use of equations (7)–(10) with right-hand sides in which there are Taylor series with members only up to the second order. To determine this time interval $[t_0, t_F]$ in the designed application software [6], the retrospective analysis of the changes of the observed trends $\mathbf{Y}$ of the dynamics of a complex system with the help of the wavelet analysis is used. As a result of this analysis of frequency changes corresponding to the local maximums of energy in the wavelet spectrum of appropriate experimental data

$$\mathbf{Y} = \begin{pmatrix} y_1(t_0) & \dots & y_n(t_0) \\ \vdots & \vdots & \vdots \\ y_1(t_F) & \dots & y_n(t_F) \end{pmatrix}, \tag{17}$$

the choice of data is made with the necessary step of discretization for further use of the identification of parameters of motion equations, defined in one of forms (7)–(10).

It should also be noted that in equations (3) and (4) (and in equations (7)–(10)), in the case of the investigation of physical systems, all values are specified in a certain system of units, particularly in the international system of units SI. Therefore, there are no problems connected with the calibration of the time scale t. In the case of the analysis of socio-economic systems by using motion equations of one of forms (7)–(10), the problem of the calibration of the time scale exists. The search strategy of the end t_F of time interval $[t_0, t_F]$ at the kth step of iteration, which is used in our algorithm, consists of a corresponding decrease of the value t_F M times in the case of fulfilling the following mathematical inequality:

$$\left(\mathbf{Z}^{\mathrm{T}}\right)^{(k)}\mathbf{Z}^{(k)} > M^2\left(\mathbf{Y}^{\mathrm{T}}\right)^{(k)}\mathbf{Y}^{(k)}.$$

Here $\mathbf{Y}^{(k)}$ is the experimental data obtained as a result of processing by using a wavelet analysis (see formula (17)) at the kth step of iteration; $\mathbf{Z}^{(k)}$ are the solutions of the one of equations systems (7)–(10) at the kth step of iteration, which are computed at the same points of the interval $[t_0, t_F]$ and the corresponding elements of the matrix (17); M is the restrictive parameter that forms the search area $[t_0, t_F]$ of the solutions of motion equations. This parameter M has been introduced because of the poorly formalized constraints of the parameters $\mathbf{P}$ in conditions in uncertainty of the time intervals of the solutions of differential equations systems (7)–(10).

4 Investigation of the dynamics of the Russian socio-economic system

In the predator–prey model, the structure of inter-element interactions of which is shown in Fig. 2, were chosen as key elements that determine the dynamics of the socio-economic system in total: X_1, denoting the incomes of consolidated budget x_1; X_2, denoting the Gross domestic product (GDP) x_2; X_3, denoting the cost of funding for science x_3; X_4, denoting the incomes x_4 of the population; X_5, denoting the outflow of capital x_5. The choice of this set of interacting elements $X_1, \dots, X_5$ is reasonable because they characterize and determine the pace of the socio-economic development of the country as a whole as well as the level of the welfare of the population.

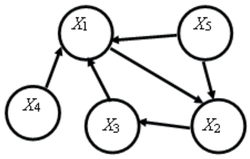

Fig. 2. The structure of elements interaction of the socio-economic system in the predator–prey model.

In accordance with the interaction structure between predators and prey represented in Fig. 2, the mathematical model can be written as the following equation system:

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= \alpha_1 x_1 x_2 - \alpha_2 x_1 x_3 - \alpha_3 x_1 x_4 - \alpha_4 x_1 x_5 - \alpha_5 x_1,\\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= -\beta_1 x_1 x_2 + \beta_2 x_2 x_3 - \beta_3 x_2 x_5 - \beta_4 x_2,\\
\frac{\mathrm{d}x_3}{\mathrm{d}t} &= \gamma_1 x_1 x_3 - \gamma_2 x_2 x_3 - \gamma_3 x_3,\\
\frac{\mathrm{d}x_4}{\mathrm{d}t} &= \delta_1 x_1 x_4 - \delta_2 x_4,\\
\frac{\mathrm{d}x_5}{\mathrm{d}t} &= \varphi_1 x_1 x_5 + \varphi_2 x_2 x_5 - \varphi_3 x_5.
\end{aligned} \tag{18}$$

In equations (18), the coefficients α_i, β_i, γ_i, δ_i and φ_i in a general case can depend on time t. Let us assume that these coefficients are constants in the investigated time interval $[t_0, t_F]$.

Summarizing the results of the studies [10, 11], it can be argued that the predator–prey model (18) adequately describes the dynamics of the Russian socio-economic system. For example, from the study performed in paper [11], it follows that model (18) correctly describes the socio-economic situation in the second half of the 1990s and the default that occurred in Russia in 1998. In addition, from the analysis performed in papers [10, 11], it follows that the crisis phenomena after the year 2000 increased, especially in 2004–2005 and 2011–2012. These dates correlate with the years of presidential elections in Russia (2004, 2008 and 2012). Apropos of 2008, it can be argued that the elections took place with the background of the global financial crisis and a local crisis in Russia in 2008–2009 that coincided with the global crisis. The results obtained in paper [10] show that in maintaining the trends that have taken place in the Russian socio-economic system up to the end of 2011, in the near future it will be particularly difficult in 2016–2017. It should be noted that in paper [10], the identification of the parameters of model (18) in accordance with algorithms (11)–(17) was performed with the use of data of government statistics [2, 18] from 2000 to 2011, inclusively.

Let us analyse the dynamics of the Russian socio-economic system on the basis of model (18) with the use of data from government statistics for the elements $X_1, \dots, X_5$ from 2005 to 2012 [2, 18]. The initial point of 2005 was studied in detail in paper [10]. During the identification of the unknown parameters of model (18), minimizing the value S (see formula (12)), let us determine the optimal parameters vector $\mathbf{P}_{\mathrm{opt}}$ with which the solutions $x_i(t)$, $i = 1, 2, \dots, 5$, of model (18) most accurately describe quantitatively and qualitatively the relevant data of government statistics in the analysed period of time. The identification of the parameters of differential equations system (18) gives the following their optimal values (19) in the time interval from 2005 to 2012:

$$\begin{aligned}
&\alpha_1 = 0.0098, && \alpha_2 = 0.00158, && \alpha_3 = -5.8111 \cdot 10^{-4},\\
&\alpha_4 = 0.0021, && \alpha_5 = 0.09122, && \beta_1 = -8.0666 \cdot 10^{-4},\\
&\beta_2 = -3.7113 \cdot 10^{-4}, && \beta_3 = 3.8624 \cdot 10^{-4}, && \beta_4 = -0.01947,\\
&\gamma_1 = 0.002517, && \gamma_2 = 9.4216 \cdot 10^{-4}, && \gamma_3 = -0.07443,\\
&\delta_1 = -3.8554 \cdot 10^{-4}, && \delta_2 = -0.0935, && \varphi_1 = 0.00857,\\
&\varphi_2 = -0.00232, && \varphi_3 = 0.259.
\end{aligned} \tag{19}$$

The dependencies $x_i(t)$, $i = 1, 2, \dots, 5$, obtained by the numerical solution of system of differential equations (18) with the coefficients (19) are represented in Fig. 3 (the dashed curves). The solid curves in Fig. 3 correspond to the data from government statistics from 2005 to 2012. The initial conditions x_i ($t_0 = 2005$ year), $i = 1, 2, \dots, 5$, were set equal to the corresponding data of the statistics in 2005.

The dotted curves in Fig. 3 correspond to the solutions $x_i(t_0)$, $i = 1, 2, \dots, 5$, of the system of differential equations (18) in the case, when for the identification of unknown parameters vector $\mathbf{P}$ the data from government statistics from 2005 to 2011, inclusively, were applied. The initial conditions, as in the previous case, were set equal to the corresponding data of statistics in 2005. In this case, the optimal values of coefficients of model (18) are as follows:

$$\begin{aligned}
&\alpha_1 = 0.001, && \alpha_2 = 0.0015, && \alpha_3 = -5.1099 \cdot 10^{-4},\\
&\alpha_4 = 0.002, && \alpha_5 = 0.0913, && \beta_1 = -8.2943 \cdot 10^{-4},\\
&\beta_2 = -4.601 \cdot 10^{-4}, && \beta_3 = 4.0486 \cdot 10^{-4}, && \beta_4 = -0.0193,\\
&\gamma_1 = 0.0024, && \gamma_2 = 9.6204 \cdot 10^{-4}, && \gamma_3 = -0.0744,\\
&\delta_1 = -4.0673 \cdot 10^{-4}, && \delta_2 = -0.0937, && \varphi_1 = 0.0091,\\
&\varphi_2 = -0.0027, && \varphi_3 = 0.2587.
\end{aligned} \tag{20}$$

The dashed curves in Fig. 3, which are extended outside the 2012 year, show the dynamics of the Russian socio-economic system after 2012 in terms of the elements $X_1, \dots, X_5$ in the case that the same trends that were established in the system at the end of 2012 are retained. Similarly, the dotted curves in Fig. 3 show the dynamics of the Russian socio-economic system in the case that the trends that were established in the system by the end of 2011 remain unchanged in the future.

The comparison of the sets of parameters (19) and (20) of model (18) shows that most significantly (approximately 10 times) only coefficient α_1 has changed, while other parameters have changed slightly. However, as a result, the solutions of model (18) have qualitatively changed their patterns in the long-term. The parameter α_1 is contained in the first equation of model (18) as a multiplier in a member of the $\alpha_1 x_1 x_2$ type. The

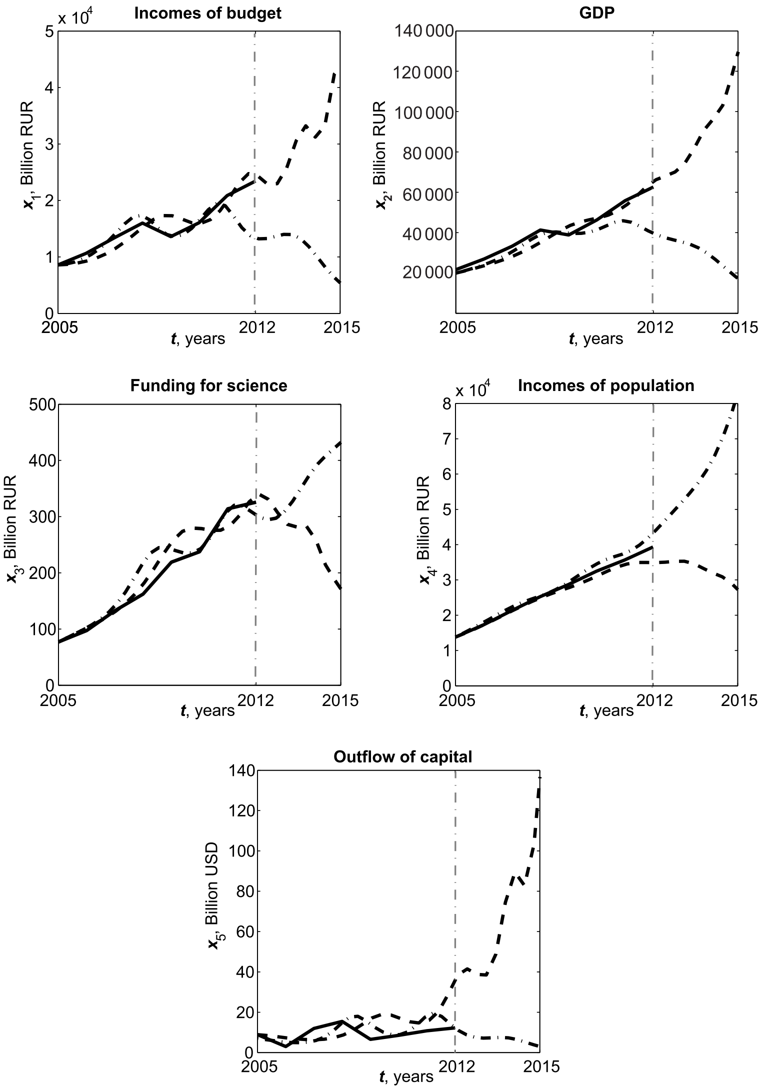

Fig. 3. The simulation results. The solid curves correspond to the data from government statistics from 2005 to 2012. The dashed curves correspond to the solution of model (18) with coefficients (19). The dotted curves correspond to the solution of model (18) with coefficients (20). Here RUR are Russian Rubles.

function $x_1(t)$ describes the dynamics of the incomes of consolidated budgets, and the function $x_2(t)$ describes the dynamics of GDP. Therefore, it is possible to conclude that after the presidential elections in Russia in 2012, the change of the socio-economic course implemented in the country has moved in the direction of the dynamic correction of the incomes of consolidated budgets and GDP. Thus, in the case of the dotted curves in Fig. 3, the decrease after 2011 of the incomes of consolidated budgets (the function x_1)

and GDP (the function x_2) and the simultaneous increase of budget spending on science funding (the function x_3) and incomes of the population (the function x_4 correspond to the occurrence of high inflation and, consequently, the printing of money. Policy revisions in the socio-economic field in 2012 likely consisted of the fact that the continued growth of the incomes of consolidated budgets and GDP (see the dashed curves in Fig. 3) will occur in the next few years. However, in this case, the tendencies of the decrease of budget spending on science funding and of the incomes of the population are observed (see the dashed curves $x_3(t)$ and $x_4(t)$ in Fig. 3 accordingly). At the same time, in the case of the conservation of established trends, the capital outflow increases after 2012 (the dashed curve $x_5(t)$ in Fig. 3). This means that in the country in the coming years, the policy of raising incomes of consolidated budgets and GDP will be implemented without investing sufficient funds in the development of the socio-economic field. Such trends in the Russian socio-economic system that emerged by the end of 2012, will lead to a highly unstable situation in the country in the coming years.

At the same time, it was noted in paper [10] that in the case of the further development of the socio-economic system of Russia in accordance with the trends that emerged at the end of 2011, it will come to the so-called blow-up regime in 2017. The blow-up regime is understood to occur when the behaviour of one or several functions $x_i(t)$ of the system state begin to grow uncontrollably during the small time interval Δt [14]. Obviously, the real systems do not have resources for a sustainable existence in the blow-up regime. The analysis of the dashed curves in Fig. 3 shows that the trend that will lead to a system under the blow-up regime, which was observed when using the data from government statistics from 2005 to 2011 (the dotted curves in Fig. 3) for the identification of model parameters, has not changed. Possibly, the trends that emerged at the end of 2012 year will lead to the failure of the socio-economic system of Russia in the blow-up regime just before 2017. This fact by rapid growth of functions $x_2(t)$ and $x_5(t)$ is shown (see the dashed curves in Fig. 3).

Thus, it is possible to conclude that in Russia, both short-term and long-term plans of socio-economic development are absent. The state's socio-economic policy will be carried out from the most recent presidential elections until the next.

It should be noted that the divergence between the statistical data and the simulation results of equations (18) is large (see Fig. 3). To provide greater accuracy, it is necessary to include terms from the Taylor series of higher orders in the right-hand sides of the equations of system (18). This leads to an increase in the complexity of solving the identification problem of the model parameters (see equations (11)–(17)). However, for the identification of key trends of the dynamic evolution of socio-economic systems, it is sufficient to use the models that are similar to equation system (18) in the right-hand sides of the equations, of which there are members of the Taylor series up to second order.

5 Conclusions

The models of the predator–prey class are widely used for research on socio-economic systems. In addition to the papers listed above, which are devoted to the predator–prey models, paper [37] should be mentioned. This research [37] applies the Verhulst-Lotka-Volterra (VLV) model to a social system in which individuals can choose how to follow different ideologies or even be ideology-free individuals. Agents are influenced by binary conversion (interpersonal contact) and unitary conversion (media influence).

The main aim of this study was to analyse the interrelations between the processes in socio-economic systems and the physical processes that are described by motion equations. As a result of this research, it is shown that the dynamics of socio-economic systems can be described by the mathematical equations that are analogues of the motion equations (or Newton's second law) that are well known in physics (particularly in classical mechanics). Therefore, it is possible to confirm that the dynamics of socio-economic systems are predictable to the same extent to which the dynamics of the physical system described by the motion equations is predictable.

The models investigated allow us to determine the trends that are developed regarding the dynamics of the socio-economic system. We need hardly mention that the modelling and analysis of the trends that take place in socio-economic systems are extremely important for the timely assessment of the effectiveness of those or other accepted administrative decisions. It is obvious that the acceptance of effective administrative decisions can help them be passed with minimal losses in crisis stages of the development of socio-economic systems. Research on the basis of using adequate mathematical models is important because it allows for the discovery of these or other undesirable trends in the socio-economic system in advance. This allows the timely acceptance of effective administrative decisions.

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