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Exact travelling wave solutions for the modified Novikov equation*

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Abstract. In this paper, by using the factorization technique and the direct integral method, some exact travelling wave solutions of the modified Novikov equation are obtained. Moreover, our results extend previously known results in the literature.

Keywords: modified Novikov equation, factorization technique, travelling wave solutions.

1 Introduction

Recently, Vladimir Novikov [9] presented a new integrable partial differential equation (called Novikov equation)

$$u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx}, \tag{1}$$

which was discovered in a symmetry classification of nonlocal PDEs with quadratic or cubic nonlinearity. By using the perturbation symmetry approach [7], Novikov found the first few symmetries and a scalar Lax pair for Eq. (1), then proved that it is integrable [9]. Hone and Wang [5] gave a matrix Lax pair for the Novikov equation and found its infinitely many conserved quantities, as well as a bi-Hamiltonian structure. Then using the matrix Lax pair found by Hone and Wang [5], Hone, Lundmark and Szmigielski [4] obtained the explicit formulas for multipeakon solutions of Eq. (1). For other studies concerned with blow-up phenomenon, Cauchy problem of Eq. (1), we refer the reader to see [3, 6, 8, 10, 13, 14].

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More recently, Zhao and Zhou [15] proposed the following modified Novikov equation:

$$u_t - u_{xxt} + 4u^4 u_x = 3uu_x u_{xx} + u^2 u_{xxx}. \tag{2}$$

By using the extended-$\tanh$ function method [1,2] and the homogeneous balance method [11], some exact travelling wave solutions to Eq. (2) are established. However, it should be noted that when the extended-$\tanh$ function method and the homogeneous balance method are applied to the modified Novikov equation (2), both need resort to computer symbol system (for example, Maple and Mathematica). In this paper, we shall investigate and search for the travelling wave solutions of the modified Novikov equation (2) by using the factorization technique proposed by Wang and Li [12]. Namely, we firstly use the factorization technique to simplify Eq. (2) into the first-order ordinary differential equations, and then solve it by direct integral method.

The rest of this paper is organized as follows. In Section 2, we introduce the main idea of the factorization technique and its applications to the reduction of Eq. (2). Some exact travelling wave solutions for Eq. (2) are obtained through direct integral in Section 3. A brief discussion is provided in Section 4.

2 The factorization technique and its application to Eq. (2)

Wang and Li [12] proposed the following factorization technique, which establishes the conditions concerning the factorization of a third-order nonlinear ordinary differential equation (ODE).

Proposition 1. *Given a nonlinear ODE of third-order*

$$f(U)U''' + g(U,U')U'' + h(U)U' + k(U) = 0. \tag{3}$$

Equation (3) *owns the factorization*

$$\big[f(U)\partial_\xi - \phi_1(U)U' - \phi_2(U)\big]\big[\partial_{\xi\xi} - \phi_3(U)\partial_\xi - \phi_4(U)\big]U = 0$$

if and only if the following expressions hold:

$$\begin{aligned}
&g(U,U') = -f(U)\phi_3(U) - \phi_1(U)U' - \phi_2(U),\\
&\phi_1(U)\phi_3(U) - f(U)\frac{\mathrm{d}\phi_3(U)}{\mathrm{d}U} = 0,\\
&k(U) = \phi_2(U)\phi_4(U)U,\\
&h(U) = \phi_2(U)\phi_3(U) - f(U)\frac{\mathrm{d}\phi_4(U)}{\mathrm{d}U}U - f(U)\phi_4(U) + \phi_1(U)\phi_4(U)U.
\end{aligned}$$

Here U' denotes the derivative of U about ξ.

Taking travelling wave transformation

$$u(x,t) = U(\xi), \quad \xi = x - ct,$$

and substituting it into Eq. (2) leads to the following third-order ODE:

$$\big(U^2 - c\big)U''' + 3UU'U'' - \big(4U^4 - c\big)U' = 0. \tag{4}$$

Comparing it with Eq. (3) yields

$$f(U) = U^2 - c, \qquad g(U, U') = 3UU', \qquad h(U) = c - 4U^4, \qquad k(U) = 0.$$

According to the factorization technique described in Proposition 1, we have

$$\begin{aligned}
&3UU' = \big(c - U^2\big)\phi_3(U) - \phi_1(U)U' - \phi_2(U),\\
&\phi_1(U)\phi_3(U) + \big(c - U^2\big)\frac{\mathrm{d}\phi_3(U)}{\mathrm{d}U} = 0,\\
&\phi_2(U)\phi_4(U)U = 0,\\
&c - 4U^4 = \phi_2(U)\phi_3(U) + \big(c - U^2\big)\frac{\mathrm{d}\phi_4(U)}{\mathrm{d}U}U + \big(c - U^2\big)\phi_4(U) + \phi_1(U)\phi_4(U)U,
\end{aligned}$$

which has the following nontrivial solution:

$$\phi_1(U) = -3U, \qquad \phi_2(U) = \phi_3(U) = 0, \qquad \phi_4(U) = \frac{2}{3}U^2 + 1, \qquad c = 2.$$

So, Eq. (4) has the following factorization:

$$\big[(U^2 - 2)\partial_\xi + 3UU'\big]\left[\partial_{\xi\xi} - \left(\frac{2}{3}U^2 + 1\right)\right]U = 0. \tag{5}$$

It is easily seen that some special solutions of (5) could be obtained by solving the following second-order differential equation

$$U'' = U\left(\frac{2}{3}U^2 + 1\right). \tag{6}$$

Integrating Eq. (6), we obtain

$$U'^2 = \frac{1}{3}\big(U^4 + 3U^2 + h\big), \tag{7}$$

where h is an arbitrary integral constant.

3 Some exact travelling wave solutions for Eq. (2)

In this section, we will proceed to derive some exact travelling wave solutions for Eq. (2). Obviously, it is easy to see that the problem of finding particular solutions to Eq. (2) can be addressed by solving the reduced Eq. (7).

Making transformation $\phi = U^2 \geqslant 0$ and submitting it into (7) yields

$$\phi'^2 = \frac{4}{3}\phi\big(\phi^2 + 3\phi + h\big). \tag{8}$$

Denote that $F(\phi) = \phi^2 + 3\phi + h$, whose complete discriminant is given by

$$\Delta = 9 - 4h.$$

According to the signs of Δ, we distinguish three cases to discuss the travelling wave solutions for Eq. (2).

Case 1. $\Delta = 0$ (namely $h = 9/4$). In this case, we have $F(\phi) = (\phi + 3/2)^2$. From (8) we get

$$\pm\frac{2}{\sqrt{3}}(\xi - \xi_0) = \frac{4}{\sqrt{6}} \arctan\sqrt{\frac{2}{3}\phi},$$

which yields

$$\phi_1(\xi) = \frac{3}{2}\tan^2\frac{\sqrt{2}}{2}(\xi - \xi_0),$$

where ξ_0 is an arbitrary integral constant. So, Eq. (2) admits triangle function forms of travelling wave solutions as follows:

$$u_1(x,t) = \pm\frac{\sqrt{6}}{2}\tan\frac{\sqrt{2}}{2}(x - 2t - \xi_0). \tag{9}$$

Case 2. $\Delta > 0$ (namely $h < 9/4$). In this case, we have $F(\phi) = (\phi - \alpha)(\phi - \beta)$, where

$$\alpha = \frac{-3 + \sqrt{9 - 4h}}{2}, \qquad \beta = \frac{-3 - \sqrt{9 - 4h}}{2}. \tag{10}$$

If $h = 0$, then we have $\alpha = 0$, $\beta = -3$, and so (8) becomes

$$\phi'^2 = \frac{4}{3}\phi^2(\phi + 3). \tag{11}$$

Integrating (11) directly leads to

$$\pm\frac{2}{\sqrt{3}}(\xi - \xi_0) = \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{\phi + 3} - \sqrt{3}}{\sqrt{\phi + 3} + \sqrt{3}}\right|. \tag{12}$$

Note that $\phi \geqslant 0$, then consequently from (12) we can obtain

$$\phi_2(\xi) = -3 + 3\coth^2(\xi - \xi_0) = 3\operatorname{csch}^2(\xi - \xi_0),$$

where ξ_0 is an arbitrary integral constant. This implies that Eq. (2) admits the following singular travelling wave solutions:

$$u_2(x,t) = \pm\sqrt{3}\operatorname{csch}(x - 2t - \xi_0). \tag{13}$$

If $h < 0$, then from (10) we have $\beta < 0 < \alpha$. Making variable transformation

$$\phi = \alpha\sec^2\theta \tag{14}$$

and submitting it into (8), we have

$$\pm\frac{2}{\sqrt{3}}(\xi-\xi_0)=\int\frac{2\alpha\sec^2\theta\tan\theta\,\mathrm{d}\theta}{\alpha\sec\theta\tan\theta\sqrt{\alpha-\beta+\alpha\tan^2\theta}},$$

which can be reduced to

$$\pm\sqrt{\frac{\alpha-\beta}{3}}(\xi-\xi_0)=\int\frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}},\tag{15}$$

where $k^2=\beta/(\beta-\alpha)$. Recall the definitions of Jacobi elliptic sine or cosine function, we can obtain from (15) and (14) that

$$\phi_3(\xi)=\alpha cn^{-2}\left(\sqrt{\frac{\alpha-\beta}{3}}(\xi-\xi_0),k\right),$$

where ξ_0 is an arbitrary integral constant. Therefore, Eq. (2) admits the following singular periodic wave solutions:

$$u_3(x,t)=\pm\sqrt{\alpha}cn^{-1}\left(\sqrt{\frac{\alpha-\beta}{3}}(x-2t-\xi_0),k\right).\tag{16}$$

If $0<h<9/4$, then from (10) we have $\beta<\alpha<0$. Making variable transformation

$$\phi=-\alpha\tan^2\theta.$$

Implementing similar arguments as above, one can get

$$\phi_4(\xi)=-\alpha\frac{sn^2(\sqrt{(\alpha-\beta)/3}(\xi-\xi_0),k)}{cn^2(\sqrt{(\alpha-\beta)/3}(\xi-\xi_0),k)},$$

where $k^2=(\beta-\alpha)/\alpha$. Thus, it implies that Eq. (2) still admits the singular periodic wave solutions as

$$u_4(x,t)=\pm\frac{\sqrt{-\alpha}sn(\sqrt{(\alpha-\beta)/3}(x-2t-\xi_0),k)}{cn(\sqrt{(\alpha-\beta)/3}(x-2t-\xi_0),k)}.$$

Case 3. $\Delta<0$ (that is, $h>9/4$). In this case, we can take another change of the variable

$$\phi=\sqrt{h}\tan^2\frac{\theta}{2}.\tag{17}$$

Then from (8) we have

$$\pm\frac{2}{\sqrt{3}}(\xi-\xi_0)=\int\frac{\sqrt{h}\sec^2(\theta/2)\tan(\theta/2)\,\mathrm{d}\theta}{h^{3/4}\tan(\theta/2)\sqrt{\tan^4(\theta/2)+3h^{-1/2}\tan^2(\theta/2)+1}},\tag{18}$$

which can be simplified as

$$\pm\sqrt{\frac{4}{3}h^{1/2}}(\xi-\xi_0)=\int\frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}},\tag{19}$$

where $k^2=1/2-3/(4\sqrt{h})$. From (19) and (17) we can obtain

$$\phi_5(\xi)=h^{1/2}\frac{1-cn(\sqrt{(4/3)h^{1/2}}(\xi-\xi_0),k)}{1+cn(\sqrt{(4/3)h^{1/2}}(\xi-\xi_0),k},$$

where ξ_0 is an arbitrary integral constant. Therefore, Eq. (2) admits the following elliptic periodic wave solutions

$$u_5(x,t)=\pm h^{1/4}\left(\frac{1-cn(\sqrt{(4/3)h^{1/2}}(x-2t-\xi_0),k)}{1+cn(\sqrt{(4/3)h^{1/2}}(x-2t-\xi_0),k)}\right)^{1/2}.$$

Remark 1. Since $\tan^2(x+\pi/2)=\cot^2 x$, then from (9) we can find that the functions

$$u_1^*(x,t)=\pm\frac{\sqrt{6}}{2}\cot\frac{\sqrt{2}}{2}(x-2t-\xi_0)\tag{20}$$

are also the exact solutions of Eq. (2). Note that $\sec^2 x-1=\tan^2 x, \csc^2 x-1=\cot^2 x$, it is easy to see that the above solutions (9) and (20) are in agreement with the solutions (42) (or (16)), (40) obtained in [15], respectively. Furthermore, solution (13) agrees well with solutions (18) and (19) described in [15]. However, to the best of our knowledge, the solutions just like $u_4(x,t)$ and $u_5(x,t)$ are new and could not be found in [15].

Remark 2. By comparing with the method used in [15], which requires the aid of computer symbolic computation, the advantages of our method are very simple and direct. Thus, it also reminds us that before searching for exact solutions of various nonlinear wave equations by using symbolic computation we should remember it will be better to use the direct method.

4 Conclusions

In summary, we investigate and obtain some exact travelling wave solutions of the modified Novikov equation by applying the factorization technique and direct integral method. It is shown that our results are new and extend some previously known results in the literature. However, it should also be noted that we have only obtained some particular travelling wave solutions to the modified Novikov equation. As for more general exact solutions, we believe that it can be proceeded by relaxing some restrictions mentioned in the present work. These are being pursued currently.

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