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New synchronization criteria for an array of neural networks with hybrid coupling and time-varying delays*

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Abstract. This paper is concerned with the global exponential synchronization for an array of hybrid coupled neural networks with time-varying leakage delay, discrete and distributed delays. Applying a novel Lyapunov functional and the property of outer coupling matrices of the neural networks, sufficient conditions are obtained for the global exponential synchronization of the system. The derived synchronization criteria are closely related with the time-varying delays and the coupling structure of the networks. The maximal allowable upper bounds of the time-varying delays can be obtained guaranteeing the global synchronization for the neural networks. The method we adopt in this paper is different from the commonly used linear matrix inequality (LMI) technique, and our synchronization conditions are new, which are easy to check in comparison with the previously reported LMI-based ones. Some examples are given to show the effectiveness of the obtained theoretical results.

Keywords: array of neural networks, synchronization, hybrid coupling, time-varying delays.

1 Introduction

Since small-world and scale-free complex networks were proposed in [6] and [43], complex dynamical networks, which are a set of interconnected nodes with specific dynamics, have received increasing attention from various fields of science and engineering such as, for example, the World Wide Web, electrical power grids, communication networks, the Internet, global economic markets, and so on. Many interesting behaviors have been observed from complex dynamical networks, e.g., synchronization, consensus, self-organization, and spatiotemporal chaos spiral waves. Synchronization, as an important collective behavior of complex dynamical networks, has been widely investigated in the last two decades (see, for example, [3, 7, 10, 26, 27, 32, 34, 35, 49, 51]).

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Due to the complexity of neural networks, coupling of neurons should be taken into account. Coupled neural networks (CNNs), as a special kind of complex networks, have been found to exhibit more complicated and unpredictable behaviors than a single neural network. Particularly, synchronization in an array of CNNs, which is one of the hot research fields of complex networks, has been a challenging issue due to its potential applications in many areas such as secure communication, information science, chaos generator design, and harmonic oscillation generation. On the other hand, in the applications of neural networks, there exist unavoidably time delays due to the finite information processing speed and the finite switching speed of amplifiers. It is well known that time delay can cause oscillation and instability of neural networks. Therefore, various synchronization criteria for CNNs with time delays have been investigated in the literature [8, 9, 11, 12, 15, 19, 25, 28, 30, 31, 33, 36, 37, 38, 39, 40, 42, 45, 46, 47, 48, 50, 51, 52, 53] and references therein. To mention a few representative works, the synchronization problems for an array of neural networks with hybrid coupling and constant delay or interval time-varying delay were investigated in [8, 50]. In [31], the global exponential synchronization was investigated for an array of asymmetric neural networks with time-varying delays and nonlinear coupling. Cao and Li [9] presented cluster synchronization criteria for an array of hybrid coupled neural networks with delay. Park et al. [30] obtained delay-dependent synchronization conditions for coupled discrete-time neural networks with interval time-varying delays in network coupling.

So far, very little attention has been paid to neural networks with time delay in leakage (or "forgetting") term (see [1,2,4,5,12,13,14,20,21,28,29]). This is due to some theoretical and technical difficulties. In fact, in electronic implementation of neural networks, the self-decay process of neurons is not instantaneous. When a neuron disconnects from the neural network and external inputs, it takes time to reset electrical potential to stationary state. In order to describe the phenomena, Gopalsamy [16] studied the neural network model with time delay in the stabilizing negative feedback term (i.e., leakage term), and found that time delay in the leakage term has a tendency to destabilize a system. On the other hand, it has been observed that neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths. It is desired to model them by introducing continuously distributed delays over a certain duration of time such that the distant past has less influence compared with the recent behavior of the state (see [25, 39]). However, to the best of our knowledge, there are no global exponential synchronization results about an array of hybrid coupled neural networks with time-varying leakage delay, discrete and distributed delays.

In previous works, most of the synchronization criteria have been derived based on the LMI method (see [8, 9, 12, 25, 28, 30, 31, 36, 39, 40, 41, 45, 46, 47, 50, 52]). Although interesting from the theoretical viewpoint, the LMI method would bring some slack variables and the presence of too many slack variables increases computation burden and restricts applications of the synchronization conditions. In addition, the synchronization criteria for CNNs based on the LMI method are usually composed of C_N^2 LMIs (see [12, 31, 50]), where N is the number of coupled nodes, and C_N^2 is the combinatorial number. Thus, it involves a large number of calculations. In order to avoid the heavy workload to verify the synchronization conditions based on LMI method, in this paper, another effective method will be used to discuss synchronization of CNNs with time delays. This approach is based on the work of Hajnal [17], in which scrambling matrix was proposed to tackle the convergence of products of stochastic matrices. Similar method has also been used to study consensus problem in continuous time networks (see [18, 23]) and synchronization problem (see [24, 44]). Whereas, few people extended this approach to synchronization of complex neural networks with mixed time delays. Motivated by the works of Zhang et al. [50], Park et al. [28], Liu et al. [24], and the discussions above, the purpose of this paper is to adopt a novel method which applies the property of outer coupling matrices of the neural networks, and present some new sufficient conditions for the global exponential synchronization of hybrid coupled neural networks with time-varying mixed delays. To this end, we consider the following differential equation system:

$$\begin{aligned}\dot{x}_i(t) &= -Ax_i\big(t-\tau(t)\big) + W_1 f\big(x_i(t)\big) + W_2 f\big(x_i\big(t-h(t)\big)\big) \\ &\quad + W_3 \int\limits_{t-\sigma(t)}^{t} f\big(x_i(s)\big)\,\mathrm{d}s + u(t) + \alpha_1 \sum_{j=1}^{N} g_{ij}^{(1)} \varGamma_1 x_j(t) \\ &\quad + \alpha_2 \sum_{j=1}^{N} g_{ij}^{(2)} \varGamma_2 x_j\big(t-h(t)\big) + \alpha_3 \sum_{j=1}^{N} g_{ij}^{(3)} \varGamma_3 \int\limits_{t-\sigma(t)}^{t} x_j(s)\,\mathrm{d}s, \end{aligned} \tag{1}$$

where $i = 1, 2, \ldots, N$, N is the number of coupled nodes, $x_i(t) = (x_i^1(t), x_i^2(t), \ldots, x_i^n(t))^{\mathrm{T}} \in \mathbb{R}^n$ is the neuron state vector of the ith node, n denotes the number of neurons in a neural network, $f(x_i(\cdot)) = (f_1(x_i^1(\cdot)), f_2(x_i^2(\cdot)), \ldots, f_n(x_i^n(\cdot)))^{\mathrm{T}} \in \mathbb{R}^n$ and $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^{\mathrm{T}} \in \mathbb{R}^n$ denote the neuron activation function vector and the external input vector, respectively, $\tau(t)$, $h(t)$ and $\sigma(t)$ denote the time-varying leakage delay, discrete delay and distributed delay, respectively, $A = \mathrm{diag}\{a_1, a_2, \ldots, a_n\} > 0$ is the self-feedback matrix, $W_k = (w_{ij}^{(k)}) \in \mathbb{R}^{n\times n}$ $(k = 1, 2, 3)$ are the connection weight matrices, $\alpha_k > 0$ $(k = 1, 2, 3)$ are the coupling strengths, $\varGamma_k = \mathrm{diag}\{\gamma_1^{(k)}, \gamma_2^{(k)}, \ldots, \gamma_n^{(k)}\} > 0$ $(k = 1, 2, 3)$ are the constant inner coupling matrices of nodes, which describe the individual coupling between networks, $G_k = (g_{ij}^{(k)})_{N\times N}$ $(k = 1, 2, 3)$ are the outer coupling matrices representing the coupling strength and the topological structure of the networks.

Our control goal is to let system (1) achieve synchronization, that is, $\lim_{t\to\infty} |x_i^k(t) - x_j^k(t)| = 0$ for all $i, j \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, n\}$.

The organization of this paper is as follows. In Section 2, some preliminaries are given. In Section 3, new synchronization criteria are derived for system (1). In Section 4, some numerical examples are provided to illustrate the effectiveness of the obtained theoretical results. A brief remark is given in Section 5 to conclude this work.

2 Preliminaries

For convenience, we introduce several notations. $A > 0$ means that A is a symmetric positive definite matrix. I_n represents the nth-order identity matrix. For a vector

$x = (x_1, x_2, \dots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, $\|x\| = \max_{1\leqslant i\leqslant n} |x_i|$. For $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $\|A\| = \max_{1\leqslant i\leqslant n}\{\sum_{j=1}^{n} |a_{ij}|\}$.

Throughout this paper, we make the following assumptions.

(H1) The time delays satisfy that

$$0 \leqslant \tau(t) \leqslant \tau, \qquad 0 \leqslant h(t) \leqslant h, \qquad 0 \leqslant \sigma(t) \leqslant \sigma,$$

where τ, h and σ are known constants.

(H2) The outer coupling matrices G_k ($k = 1, 2, 3$) satisfy the diffusive coupling connections:

$$g_{ij}^{(k)} \geqslant 0 \quad (i \neq j), \qquad g_{ii}^{(k)} = -\sum_{\substack{j=1,\\ j\neq i}}^{N} g_{ij}^{(k)} \quad (i, j = 1, 2, \dots, N).$$

(H3) $f(x)$ is Lipschitz continuous with the Lipschitz constant $L > 0$, that is,

$$\big\|f(x) - f(y)\big\| \leqslant L\|x - y\|.$$

Next, we give a useful lemma.

Lemma 1. (*See* [22].) *Assume that $V(t)$ is a nonnegtive continuous function on $[t^*, t]$, and satisfies the inequality*

$$D^+\big(V(t)\big) \leqslant -aV(t) + b\bar{V}(t),$$

where $a > b > 0$ and $\bar{V}(t) = \sup_{t-\tau\leqslant s\leqslant t} V(s)$, $\tau > 0$ is a constant. Then

$$V(t) \leqslant \bar{V}(t^*)\mathrm{e}^{-\lambda(t-t^*)}$$

for $t \geqslant t^*$, where λ is the unique positive root of the transcendental equation $\lambda = a - b\mathrm{e}^{\lambda\tau}$.

Remark 1. The outer coupling matrix G_k reflects the topology structure which can be well defined by a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the node set $\mathcal{V}$ and the edge set $\mathcal{E}$. Each subsystem is viewed as a node, and if there exists a link from node j to i, then $g_{ij}^{(k)} > 0$, otherwise, $g_{ij}^{(k)} = 0$ ($i \neq j$). In the synchronization analysis of CNNs (1), if all the states are synchronous, a restriction condition must be imposed on the coupling matrices $G_k = (g_{ij}^{(k)})_{N\times N}$ ($k = 1, 2, 3$). This is the fundamental reason why a zero-row-sum condition $\sum_{j=1}^{N} g_{ij}^{(k)} = 0$ is required. A usually used zero-row-sum condition is expressed as $g_{ii}^{(k)} = -\sum_{j=1, j\neq i}^{N} g_{ij}^{(k)}$.

Remark 2. Nowadays, the synchronization analyses of CNNs usually require that the outer coupling matrix is symmetric [8, 12, 25, 28, 30, 50] or irreducible [33, 46, 47]. The underlying reason is the requirement of mathematical techniques, for example, the LMI-based synchronization results are on the basis of Kronecker product expression, in which the symmetric or irreducible feature of coupling matrix plays an important role in the derivation. In this paper, without the requirement of symmetric or irreducible conditions on outer coupling matrices, some global exponential synchronization criteria are established for an array of hybrid coupled neural networks with time-varying mixed delays, which are more flexible.

3 Global exponential synchronization criteria

Let $x_i(t) = (x_i^1(t), x_i^2(t), \dots, x_i^n(t))^{\mathrm{T}}$ be the solution of system (1) with the initial value $\varphi_i(\theta)$, $\theta \in [-\max\{\tau, h, \sigma\}, 0]$. Let $\bar{i}(t)$, $\underline{i}(t)$ and $\bar{k}(t)$ be the indexes satisfying

$$x^k_{\bar{i}(t)}(t) = \max_{1\leqslant i\leqslant N}\{x_i^k(t)\}, \qquad x^k_{\underline{i}(t)}(t) = \min_{1\leqslant i\leqslant N}\{x_i^k(t)\},$$
$$\|y(t)\| = \max_{1\leqslant k\leqslant n}|y_k(t)| = y_{\bar{k}(t)}(t) = x^{\bar{k}(t)}_{\bar{i}(t)}(t) - x^{\bar{k}(t)}_{\underline{i}(t)}(t),$$

where $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^{\mathrm{T}}$, $y_k(t) = x^k_{\bar{i}(t)}(t) - x^k_{\underline{i}(t)}(t)$, $\bar{i}(t), \underline{i}(t) \in \{1, 2, \dots, N\}$, $\bar{k}(t) \in \{1, 2, \dots, n\}$.

Denote

$$F(G_k) = \min_{\substack{1\leqslant i,j\leqslant N\\ i\neq j}}\left\{g_{ij}^{(k)} + g_{ji}^{(k)} + \sum_{\substack{l=1,\\ l\neq i,j}}^{N}\min\{g_{il}^{(k)}, g_{jl}^{(k)}\}\right\},$$
$$H(G_k) = \max_{1\leqslant i\leqslant N}\left\{\sum_{\substack{j=1\\ j\neq i}}^{N} g_{ij}^{(k)}\right\},$$
$$p = \|W_1\|L + \|W_2\|L + \|W_3\|L\sigma,$$
$$q = 2\alpha_1\bar{\gamma}^{(1)}H(G_1) + 2\alpha_2\bar{\gamma}^{(2)}H(G_2) + 2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma,$$
$$0 < \underline{a} = \min_{1\leqslant i\leqslant n}\{a_i\} \leqslant \max_{1\leqslant i\leqslant n}\{a_i\} = \bar{a},$$
$$0 < \underline{\gamma}^{(k)} = \min_{1\leqslant i\leqslant n}\{\gamma_i^{(k)}\} \leqslant \max_{1\leqslant i\leqslant n}\{\gamma_i^{(k)}\} = \bar{\gamma}^{(k)} \quad (k = 1, 2, 3).$$

Theorem 1. *Under assumptions* (H1)–(H3)*, if*

$$\underline{a} + \alpha_1\underline{\gamma}^{(1)}F(G_1) > \bar{a}\tau(\bar{a} + p + q) + p + 2\alpha_2\bar{\gamma}^{(2)}H(G_2) + 2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma, \tag{2}$$

then system (1) *can achieve exponential synchronization.*

Proof. Define a Lyapunov functional:

$$V(t) = \|y(t)\| = x^{\bar{k}(t)}_{\bar{i}(t)}(t) - x^{\bar{k}(t)}_{\underline{i}(t)}(t).$$

Let t_l $(l = 1, 2, \dots)$ be critical values such that at least one of the indexes $\bar{i}(t)$, $\underline{i}(t)$ and $\bar{k}(t)$ in $V(t)$ switch. We suppose

$$V(t) = x^{\bar{k}_l}_{\bar{i}_l}(t) - x^{\bar{k}_l}_{\underline{i}_l}(t), \quad t \in [t_l, t_{l+1}),\ l = 1, 2, \dots.$$

Here $\bar{i}_l$, $\underline{i}_l$ and $\bar{k}_l$ are constants. In what follows, we will calculate the right upper derivative of $V(t)$ along the trajectories of (1).

Since each $x_i^j(t)$ $(i = 1, 2, \dots, N;\ j = 1, 2, \dots, n)$ is differentiable in $(0, +\infty)$, we have

$$D^+\big(V(t)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\big(x^{\bar{k}_l}_{\bar{i}_l}(t) - x^{\bar{k}_l}_{\underline{i}_l}(t)\big), \quad t \in (t_l, t_{l+1}),$$

and

$$D^+\big(V(t_l)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\big(x_{\bar{i}_l}^{\bar{k}_l}(t) - x_{\underline{i}_l}^{\bar{k}_l}(t)\big)\big|_{t=t_l}.$$

Therefore, for $t \in (0, +\infty)$,

$$D^+\big(V(t)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\big(x_{\bar{i}(t)}^{\bar{k}(t)}(t) - x_{\underline{i}(t)}^{\bar{k}(t)}(t)\big),$$

in which the indexes $\bar{i}(t)$, $\underline{i}(t)$ and $\bar{k}(t)$ are regarded as constants.

By the above analysis, we have

$$\begin{aligned}
&D^+\big(V(t)\big)\\
&\quad= -a_{\bar{k}(t)}\big[x_{\bar{i}(t)}^{\bar{k}(t)}\big(t-\tau(t)\big) - x_{\underline{i}(t)}^{\bar{k}(t)}\big(t-\tau(t)\big)\big] + \sum_{j=1}^{n} w_{\bar{k}(t)j}^{(1)}\big[f_j\big(x_{\bar{i}(t)}^{j}(t)\big) - f_j\big(x_{\underline{i}(t)}^{j}(t)\big)\big]\\
&\qquad+ \sum_{j=1}^{n} w_{\bar{k}(t)j}^{(2)}\big[f_j\big(x_{\bar{i}(t)}^{j}\big(t-h(t)\big)\big) - f_j\big(x_{\underline{i}(t)}^{j}\big(t-h(t)\big)\big)\big]\\
&\qquad+ \sum_{j=1}^{n} w_{\bar{k}(t)j}^{(3)} \int\limits_{t-\sigma(t)}^{t} \big[f_j\big(x_{\bar{i}(t)}^{j}(s)\big) - f_j\big(x_{\underline{i}(t)}^{j}(s)\big)\big]\,\mathrm{d}s\\
&\qquad+ \alpha_1\gamma_{\bar{k}(t)}^{(1)} \sum_{j=1}^{N}\big(g_{\bar{i}(t)j}^{(1)} - g_{\underline{i}(t)j}^{(1)}\big)x_j^{\bar{k}(t)}(t) + \alpha_2\gamma_{\bar{k}(t)}^{(2)} \sum_{j=1}^{N}\big(g_{\bar{i}(t)j}^{(2)} - g_{\underline{i}(t)j}^{(2)}\big)x_j^{\bar{k}(t)}\big(t-h(t)\big)\\
&\qquad+ \alpha_3\gamma_{\bar{k}(t)}^{(3)} \sum_{j=1}^{N}\big(g_{\bar{i}(t)j}^{(3)} - g_{\underline{i}(t)j}^{(3)}\big) \int\limits_{t-\sigma(t)}^{t} x_j^{\bar{k}(t)}(s)\,\mathrm{d}s\\
&\quad\leqslant -a_{\bar{k}(t)}\big[x_{\bar{i}(t)}^{\bar{k}(t)}\big(t-\tau(t)\big) - x_{\underline{i}(t)}^{\bar{k}(t)}\big(t-\tau(t)\big)\big]\\
&\qquad+ \sum_{j=1}^{n}\big|w_{\bar{k}(t)j}^{(1)}\big|LV(t) + \sum_{j=1}^{n}\big|w_{\bar{k}(t)j}^{(2)}\big|LV\big(t-h(t)\big) + \sum_{j=1}^{n}\big|w_{\bar{k}(t)j}^{(3)}\big|L\sigma \sup_{t-\sigma\leqslant s\leqslant t} V(s)\\
&\qquad+ \alpha_1\gamma_{\bar{k}(t)}^{(1)} \sum_{j=1}^{N}\big(g_{\bar{i}(t)j}^{(1)} - g_{\underline{i}(t)j}^{(1)}\big)x_j^{\bar{k}(t)}(t) + \alpha_2\gamma_{\bar{k}(t)}^{(2)} \sum_{j=1}^{N}\big(g_{\bar{i}(t)j}^{(2)} - g_{\underline{i}(t)j}^{(2)}\big)x_j^{\bar{k}(t)}\big(t-h(t)\big)\\
&\qquad+ \alpha_3\gamma_{\bar{k}(t)}^{(3)} \sum_{j=1}^{N}\big(g_{\bar{i}(t)j}^{(3)} - g_{\underline{i}(t)j}^{(3)}\big) \int\limits_{t-\sigma(t)}^{t} x_j^{\bar{k}(t)}(s)\,\mathrm{d}s. \qquad (3)
\end{aligned}$$

By Newton–Leibniz formula and (3), we have

$$\begin{aligned}
&-a_{\bar{k}(t)}\big[x_{\bar{i}(t)}^{\bar{k}(t)}\big(t-\tau(t)\big) - x_{\underline{i}(t)}^{\bar{k}(t)}\big(t-\tau(t)\big)\big]\\
&\quad= -a_{\bar{k}(t)}\big[x_{\bar{i}(t)}^{\bar{k}(t)}(t) - x_{\underline{i}(t)}^{\bar{k}(t)}(t)\big] + a_{\bar{k}(t)} \int\limits_{t-\tau(t)}^{t} \big[\dot{x}_{\bar{i}(t)}^{\bar{k}(t)}(s) - \dot{x}_{\underline{i}(t)}^{\bar{k}(t)}(s)\big]\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leqslant -a_{\bar{k}(t)}V(t) + a_{\bar{k}(t)} \int\limits_{t-\tau(t)}^{t} \Bigg\{ -a_{\bar{k}(t)}\big[x_{\bar{i}(t)}^{\bar{k}(t)}\big(s-\tau(s)\big) - x_{\underline{i}(t)}^{\bar{k}(t)}\big(s-\tau(s)\big)\big] \\
&\quad + \sum_{j=1}^{n} \big|w_{\bar{k}(t)j}^{(1)}\big| LV(s) + \sum_{j=1}^{n} \big|w_{\bar{k}(t)j}^{(2)}\big| LV\big(s-h(s)\big) + \sum_{j=1}^{n} \big|w_{\bar{k}(t)j}^{(3)}\big| L\sigma \sup_{s-\sigma\leqslant s'\leqslant s} V(s') \\
&\quad + \alpha_1\gamma_{\bar{k}(t)}^{(1)} \sum_{j=1}^{N} \big(g_{\bar{i}(t)j}^{(1)} - g_{\underline{i}(t)j}^{(1)}\big) x_j^{\bar{k}(t)}(s) + \alpha_2\gamma_{\bar{k}(t)}^{(2)} \sum_{j=1}^{N} \big(g_{\bar{i}(t)j}^{(2)} - g_{\underline{i}(t)j}^{(2)}\big) x_j^{\bar{k}(t)}\big(s-h(s)\big) \\
&\quad + \alpha_3\gamma_{\bar{k}(t)}^{(3)} \sum_{j=1}^{N} \big(g_{\bar{i}(t)j}^{(3)} - g_{\underline{i}(t)j}^{(3)}\big) \int\limits_{s-\sigma(s)}^{s} x_j^{\bar{k}(t)}(s')\,\mathrm{d}s' \Bigg\}\,\mathrm{d}s \\
&\leqslant -a_{\bar{k}(t)}V(t) + a_{\bar{k}(t)}^2\tau \sup_{t-2\tau\leqslant s\leqslant t} V(s) + a_{\bar{k}(t)}\|W_1\|L\tau \sup_{t-\tau\leqslant s\leqslant t} V(s) \\
&\quad + a_{\bar{k}(t)}\|W_2\|L\tau \sup_{t-\tau-h\leqslant s\leqslant t} V(s) + a_{\bar{k}(t)}\|W_3\|L\tau\sigma \sup_{t-\tau-\sigma\leqslant s\leqslant t} V(s) \\
&\quad + a_{\bar{k}(t)} \int\limits_{t-\tau(t)}^{t} \Bigg\{ \alpha_1\gamma_{\bar{k}(t)}^{(1)} \sum_{j=1}^{N} \big(g_{\bar{i}(t)j}^{(1)} - g_{\underline{i}(t)j}^{(1)}\big) x_j^{\bar{k}(t)}(s) + \alpha_2\gamma_{\bar{k}(t)}^{(2)} \sum_{j=1}^{N} \big(g_{\bar{i}(t)j}^{(2)} - g_{\underline{i}(t)j}^{(2)}\big) \\
&\quad \times x_j^{\bar{k}(t)}\big(s-h(s)\big) + \alpha_3\gamma_{\bar{k}(t)}^{(3)} \sum_{j=1}^{N} \big(g_{\bar{i}(t)j}^{(3)} - g_{\underline{i}(t)j}^{(3)}\big) \int\limits_{s-\sigma(s)}^{s} x_j^{\bar{k}(t)}(s')\,\mathrm{d}s' \Bigg\}\,\mathrm{d}s. \qquad (4)
\end{aligned}
$$

By (H2), it follows that

$$
\begin{aligned}
&\alpha_1\gamma_{\bar{k}(t)}^{(1)} \sum_{j=1}^{N} \big(g_{\bar{i}(t)j}^{(1)} - g_{\underline{i}(t)j}^{(1)}\big) x_j^{\bar{k}(t)}(s) \\
&\quad = \alpha_1\gamma_{\bar{k}(t)}^{(1)} \Bigg[\sum_{\substack{j=1 \\ j\neq\bar{i}(t)}}^{N} g_{\bar{i}(t)j}^{(1)} x_j^{\bar{k}(t)}(s) + g_{\bar{i}(t)\bar{i}(t)}^{(1)} x_{\bar{i}(t)}^{\bar{k}(t)}(s) \Bigg] \\
&\qquad - \alpha_1\gamma_{\bar{k}(t)}^{(1)} \Bigg[\sum_{\substack{j=1 \\ j\neq\underline{i}(t)}}^{N} g_{\underline{i}(t)j}^{(1)} x_j^{\bar{k}(t)}(s) + g_{\underline{i}(t)\underline{i}(t)}^{(1)} x_{\underline{i}(t)}^{\bar{k}(t)}(s) \Bigg] \\
&\quad = \alpha_1\gamma_{\bar{k}(t)}^{(1)} \Bigg[\sum_{\substack{j=1 \\ j\neq\bar{i}(t)}}^{N} g_{\bar{i}(t)j}^{(1)} x_j^{\bar{k}(t)}(s) - \sum_{\substack{j=1 \\ j\neq\bar{i}(t)}}^{N} g_{\bar{i}(t)j}^{(1)} x_{\bar{i}(t)}^{\bar{k}(t)}(s) \Bigg] \\
&\qquad - \alpha_1\gamma_{\bar{k}(t)}^{(1)} \Bigg[\sum_{\substack{j=1 \\ j\neq\underline{i}(t)}}^{N} g_{\underline{i}(t)j}^{(1)} x_j^{\bar{k}(t)}(s) - \sum_{\substack{j=1 \\ j\neq\underline{i}(t)}}^{N} g_{\underline{i}(t)j}^{(1)} x_{\underline{i}(t)}^{\bar{k}(t)}(s) \Bigg]
\end{aligned}
$$

$$= \alpha_1 \gamma^{(1)}_{\bar{k}(t)} \Bigg\{ \sum_{\substack{j=1 \\ j \neq \bar{i}(t)}}^{N} g^{(1)}_{\bar{i}(t)j} \big[x^{\bar{k}(t)}_j (s) - x^{\bar{k}(t)}_{\bar{i}(t)} (s) \big] - \sum_{\substack{j=1 \\ j \neq \underline{i}(t)}}^{N} g^{(1)}_{\underline{i}(t)j} \big[x^{\bar{k}(t)}_j (s) - x^{\bar{k}(t)}_{\underline{i}(t)} (s) \big] \Bigg\}$$
$$\leqslant 2\alpha_1 \bar{\gamma}^{(1)} H(G_1) V(s). \tag{5}$$

Similarly, we have

$$\alpha_2 \gamma^{(2)}_{\bar{k}(t)} \sum_{j=1}^{N} \big(g^{(2)}_{\bar{i}(t)j} - g^{(2)}_{\underline{i}(t)j} \big) x^{\bar{k}(t)}_j \big(s - h(s) \big) \leqslant 2\alpha_2 \bar{\gamma}^{(2)} H(G_2) V\big(s - h(s) \big), \tag{6}$$

and

$$\alpha_3 \gamma^{(3)}_{\bar{k}(t)} \sum_{j=1}^{N} \big(g^{(3)}_{\bar{i}(t)j} - g^{(3)}_{\underline{i}(t)j} \big) \int\limits_{s-\sigma(s)}^{s} x^{\bar{k}(t)}_j (s') \, \mathrm{d}s'$$
$$\leqslant 2\alpha_3 \bar{\gamma}^{(3)} H(G_3) \sigma(s) \sup_{s-\sigma(s) \leqslant s' \leqslant s} V(s'). \tag{7}$$

Substituting (5)–(7) into (4), we get

$$\begin{aligned}
&-a_{\bar{k}(t)} \big[x^{\bar{k}(t)}_{\bar{i}(t)} \big(t - \tau(t) \big) - x^{\bar{k}(t)}_{\underline{i}(t)} \big(t - \tau(t) \big) \big] \\
&\quad \leqslant -a_{\bar{k}(t)} V(t) + a^2_{\bar{k}(t)} \tau \sup_{t-2\tau \leqslant s \leqslant t} V(s) + a_{\bar{k}(t)} \|W_1\| L\tau \sup_{t-\tau \leqslant s \leqslant t} V(s) \\
&\qquad + a_{\bar{k}(t)} \|W_2\| L\tau \sup_{t-\tau-h \leqslant s \leqslant t} V(s) + a_{\bar{k}(t)} \|W_3\| L\tau\sigma \sup_{t-\tau-\sigma \leqslant s \leqslant t} V(s) \\
&\qquad + 2a_{\bar{k}(t)} \int\limits_{t-\tau(t)}^{t} \Big\{ \alpha_1 \bar{\gamma}^{(1)} H(G_1) V(s) + \alpha_2 \bar{\gamma}^{(2)} H(G_2) V\big(s - h(s) \big) \\
&\qquad + \alpha_3 \bar{\gamma}^{(3)} H(G_3) \sigma(s) \sup_{s-\sigma(s) \leqslant s' \leqslant s} V(s') \Big\} \, \mathrm{d}s \\
&\quad \leqslant -\underline{a} V(t) + \bar{a}^2 \tau \sup_{t-2\tau \leqslant s \leqslant t} V(s) + \bar{a} \|W_1\| L\tau \sup_{t-\tau \leqslant s \leqslant t} V(s) \\
&\qquad + \bar{a} \|W_2\| L\tau \sup_{t-\tau-h \leqslant s \leqslant t} V(s) + \bar{a} \|W_3\| L\tau\sigma \sup_{t-\tau-\sigma \leqslant s \leqslant t} V(s) \\
&\qquad + 2\bar{a}\tau \big\{ \alpha_1 \bar{\gamma}^{(1)} H(G_1) \sup_{t-\tau \leqslant s \leqslant t} V(s) + \alpha_2 \bar{\gamma}^{(2)} H(G_2) \sup_{t-\tau-h \leqslant s \leqslant t} V(s) \\
&\qquad + \alpha_3 \bar{\gamma}^{(3)} H(G_3) \sigma \sup_{t-\tau-\sigma \leqslant s \leqslant t} V(s) \big\} \\
&\quad \leqslant -\underline{a} V(t) + \bar{a}\tau \big[\bar{a} + \|W_1\| L + \|W_2\| L + \|W_3\| L\sigma + 2\alpha_1 \bar{\gamma}^{(1)} H(G_1) \\
&\qquad + 2\alpha_2 \bar{\gamma}^{(2)} H(G_2) + 2\alpha_3 \bar{\gamma}^{(3)} H(G_3) \sigma \big] \sup_{t-2\tau-h-\sigma \leqslant s \leqslant t} V(s).
\end{aligned} \tag{8}$$

Noting that we can always choose $\bar{i}(t)$ and $\underline{i}(t)$ satisfying $\bar{i}(t) \neq \underline{i}(t)$ for $N \geqslant 2$, we further derive from (5) that

$$
\begin{aligned}
&\alpha_1\gamma^{(1)}_{\bar{k}(t)}\sum_{j=1}^{N}\big(g^{(1)}_{\bar{i}(t)j} - g^{(1)}_{\underline{i}(t)j}\big)x^{\bar{k}(t)}_j(t)\\
&\quad= \alpha_1\gamma^{(1)}_{\bar{k}(t)}\Bigg\{\sum_{\substack{j=1\\ j\neq\bar{i}(t)}}^{N} g^{(1)}_{\bar{i}(t)j}\big[x^{\bar{k}(t)}_j(t) - x^{\bar{k}(t)}_{\bar{i}(t)}(t)\big] - \sum_{\substack{j=1\\ j\neq\underline{i}(t)}}^{N} g^{(1)}_{\underline{i}(t)j}\big[x^{\bar{k}(t)}_j(t) - x^{\bar{k}(t)}_{\underline{i}(t)}(t)\big]\Bigg\}\\
&\quad= \alpha_1\gamma^{(1)}_{\bar{k}(t)}\Bigg\{g^{(1)}_{\bar{i}(t)\underline{i}(t)}\big[x^{\bar{k}(t)}_{\underline{i}(t)}(t) - x^{\bar{k}(t)}_{\bar{i}(t)}(t)\big] - g^{(1)}_{\underline{i}(t)\bar{i}(t)}\big[x^{\bar{k}(t)}_{\bar{i}(t)}(t) - x^{\bar{k}(t)}_{\underline{i}(t)}(t)\big]\\
&\qquad+ \sum_{\substack{j=1\\ j\neq\bar{i}(t),\underline{i}(t)}}^{N} g^{(1)}_{\bar{i}(t)j}\big[x^{\bar{k}(t)}_j(t) - x^{\bar{k}(t)}_{\bar{i}(t)}(t)\big] - \sum_{\substack{j=1\\ j\neq\bar{i}(t),\underline{i}(t)}}^{N} g^{(1)}_{\underline{i}(t)j}\big[x^{\bar{k}(t)}_j(t) - x^{\bar{k}(t)}_{\underline{i}(t)}(t)\big]\Bigg\}\\
&\quad= -\alpha_1\gamma^{(1)}_{\bar{k}(t)}\Bigg\{\big(g^{(1)}_{\bar{i}(t)\underline{i}(t)} + g^{(1)}_{\underline{i}(t)\bar{i}(t)}\big)\big[x^{\bar{k}(t)}_{\bar{i}(t)}(t) - x^{\bar{k}(t)}_{\underline{i}(t)}(t)\big]\\
&\qquad+ \sum_{\substack{j=1\\ j\neq\bar{i}(t),\underline{i}(t)}}^{N} \big\{g^{(1)}_{\bar{i}(t)j}\big[x^{\bar{k}(t)}_{\bar{i}(t)}(t) - x^{\bar{k}(t)}_j(t)\big] + g^{(1)}_{\underline{i}(t)j}\big[x^{\bar{k}(t)}_j(t) - x^{\bar{k}(t)}_{\underline{i}(t)}(t)\big]\big\}\Bigg\}\\
&\quad\leqslant -\alpha_1\gamma^{(1)}_{\bar{k}(t)}\Bigg\{\big(g^{(1)}_{\bar{i}(t)\underline{i}(t)} + g^{(1)}_{\underline{i}(t)\bar{i}(t)}\big)\big[x^{\bar{k}(t)}_{\bar{i}(t)}(t) - x^{\bar{k}(t)}_{\underline{i}(t)}(t)\big]\\
&\qquad+ \sum_{\substack{j=1\\ j\neq\bar{i}(t),\underline{i}(t)}}^{N} \big\{\min\big(g^{(1)}_{\bar{i}(t)j}, g^{(1)}_{\underline{i}(t)j}\big)\big[x^{\bar{k}(t)}_{\bar{i}(t)}(t) - x^{\bar{k}(t)}_{\underline{i}(t)}(t)\big]\big\}\Bigg\}\\
&\quad= -\alpha_1\gamma^{(1)}_{\bar{k}(t)}\Bigg\{g^{(1)}_{\bar{i}(t)\underline{i}(t)} + g^{(1)}_{\underline{i}(t)\bar{i}(t)} + \sum_{\substack{j=1\\ j\neq\bar{i}(t),\underline{i}(t)}}^{N} \min\big(g^{(1)}_{\bar{i}(t)j}, g^{(1)}_{\underline{i}(t)j}\big)\Bigg\}V(t)\\
&\quad\leqslant -\alpha_1\underline{\gamma}^{(1)}F(G_1)V(t).
\end{aligned}
\tag{9}
$$

Similar to (6) and (7), we get

$$
\alpha_2\gamma^{(2)}_{\bar{k}(t)}\sum_{j=1}^{N}\big(g^{(2)}_{\bar{i}(t)j} - g^{(2)}_{\underline{i}(t)j}\big)x^{\bar{k}(t)}_j\big(t-h(t)\big) \leqslant 2\alpha_2\bar{\gamma}^{(2)}H(G_2)V\big(t-h(t)\big), \tag{10}
$$

and

$$
\alpha_3\gamma^{(3)}_{\bar{k}(t)}\sum_{j=1}^{N}\big(g^{(3)}_{\bar{i}(t)j} - g^{(3)}_{\underline{i}(t)j}\big)\int_{t-\sigma(t)}^{t} x^{\bar{k}(t)}_j(s)\,\mathrm{d}s \leqslant 2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma\sup_{t-\sigma\leqslant s\leqslant t}V(s). \tag{11}
$$

It follows from (3) and (8)–(11) that

$$\begin{aligned}
&D^+\big(V(t)\big)\\
&\quad\leqslant -\underline{a}V(t) + \bar{a}\tau\big[\bar{a} + \|W_1\|L + \|W_2\|L + \|W_3\|L\sigma + 2\alpha_1\bar{\gamma}^{(1)}H(G_1)\\
&\qquad + 2\alpha_2\bar{\gamma}^{(2)}H(G_2) + 2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma\big] \sup_{t-2\tau-h-\sigma\leqslant s\leqslant t} V(s) + \|W_1\|LV(t)\\
&\qquad + \|W_2\|LV\big(t-h(t)\big) + \|W_3\|L\sigma \sup_{t-\sigma\leqslant s\leqslant t} V(s) - \alpha_1\underline{\gamma}^{(1)}F(G_1)V(t)\\
&\qquad + 2\alpha_2\bar{\gamma}^{(2)}H(G_2)V\big(t-h(t)\big) + 2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma \sup_{t-\sigma\leqslant s\leqslant t} V(s)\\
&\quad\leqslant -\big[\underline{a} - \|W_1\|L + \alpha_1\underline{\gamma}^{(1)}F(G_1)\big]V(t) + \big\{\bar{a}\tau\big[\bar{a} + \|W_1\|L + \|W_2\|L + \|W_3\|L\sigma\\
&\qquad + 2\alpha_1\bar{\gamma}^{(1)}H(G_1) + 2\alpha_2\bar{\gamma}^{(2)}H(G_2) + 2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma\big] + \|W_2\|L + \|W_3\|L\sigma\\
&\qquad + 2\alpha_2\bar{\gamma}^{(2)}H(G_2) + 2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma\big\} \sup_{t-2\tau-h-\sigma\leqslant s\leqslant t} V(s) \qquad (12)
\end{aligned}$$

for all $t \geqslant 0$. By (2) and Lemma 1, we have that $V(t)$ exponentially approaches to zero as $t \to \infty$. Therefore, system (1) can achieve exponential synchronization. The proof is complete. □

Theorem 2. *Under assumptions* (H1)–(H3)*, if*

$$\begin{aligned}
&\underline{a} + \alpha_1\underline{\gamma}^{(1)}F(G_1) + \alpha_2\underline{\gamma}^{(2)}F(G_2)\\
&\qquad > \big[\bar{a}\tau + 2\alpha_2\bar{\gamma}^{(2)}H(G_2)h\big](\bar{a} + p + q) + p + 2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma, \qquad (13)
\end{aligned}$$

then system (1) *can achieve exponential synchronization.*

Proof. We note that

$$\begin{aligned}
&\alpha_2\gamma^{(2)}_{\bar{k}(t)}\sum_{j=1}^{N}\big(g^{(2)}_{\bar{i}(t)j} - g^{(2)}_{\underline{i}(t)j}\big)x_j^{\bar{k}(t)}\big(t-h(t)\big)\\
&\quad = \alpha_2\gamma^{(2)}_{\bar{k}(t)}\sum_{j=1}^{N}\big(g^{(2)}_{\bar{i}(t)j} - g^{(2)}_{\underline{i}(t)j}\big)x_j^{\bar{k}(t)}(t)\\
&\qquad - \alpha_2\gamma^{(2)}_{\bar{k}(t)}\sum_{j=1}^{N}\big(g^{(2)}_{\bar{i}(t)j} - g^{(2)}_{\underline{i}(t)j}\big)\int_{t-h(t)}^{t}\dot{x}_j^{\bar{k}(t)}(s)\,\mathrm{d}s. \qquad (14)
\end{aligned}$$

By (9), we have

$$\alpha_2\gamma^{(2)}_{\bar{k}(t)}\sum_{j=1}^{N}\big(g^{(2)}_{\bar{i}(t)j} - g^{(2)}_{\underline{i}(t)j}\big)x_j^{\bar{k}(t)}(t) \leqslant -\alpha_2\underline{\gamma}^{(2)}F(G_2)V(t). \qquad (15)$$

Following the similar derivation in Theorem 1, we get

$$\begin{aligned}
&-\alpha_2\gamma^{(2)}_{\bar{k}(t)}\sum_{j=1}^{N}\bigl(g^{(2)}_{\bar{i}(t)j}-g^{(2)}_{\underline{i}(t)j}\bigr)\int\limits_{t-h(t)}^{t}\dot{x}_j^{\bar{k}(t)}(s)\,\mathrm{d}s\\
&\quad=-\alpha_2\gamma^{(2)}_{\bar{k}(t)}\sum_{j=1}^{N}\bigl(g^{(2)}_{\bar{i}(t)j}-g^{(2)}_{\underline{i}(t)j}\bigr)\int\limits_{t-h(t)}^{t}\Biggl\{-a_{\bar{k}(t)}x_j^{\bar{k}(t)}\bigl(s-\tau(s)\bigr)\\
&\qquad+\sum_{p=1}^{n}w^{(1)}_{\bar{k}(t)p}f_p\bigl(x_j^p(s)\bigr)+\sum_{p=1}^{n}w^{(2)}_{\bar{k}(t)p}f_p\bigl(x_j^p\bigl(s-h(s)\bigr)\bigr)\\
&\qquad+\sum_{p=1}^{n}w^{(3)}_{\bar{k}(t)p}\int\limits_{s-\sigma(s)}^{s}f_p\bigl(x_j^p(s')\bigr)\,\mathrm{d}s'+u_{\bar{k}(t)}(s)+\alpha_1\sum_{p=1}^{N}g^{(1)}_{jp}\gamma^{(1)}_{\bar{k}(t)}x_p^{\bar{k}(t)}(s)\\
&\qquad+\alpha_2\sum_{p=1}^{N}g^{(2)}_{jp}\gamma^{(2)}_{\bar{k}(t)}x_p^{\bar{k}(t)}\bigl(s-h(s)\bigr)+\alpha_3\sum_{p=1}^{N}g^{(3)}_{jp}\gamma^{(3)}_{\bar{k}(t)}\int\limits_{s-\sigma(s)}^{s}x_p^{\bar{k}(t)}(s')\,\mathrm{d}s'\Biggr\}\,\mathrm{d}s\\
&\quad\leqslant 2\alpha_2\bar{\gamma}^{(2)}\bar{a}H(G_2)h\sup_{t-h-\tau\leqslant s\leqslant t}V(s)+2\alpha_2\bar{\gamma}^{(2)}H(G_2)\|W_1\|Lh\sup_{t-h\leqslant s\leqslant t}V(s)\\
&\qquad+2\alpha_2\bar{\gamma}^{(2)}H(G_2)\|W_2\|Lh\sup_{t-2h\leqslant s\leqslant t}V(s)\\
&\qquad+2\alpha_2\bar{\gamma}^{(2)}H(G_2)\|W_3\|Lh\sigma\sup_{t-h-\sigma\leqslant s\leqslant t}V(s)\\
&\qquad+4\alpha_1\alpha_2\bar{\gamma}^{(1)}\bar{\gamma}^{(2)}H(G_1)H(G_2)h\sup_{t-h\leqslant s\leqslant t}V(s)\\
&\qquad+4(\alpha_2)^2\bigl[\bar{\gamma}^{(2)}\bigr]^2\bigl[H(G_2)\bigr]^2h\sup_{t-2h\leqslant s\leqslant t}V(s)\\
&\qquad+4\alpha_2\alpha_3\bar{\gamma}^{(2)}\bar{\gamma}^{(3)}H(G_2)H(G_3)h\sigma\sup_{t-h-\sigma\leqslant s\leqslant t}V(s)\\
&\quad\leqslant 2\alpha_2\bar{\gamma}^{(2)}H(G_2)h\bigl[\bar{a}+\|W_1\|L+\|W_2\|L+\|W_3\|L\sigma+2\alpha_1\bar{\gamma}^{(1)}H(G_1)\\
&\qquad+2\alpha_2\bar{\gamma}^{(2)}H(G_2)+2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma\bigr]\sup_{t-\tau-2h-\sigma\leqslant s\leqslant t}V(s).
\end{aligned}\tag{16}$$

It follows from (14)–(16) that

$$\begin{aligned}
&\alpha_2\gamma^{(2)}_{\bar{k}(t)}\sum_{j=1}^{N}\bigl(g^{(2)}_{\bar{i}(t)j}-g^{(2)}_{\underline{i}(t)j}\bigr)x_j^{\bar{k}(t)}\bigl(t-h(t)\bigr)\\
&\quad\leqslant-\alpha_2\underline{\gamma}^{(2)}F(G_2)V(t)+2\alpha_2\bar{\gamma}^{(2)}H(G_2)h\bigl[\bar{a}+\|W_1\|L+\|W_2\|L+\|W_3\|L\sigma\\
&\qquad+2\alpha_1\bar{\gamma}^{(1)}H(G_1)+2\alpha_2\bar{\gamma}^{(2)}H(G_2)+2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma\bigr]\sup_{t-\tau-2h-\sigma\leqslant s\leqslant t}V(s).
\end{aligned}\tag{17}$$

Combining (3), (8), (9), (11) and (17), we can deduce that

$$\begin{aligned} &D^{+}\big(V(t)\big) \\ &\quad\leqslant -\underline{a}V(t) + \bar{a}\tau\big[\bar{a} + \|W_1\|L + \|W_2\|L + \|W_3\|L\sigma + 2\alpha_1\bar{\gamma}^{(1)}H(G_1) \\ &\qquad + 2\alpha_2\bar{\gamma}^{(2)}H(G_2) + 2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma\big]\sup_{t-2\tau-h-\sigma\leqslant s\leqslant t}V(s) + \|W_1\|LV(t) \\ &\qquad + \|W_2\|LV\big(t-h(t)\big) + \|W_3\|L\sigma\sup_{t-\sigma\leqslant s\leqslant t}V(s) - \alpha_1\underline{\gamma}^{(1)}F(G_1)V(t) \\ &\qquad - \alpha_2\underline{\gamma}^{(2)}F(G_2)V(t) + 2\alpha_2\bar{\gamma}^{(2)}H(G_2)h\big[\bar{a} + \|W_1\|L + \|W_2\|L + \|W_3\|L\sigma \\ &\qquad + 2\alpha_1\bar{\gamma}^{(1)}H(G_1) + 2\alpha_2\bar{\gamma}^{(2)}H(G_2) + 2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma\big]\sup_{t-\tau-2h-\sigma\leqslant s\leqslant t}V(s) \\ &\qquad + 2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma\sup_{t-\sigma\leqslant s\leqslant t}V(s) \\ &\quad\leqslant -\big[\underline{a} - \|W_1\|L + \alpha_1\underline{\gamma}^{(1)}F(G_1) + \alpha_2\underline{\gamma}^{(2)}F(G_2)\big]V(t) + \big\{\bar{a}\tau\big[\bar{a} + \|W_1\|L \\ &\qquad + \|W_2\|L + \|W_3\|L\sigma + 2\alpha_1\bar{\gamma}^{(1)}H(G_1) + 2\alpha_2\bar{\gamma}^{(2)}H(G_2) + 2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma\big] \\ &\qquad + \|W_2\|L + \|W_3\|L\sigma + 2\alpha_2\bar{\gamma}^{(2)}H(G_2)h\big[\bar{a} + \|W_1\|L + \|W_2\|L + \|W_3\|L\sigma \\ &\qquad + 2\alpha_1\bar{\gamma}^{(1)}H(G_1) + 2\alpha_2\bar{\gamma}^{(2)}H(G_2) + 2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma\big] \\ &\qquad + 2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma\big\}\sup_{t-2\tau-2h-\sigma\leqslant s\leqslant t}V(s). \end{aligned} \tag{18}$$

By (13) and Lemma 1, $V(t)$ exponentially approaches to zero as $t \to \infty$. Therefore, system (1) can achieve exponential synchronization. The proof is complete. □

Remark 3. The differences between Theorems 1 and 2 lie in the fact that the term $\alpha_2\gamma^{(2)}_{\bar{k}(t)}\sum_{j=1}^{N}(g^{(2)}_{\bar{i}(t)j} - g^{(2)}_{\underline{i}(t)j})x_j^{\bar{k}(t)}(t-h(t))$ is treated differently in the proof procedure although the selected Lyapunov functional is same. That is, different inequality treatment leads to different expressions (2) and (13). Since both Theorems 1 and 2 are sufficient conditions, in general, it is difficult to judge which criterion is superior.

Remark 4. Based on Theorem 1, we have that under assumptions (H1)–(H3), if $\underline{a} + \alpha_1\underline{\gamma}^{(1)}F(G_1) > p + 2\alpha_2\bar{\gamma}^{(2)}H(G_2)$, then system (1) can achieve exponential synchronization for sufficiently small delay upper bounds τ and σ. For example, for given constant σ satisfying that $\underline{a} + \alpha_1\underline{\gamma}^{(1)}F(G_1) > p + 2\alpha_2\bar{\gamma}^{(2)}H(G_2) + 2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma$, we can obtain the maximal allowable upper bound of the leakage delay, that is,

$$\tau < \frac{\underline{a} + \alpha_1\underline{\gamma}^{(1)}F(G_1) - p - 2\alpha_2\bar{\gamma}^{(2)}H(G_2) - 2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma}{\bar{a}(\bar{a} + p + q)}. \tag{19}$$

In another word, large leakage delay could have a negative impact on the synchronization of system (1), and cause non-synchronization of the system.

In a similar way, from Theorem 2, for given constants τ and σ satisfying $\underline{a} + \alpha_1\underline{\gamma}^{(1)}F(G_1) + \alpha_2\underline{\gamma}^{(2)}F(G_2) > p + 2\alpha_3\bar{\gamma}^{(3)}H(G_3)\sigma + \bar{a}\tau(\bar{a} + p + q)$, we can derive the maximal allowable upper bound of the time-varying delay $h(t)$ guaranteeing the exponential synchronization of system (1):

$$h < \frac{\underline{a} + \alpha_1 \underline{\gamma}^{(1)} F(G_1) + \alpha_2 \underline{\gamma}^{(2)} F(G_2) - p - 2\alpha_3 \bar{\gamma}^{(3)} H(G_3)\sigma - \bar{a}\tau(\bar{a} + p + q)}{2\alpha_2 \bar{\gamma}^{(2)} H(G_2)(\bar{a} + p + q)}.$$

Remark 5. From (19), we have that, if

$$F(G_1) = \min_{\substack{1 \leqslant i,j \leqslant N \\ i \neq j}} \left\{ g_{ij}^{(1)} + g_{ji}^{(1)} + \sum_{\substack{l=1 \\ l \neq i,j}}^{N} \min\{g_{il}^{(1)}, g_{jl}^{(1)}\} \right\} > 0,$$

which means that the outer coupling matrix G_1 is of scrambling property, i.e., for any indices i, j $(i \neq j)$, either of the following conditions satisfies: (i) $g_{ij}^{(1)} + g_{ji}^{(1)} > 0$; (ii) There is an index $l \in \{1, 2, \ldots, N\} \setminus \{i, j\}$ such that $g_{il}^{(1)} > 0$ and $g_{jl}^{(1)} > 0$, then the coupled network (1) can achieve exponential synchronization for all matrix A when the coupling strength α_1 is large enough and the delay upper bound τ satisfies (19).

Remark 6. In previous works, most of the synchronization criteria have been derived based on the LMI method. It is well known that, one of the hot issues in the synchronization field is to reduce the computing complexity of the synchronization criteria. In the aspects of analytic technique and the synchronization criteria, the present results are different from the existing researches. Recalling (2) and (13), our synchronization conditions only involve the parameters of system (1) to be computed, thus, they would be easy to check in comparison with those previously reported LMIs (see [8, 9, 12, 25, 28, 30, 31, 36, 39, 40, 45, 46, 47, 50, 52]). On the other hand, from the proofs of Theorems 1 and 2, one can see that the signs of elements of connection weight matrices are neglected in the derivation of inequalities, that is, (2) and (13) are related with $\|W_k\|$ $(k = 1, 2, 3)$. In this point, our synchronization criteria may be conservative. Nevertheless, our results provide a new, convenient, and efficient approach to study the synchronization for complex neural networks with hybrid coupling and mixed time delays.

4 Numerical simulations

In this section, we provide several numerical examples to illustrate the feasibility of the theoretical results.

Example 1. Consider the following 2-neuron delayed neural network without coupling

$$\begin{aligned} \dot{x}(t) &= -Ax\bigl(t - \tau(t)\bigr) + W_1 f\bigl(x(t)\bigr) + W_2 f\bigl(x\bigl(t - h(t)\bigr)\bigr) \\ &\quad + W_3 \int\limits_{t-\sigma(t)}^{t} f\bigl(x(s)\bigr)\,\mathrm{d}s + u(t), \end{aligned} \tag{20}$$

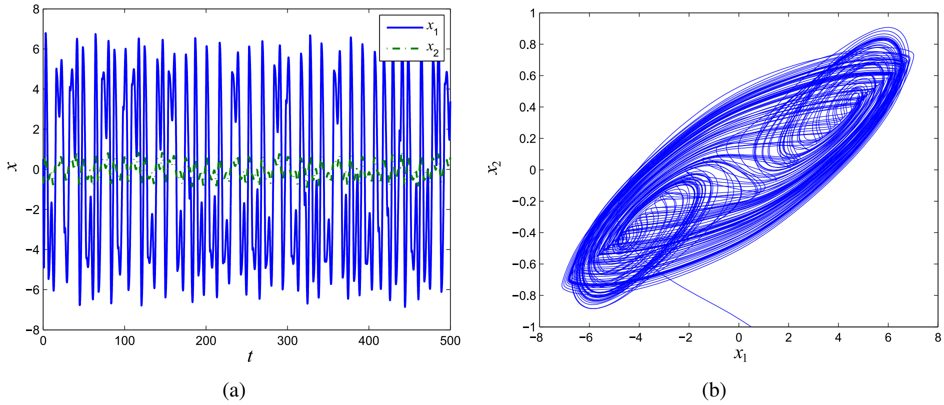


Figure 1. (a) Chaotic trajectory of (20) with initial values $\phi_1(s) = 0.5$, $\phi_2(s) = -1$. (b) Phase portrait of the strange attractor.

in which

$$\begin{aligned} &x(t) = \big(x_1(t), x_2(t)\big)^{\mathrm{T}}, \qquad f\big(x(t)\big) = \big(f_1\big(x_1(t)\big), f_2\big(x_2(t)\big)\big)^{\mathrm{T}}, \\ &f_i(x_i) = \tanh(x_i) \quad (i = 1, 2), \qquad u(t) = (0, 0)^{\mathrm{T}}, \qquad \tau(t) \equiv 0.1, \\ &h(t) = \sigma(t) = 0.1 - 0.1\mathrm{e}^{-t}, \qquad A = I_2, \\ &W_1 = \begin{bmatrix} 1.8 & 10 \\ 0.1 & 1.8 \end{bmatrix}, \qquad W_2 = \begin{bmatrix} -1.5 & 0.1 \\ 0.1 & -1.5 \end{bmatrix}, \qquad W_3 = \begin{bmatrix} 0.1 & 0.01 \\ 0.02 & 0.1 \end{bmatrix}. \end{aligned} \tag{21}$$

Through numerical simulations, we find that system (20) with (21) admits chaotic behavior. The dynamical chaotic behavior of system (20) with initial values $\phi_1(s) = 0.5$, $\phi_2(s) = -1$ is exhibited in Fig. 1.

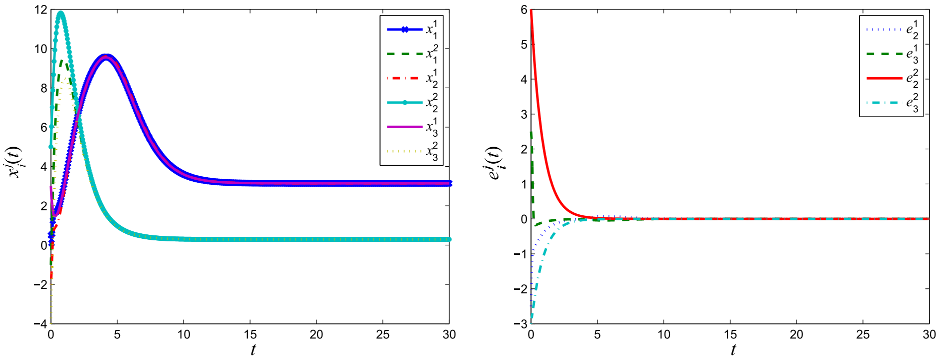
Example 2. Consider the neural network (1) with hybrid coupling. Choose the following parameters:

$$\varGamma_1 = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}, \qquad \varGamma_l = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \qquad G_k = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \tag{22}$$

$$\alpha_k = 1 \quad (l = 2, 3;\ k = 1, 2, 3). \tag{23}$$

The other functions and parameters are the same as those in (21). It is clear that $L = 1$, $\tau = h = \sigma = 0.1$. It is not difficult to verify that system (1) with (21) and (23) satisfies assumptions (H1)–(H3) and the inequality (2). According to Theorem 1, system (1) can achieve exponential synchronization. Numerical simulations illustrate the fact (Fig. 2). The states of network are shown in Fig. 2a, and the synchronization errors are illustrated in Fig. 2b, where $e_i^j(t) = x_i^j(t) - x_1^j(t)$ $(i = 2, 3;\ j = 1, 2)$. Moreover, by (2), we can calculate that the maximal allowable upper bound of the leakage delay is $\tau = 0.1415$.

If we apply the result of Theorem 3.1 in [12] to this example, and choose the decay rate $\alpha = 0$, using the LMI Toolbox in Matlab and solving the LMIs in Theorem 3.1,



(a) (b)

Figure 2. (a) Transient behaviors of the state variables $x_i^j(t)$ ($i = 1,2,3$; $j = 1,2$) in Example 2. (b) Synchronization errors $e_i^j(t)$ ($i = 2,3$; $j = 1,2$) in Example 2.

we cannot obtain a strictly feasible solution. As a result, Theorem 3.1 in [12] cannot be applied to this example.

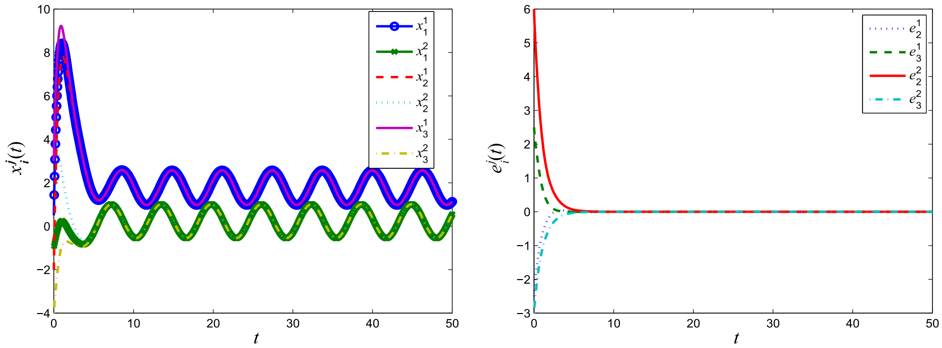
Example 3. Consider the neural network (1) with nonsymmetric coupling

$$\begin{aligned}
&f_i(x_i) = \tanh(x_i) \quad (i = 1,2), \qquad u(t) = (2 + \sin t, \cos t)^{\mathrm{T}}, \\
&\quad A = \varGamma_2 = \varGamma_3 = I_2, \qquad \varGamma_1 = 6I_2, \\
&W_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.6 \end{bmatrix}, \qquad W_2 = \begin{bmatrix} -0.5 & 0.1 \\ 0.1 & -0.4 \end{bmatrix}, \qquad W_3 = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}, \\
&G_k = \begin{bmatrix} -3 & 1 & 2 \\ 2 & -4 & 2 \\ 1 & 3 & -4 \end{bmatrix}, \qquad \alpha_k = 1 \quad (k = 1,2,3), \\
&\sigma(t) = \tau(t) = 0.1\big|\sin(t)\big|.
\end{aligned} \tag{24}$$

It is easy to verify that system (1) with (24) satisfies inequality (2). According to Theorem 1, system (1) can achieve exponential synchronization. Further, we can obtain that the conditions of Theorem 2 are satisfied if $h(t) \leqslant 0.0443$. We now choose $h(t) = 0.04|\cos(t)|$, the synchronization performance is illustrated in Fig. 3, where Fig. 3a shows the time responses of state vector of system (1), Fig. 3b depicts the synchronization errors, where $e_i^j(t) = x_i^j(t) - x_1^j(t)$ ($i = 2,3$; $j = 1,2$).

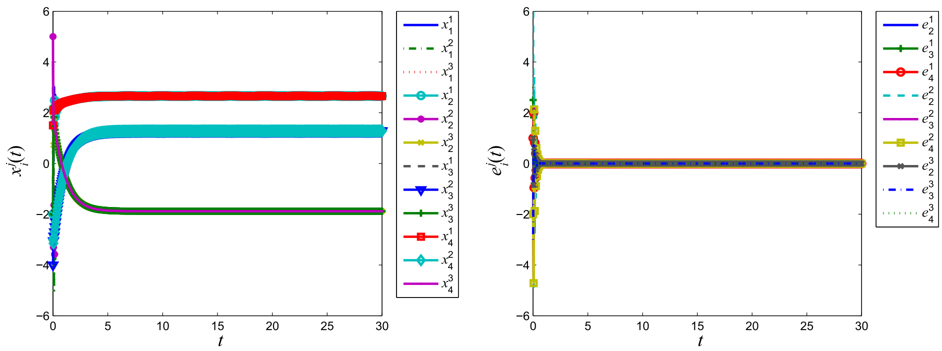
Example 4. Consider the neural network (1) with nonsymmetric and reducible coupling

$$\begin{aligned}
&f_i(x_i) = \tanh(x_i) \quad (i = 1,2), \qquad u(t) = (3,1,-2)^{\mathrm{T}}, \\
&\quad A = \varGamma_2 = \varGamma_3 = I_3, \qquad \varGamma_1 = 6I_3, \\
&\quad \alpha_k = 1 \quad (k = 1,2,3), \qquad \tau(t) \equiv 0.05, \\
&\quad \sigma(t) = 0.1\big|\sin(t)\big|, \qquad h(t) = 0.01\big|\cos(t)\big|,
\end{aligned} \tag{25a}$$



(a) (b)

Figure 3. (a) Transient behaviors of the state variables $x_i^j(t)$ ($i = 1,2,3$; $j = 1,2$) in Example 3. (b) Synchronization errors $e_i^j(t)$ ($i = 2,3$; $j = 1,2$) in Example 3.



(a) (b)

Figure 4. (a) Transient behaviors of the state variables $x_i^j(t)$ ($i = 1,2,3$; $j = 1,2$) in Example 4. (b) Synchronization errors $e_i^j(t)$ ($i = 2,3$; $j = 1,2$) in Example 4.

$$W_1 = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.1 & 0.6 & -0.2 \\ 0.2 & 0.2 & -0.3 \end{bmatrix}, \qquad W_2 = \begin{bmatrix} -0.5 & 0.1 & 0.1 \\ 0.1 & -0.4 & 0.3 \\ 0.1 & -0.3 & 0.4 \end{bmatrix},$$
$$W_3 = \begin{bmatrix} 0.1 & 0.1 & 0.5 \\ 0.2 & 0.1 & 0.2 \\ 0.3 & -0.1 & 0.2 \end{bmatrix}, \qquad G_k = \begin{bmatrix} -4 & 2 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \tag{25b}$$

It is easy to verify that system (1) with (25) satisfies inequality (13). By Theorem 2, system (1) can achieve exponential synchronization. The synchronization performance is illustrated in Fig. 4, where Fig. 4a shows the time responses of state vector of system (1), Fig. 4b depicts the synchronization errors, where $e_i^j(t) = x_i^j(t) - x_1^j(t)$ ($i = 2,3,4$; $j = 1,2,3$).

Remark 7. In Example 2 (4), system (1) satisfies the condition of Theorem 1 (2), but does not satisfy that of Theorem 2 (1). In Example 3, system (1) satisfies the conditions of both Theorems 1 and 2. Hence, as pointed out in Remark 3, it is difficult to judge which criterion is superior.

5 Conclusions

In this paper, we incorporated time-varying leakage delay, discrete and distributed time delays into an array of neural networks with nonsymmetric hybrid coupling. By employing a novel Lyapunov functional and the property of outer coupling matrices of the neural networks, sufficient conditions were obtained for the global exponential synchronization for system (1), which are closely related with the time-varying delays and the coupling structure of the networks. The maximal allowable upper bounds of the time-varying delays can be obtained guaranteeing the exponential synchronization for the neural networks. The method we adopted in this paper is different from the commonly used LMI technique, and our synchronization criteria provide a new, convenient, and efficient approach to study the synchronization for complex neural networks with hybrid coupling and mixed time delays. Some numerical examples were given to illustrate the feasibility and effectiveness of our theoretical results.

We would like to point out that the techniques used in this paper may be applied to study the synchronization for CNNs with multiple time delays and switching topology. On the other hand, we assumed the activation functions are global Lipschitz continuous, and how to tackle the synchronization problems with generalized activation functions is our future work.

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