ISSN 1392-5113 Nonlinear Analysis: Modelling and Control, 2016, Vol. 21, No. 5, 614–634
http://dx.doi.org/10.15388/NA.2016.5.4

# New fixed point results in $b$-rectangular metric spaces*

**Jamal Rezaei Roshan**[a], **Vahid Parvaneh**[b], **Zoran Kadelburg**[c], **Nawab Hussain**[d]

[a]Department of Mathematics, Qaemshahr Branch,
Islamic Azad University,
Qaemshahr, Iran
jmlroshan@gmail.com

[b]Department of Mathematics, Gilan-E-Gharb Branch,
Islamic Azad University,
Gilan-E-Gharb, Iran
zam.dalahoo@gmail.com

[c]Faculty of Mathematics, University of Belgrade,
Beograd, Serbia
kadelbur@matf.bg.ac.rs

[d]Department of Mathematics, King Abdulaziz University,
P.O. Box 80203, Jeddah, 21589, Saudi Arabia
nhusain@kau.edu.sa

**Received:** December 21, 2014 / **Revised:** May 5, 2015 / **Published online:** June 23, 2016

**Abstract.** In this work, the class of $b$-rectangular metric spaces is introduced, some fixed point results dealing with rational type contractions and almost generalized weakly contractive mappings are obtained. Certain examples are given to support the results.

**Keywords:** generalized metric space, rectangular space, $b$-metric space, Geraghty condition, altering distance function.

## 1 Introduction

The following definition was given by Branciari.

**Definition 1.** (See [5].) Let $X$ be a nonempty set, and let $d : X \times X \to [0, +\infty)$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from $x$ and $y$:

($r_1$) $d(x, y) = 0$ iff $x = y$;
($r_2$) $d(x, y) = d(y, x)$;
($r_3$) $d(x, y) \leqslant d(x, u) + d(u, v) + d(v, y)$ (rectangular inequality).

Then $(X, d)$ is called a generalized metric space (g.m.s.).

*This article was funded by the Deanship of Scientific Research (DSR) and King Abdulaziz University, Jeddah.

The above definition introduces one of the generalizations of metric spaces that became known as generalized metric spaces (g.m.s.) or rectangular spaces. Several authors (see the references cited in [19, 20]) proved various (common) fixed point results in such spaces. It may be noticed that, obviously, each metric space is a g.m.s., but a g.m.s. might not be metrizable (see [22, 30]). In particular, its topology may not be Hausdorff, as an example given in [26, 27] shows (see further Example 2). The concept of generalized metric space is similar to that of metric space. However, it is very difficult to treat this concept because generalized metric space does not necessarily have the topology, which is compatible with $d$ (see [30, Ex. 7]). So, this concept is very interesting to researchers.

On the other hand, $b$-metric spaces were introduced by Bakhtin [2] and then extensively used by Czerwik [6, 7].

**Definition 2.** (See [6].) Let $X$ be a (nonempty) set, and $s \geqslant 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is a $b$-metric if, for all $x, y, z \in X$, the following conditions are satisfied:

($b_1$) $d(x, y) = 0$ iff $x = y$;
($b_2$) $d(x, y) = d(y, x)$;
($b_3$) $d(x, z) \leqslant s[d(x, y) + d(y, z)]$.

The pair $(X, d)$ is called a $b$-metric space.

It is important to notice that $b$-metric spaces are also not metrizable. In particular, a $b$-metric might not be a continuous function of its variables (see [15, Ex. 2]). There is a vast literature concerning fixed point problems for single and multivalued mappings in $b$-metric spaces (see, e.g., [1, 14, 16, 17, 23, 24, 25, 28, 29] and the references cited therein).

In this paper, $b$-generalized metric spaces are introduced having a combination of properties of g.m.s's and $b$-metric spaces. Some fixed point results dealing with rational type contractions and almost generalized weakly contractive mappings are obtained. Examples are given to support these results.

## 2 Definition and basic properties

**Definition 3.** Let $X$ be a nonempty set, $s \geqslant 1$ be a given real number, and let $d : X \times X \to [0, +\infty)$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from $x$ and $y$:

($b_{r1}$) $d(x, y) = 0$ iff $x = y$;
($b_{r2}$) $d(x, y) = d(y, x)$;
($b_{r3}$) $d(x, y) \leqslant s[d(x, u) + d(u, v) + d(v, y)]$ ($b$-rectangular inequality).

Then $(X, d)$ is called a $b$-rectangular or a $b$-generalized metric space ($b$-g.m.s.).

The following gives some easy examples of $b$-g.m.s's.

*Example 1.* Let $(X,\rho)$ be a g.m.s., and $p \geqslant 1$ be a real number. Let $d(x,y) = (\rho(x,y))^p$. Evidently, from the convexity of function $f(x) = x^p$ for $x \geqslant 0$ and by Jensen inequality we have

$$(a+b+c)^p \leqslant 3^{p-1}\big(a^p + b^p + c^p\big)$$

for nonnegative real numbers $a$, $b$, $c$. So, it is easy to obtain that $(X,d)$ is a $b$-g.m.s with $s \leqslant 3^{p-1}$.

Convergent and Cauchy sequences in $b$-g.m.s., completeness, as well as open balls $B_r(d)$, can be introduced in a standard way. However, the following example, constructed according to [27, Ex. 1.1], shows some properties of $b$-generalized metrics not shared by standard metrics.

*Example 2.* Let $A = \{0,2\}$, $B = \{1/n\colon\ n \in \mathbb{N}\}$ and $X = A \cup B$. Define $\rho : X \times X \to [0,+\infty)$ as follows:

$$\rho(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y \text{ and } \{x,y\} \subset A \text{ or } \{x,y\} \subset B, \\ y, & x \in A,\ y \in B, \\ x, & x \in B,\ y \in A. \end{cases}$$

Then $(X,\rho)$ is a complete g.m.s. Now, taking $d(x,y) = \rho(x,y)^2$, according to Example 1, we obtain a $b$-g.m.s. $(X,d)$ with $s = 3$. It can be shown that:

1. The sequence $\{1/n\}_{n\in\mathbb{N}}$ converges to both 0 and 2;
2. $\lim_{n\to\infty} 1/n = 0$, but $1 = \lim_{n\to\infty} d(1/n,1/2) \neq d(0,1/2) = 1/4$; hence, $d$ is not a continuous function.

As shown in the previous example, a sequence in a $b$-g.m.s. may have two limits. However, there is a special situation, where this is not possible, and this will be useful in some proofs. The following lemma is a variant of [18, Lemma 1.10] and [19, Lemma 1].

**Lemma 1.** *Let $(X,d)$ be a b-g.m.s., and let $\{x_n\}$ be a Cauchy sequence in $X$ such that $x_m \neq x_n$ whenever $m \neq n$. Then $\{x_n\}$ can converge to at most one point.*

*Proof.* Suppose, to the contrary, that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} x_n = y$ and $x \neq y$. Since $x_m$ and $x_n$ are distinct elements, as well as $x$ and $y$, it is clear that there exists $\ell \in \mathbb{N}$ such that $x$ and $y$ are different from $x_n$ for all $n > l$. For $m,n > \ell$, the rectangular inequality implies that

$$d(x,y) \leqslant s\big[d(x,x_m) + d(x_m,x_n) + d(x_n,y)\big].$$

Taking the limit as $m,n \to \infty$, it follows that $d(x,y) = 0$, i.e., $x = y$. A contradiction. □

We will also need the following simple lemma about the convergent sequences in the proof of our main results.

**Lemma 2.** *Let $(X,d)$ be a b-g.m.s.*

(a) *Suppose that sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* $X$ *are such that* $x_n \to x$ *and* $y_n \to y$ *as* $n \to \infty$ *with* $x \neq y$, *and* $x_n \neq x$, $y_n \neq y$ *for* $n \in \mathbb{N}$. *Then we have*

$$\frac{1}{s}d(x,y) \leqslant \liminf_{n\to\infty} d(x_n,y_n) \leqslant \limsup_{n\to\infty} d(x_n,y_n) \leqslant sd(x,y).$$

(b) *If* $y \in X$ *and* $\{x_n\}$ *is a Cauchy sequence in* $X$ *with* $x_n \neq x_m$ *for infinitely many* $m, n \in \mathbb{N}$, $n \neq m$, *converging to* $x \neq y$, *then*

$$\frac{1}{s}d(x,y) \leqslant \liminf_{n\to\infty} d(x_n,y) \leqslant \limsup_{n\to\infty} d(x_n,y) \leqslant sd(x,y)$$

*for all* $x \in X$.

*Proof.* (a) Using the $b$-rectangular inequality in the given $b$-g.m.s, it is easy to see that

$$d(x,y) \leqslant sd(x,x_n) + sd(x_n,y_n) + sd(y_n,y)$$

and

$$d(x_n,y_n) \leqslant sd(x_n,x) + sd(x,y) + sd(y,y_n).$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality, we obtain the desired result.

(b) If $y \in X$, then, for infinitely many $m, n \in \mathbb{N}$,

$$d(x,y) \leqslant sd(x,x_n) + sd(x_n,x_m) + sd(x_m,y)$$

and

$$d(x_n,y) \leqslant sd(x_n,x_m) + sd(x_m,x) + sd(x,y). \qquad \square$$

In this paper, by an ordered $b$-generalized metric space we will understand a triple $(X, \preccurlyeq, d)$, where $(X, \preccurlyeq)$ is a partially ordered set, and $(X, d)$ is a $b$-g.m.s.

# 3 Main results

## 3.1 Results under Geraghty-type conditions

In 1973, Geraghty [13] proved a fixed point result, generalizing the Banach contraction principle. Several authors proved later various results using Geraghty-type conditions. Fixed point results of this kind in $b$-metric spaces were obtained by Đukić et al. in [9].

Following [9], for a real number $s > 1$ (the case $s = 1$ is easy and well known), let $\mathcal{F}_s$ denote the class of all functions $\beta : [0,\infty) \to [0, 1/s)$ satisfying the following condition:

$$\limsup_{n\to\infty} \beta(t_n) = \frac{1}{s} \quad \text{implies} \quad \lim_{n\to\infty} t_n = 0.$$

**Theorem 1.** *Let $(X, \preccurlyeq, d)$ be a complete ordered b-g.m.s. with parameter $s > 1$. Let $f : X \to X$ be an increasing mapping with respect to $\preccurlyeq$ such that there exists an element $x_0 \in X$ with $x_0 \preccurlyeq fx_0$. Suppose that*

$$d(fx, fy) \leqslant \beta\bigl(d(x, y)\bigr)M(x, y) \tag{1}$$

*for some $\beta \in \mathcal{F}_s$ and all comparable elements $x, y \in X$, where*

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)}, \frac{d(x, fx)d(y, fy)}{1 + d(x, y)}, \frac{d(x, fx)d(x, fy)}{1 + d(x, fy) + d(y, fx)}\right\}.$$

*If $f$ is continuous, then $f$ has a fixed point. Moreover, the set of fixed points of $f$ is well ordered if and only if $f$ has one and only one fixed point.*

*Proof.* Starting with the given $x_0$, put $x_n = f^n x_0$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x_n = fx_n$. Thus, $x_n$ is a fixed point of $f$. Therefore, we will assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since $x_0 \preccurlyeq fx_0$ and $f$ is an increasing function, we obtain by induction that

$$x_0 \preccurlyeq fx_0 \preccurlyeq f^2x_0 \preccurlyeq \cdots \preccurlyeq f^n x_0 \preccurlyeq f^{n+1}x_0 \preccurlyeq \cdots .$$

*Step I.* We will show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Since $x_n \preccurlyeq x_{n+1}$ for each $n \in \mathbb{N}$, then by (1) we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \leqslant \beta\bigl(d(x_{n-1}, x_n)\bigr)M(x_{n-1}, x_n) \\ &\leqslant \frac{1}{s}d(x_{n-1}, x_n) \leqslant d(x_{n-1}, x_n), \end{aligned} \tag{2}$$

because

$$\begin{aligned} &M(x_{n-1}, x_n) \\ &\quad = \max\biggl\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, fx_{n-1})d(x_n, fx_n)}{1 + d(fx_{n-1}, fx_n)}, \frac{d(x_{n-1}, fx_{n-1})d(x_n, fx_n)}{1 + d(x_{n-1}, x_n)}, \\ &\qquad \frac{d(x_{n-1}, fx_{n-1})d(x_{n-1}, fx_n)}{1 + d(x_{n-1}, fx_n) + d(x_n, fx_{n-1})}\biggr\} \\ &\quad = \max\biggl\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1 + d(x_n, x_{n+1})}, \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)}, \\ &\qquad \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1})}{1 + d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}\biggr\} \\ &\quad \leqslant \max\biggl\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1})}{1 + d(x_{n-1}, x_{n+1})}\biggr\} \\ &\quad \leqslant \max\bigl\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\bigr\}. \end{aligned}$$

If $\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$, then from (2) we have

$$\begin{aligned} d(x_n,x_{n+1}) &\leqslant \beta\big(d(x_n,x_{n-1})\big)M(x_n,x_{n-1}) \\ &< \frac{1}{s}d(x_n,x_{n+1}) < d(x_n,x_{n+1}), \end{aligned}$$

which is a contradiction. Hence, $\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\} = d(x_{n-1},x_n)$, so, from (2),

$$d(x_n,x_{n+1}) \leqslant \beta\big(d(x_n,x_{n-1})\big)M(x_{n-1},x_n) \leqslant d(x_{n-1},x_n).$$

Therefore, the sequence $\{d(x_n,x_{n+1})\}$ is decreasing. Then there exists $r \geqslant 0$ such that $\lim_{n\to\infty} d(x_n,x_{n+1}) = r$. Suppose that $r > 0$. Then, letting $n \to \infty$ in (2), we have $r \leqslant r/s$, which is impossible (since $s > 1$). Hence, $r = 0$, that is,

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{3}$$

*Step II.* Suppose first that $x_n = x_m$ for some $n > m$, so, we have $x_{n+1} = fx_n = fx_m = x_{m+1}$. By continuing this process, $x_{n+k} = x_{m+k}$ for $k \in \mathbb{N}$. Then (1) and Step I imply that

$$\begin{aligned} d(x_m,x_{m+1}) = d(x_n,x_{n+1}) &\leqslant \beta\big(d(x_{n-1},x_n)\big)M(x_{n-1},x_n) \\ &\leqslant \beta\big(d(x_{n-1},x_n)\big)\max\big\{d(x_{n-1},x_n),d(x_n,x_{n+1})\big\}. \end{aligned}$$

If $\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$, then we have

$$d(x_m,x_{m+1}) \leqslant \beta\big(d(x_{n-1},x_n)\big)d(x_n,x_{n+1}) < d(x_n,x_{n+1}),$$

a contradiction. If $\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\} = d(x_{n-1},x_n)$, then we have

$$\begin{aligned} d(x_m,x_{m+1}) < d(x_{n-1},x_n) &\leqslant \beta\big(d(x_{n-2},x_{n-1})\big)M(x_{n-2},x_{n-1}) \\ &\leqslant \beta\big(d(x_{n-2},x_{n-1})\big)\max\big\{d(x_{n-2},x_{n-1}),d(x_{n-1},x_n)\big\} \\ &< d(x_{n-2},x_{n-1}) < \cdots < d(x_m,x_{m+1}), \end{aligned}$$

a contradiction. Thus, in what follows, we can assume that $x_n \neq x_m$ for $n \neq m$. Then we can prove that the sequence $\{x_n\}$ is a $b$-g.m.s. Cauchy sequence.

Using the $b$-rectangular inequality and by (1), we have

$$\begin{aligned} d(x_n,x_m) &\leqslant sd(x_n,x_{n+1}) + sd(x_{n+1},x_{m+1}) + sd(x_{m+1},x_m) \\ &\leqslant sd(x_n,x_{n+1}) + s\beta\big(d(x_n,x_m)\big)M(x_n,x_m) + sd(x_m,x_{m+1}). \end{aligned} \tag{4}$$

Here

$$\begin{aligned} d(x_n,x_m) &\leqslant M(x_n,x_m) \\ &= \max\bigg\{d(x_n,x_m), \frac{d(x_n,fx_n)d(x_m,fx_m)}{1+d(fx_n,fx_m)}, \frac{d(x_n,fx_n)d(x_m,fx_m)}{1+d(x_n,x_m)}, \\ &\qquad \frac{d(x_n,fx_n)d(x_n,fx_m)}{1+d(x_n,fx_m)+d(x_m,fx_n)}\bigg\}. \end{aligned}$$

Taking the upper limit as $m, n \to \infty$ in the above inequality and using (3), we get

$$\limsup_{m,n\to\infty} M(x_n, x_m) = \limsup_{m,n\to\infty} d(x_n, x_m).$$

Hence, letting $m, n \to \infty$ in (4), we obtain

$$\limsup_{m,n\to\infty} d(x_n, x_m) \leqslant \limsup_{m,n\to\infty} s\beta\bigl(d(x_n, x_m)\bigr) \limsup_{m,n\to\infty} d(x_n, x_m).$$

Now, we claim that $\lim_{m,n\to\infty} d(x_n, x_m) = 0$. If, on the contrary, $\limsup_{m,n\to\infty} d(x_n, x_m) \neq 0$, then we get

$$\frac{1}{s} \leqslant \limsup_{m,n\to\infty} \beta\bigl(d(x_n, x_m)\bigr).$$

Since $\beta \in \mathcal{F}_s$, we deduce that $\lim_{m,n\to\infty} d(x_n, x_m) = 0$, which is a contradiction. Consequently, $\{x_n\}$ is a $b$-g.m.s. Cauchy sequence in $X$. Since $(X, d)$ is $b$-g.m.s. complete, the sequence $\{x_n\}$ $b$-g.m.s-converges to some $z \in X$, that is, $\lim_{n\to\infty} d(x_n, z) = 0$.

*Step III.* Now, we show that $z$ is a fixed point of $f$. Suppose that, on the contrary, $fz \neq z$. Then, by Lemma 1, it follows that $x_n$ differs from both $fz$ and $z$ for $n$ sufficiently large. Hence, we can apply the $b$-rectangular inequality to obtain

$$d(fz, z) \leqslant sd(fz, fx_n) + sd(fx_n, fx_{n+1}) + sd(fx_{n+1}, z).$$

Letting $n \to \infty$ and using the continuity of $f$, we have $fz = z$. Thus, $z$ is a fixed point of $f$.

Finally, suppose that the set of fixed points of $f$ is well ordered. Assume, on the contrary, that $u$ and $v$ are two fixed points of $f$ such that $u \neq v$. Then by (1) we have

$$\begin{aligned} d(u, v) = d(fu, fv) &\leqslant \beta\bigl(d(u, v)\bigr) M(u, v, a) \\ &= \beta\bigl(d(u, v)\bigr) d(u, v) < \frac{1}{s} d(u, v), \end{aligned}$$

because

$$\begin{aligned} &M(u, v) \\ &\quad = \max\left\{ d(u, v), \frac{d(u, fu)d(v, fv)}{1 + d(fu, fv)}, \frac{d(u, fu)d(v, fv)}{1 + d(u, v)}, \frac{d(u, fu)d(u, fv)}{1 + d(u, fv) + d(v, fu)} \right\} \\ &\quad = \max\bigl\{d(u, v), 0\bigr\} = d(u, v). \end{aligned}$$

So, we get $d(u, v) < d(u, v)/s$, a contradiction. Hence, $u = v$, and $f$ has a unique fixed point. Conversely, if $f$ has a unique fixed point, then the set of fixed points of $f$ is a singleton, and hence, it is well ordered. □

Note that the continuity of $f$ in Theorem 1 can be replaced by another property.

**Theorem 2.** *Under the hypotheses of Theorem* 1*, without the continuity assumption on $f$, assume that whenever $\{x_n\}$ is a nondecreasing sequence in $X$ such that $x_n \to u$, one has $x_n \preccurlyeq u$ for all $n \in \mathbb{N}$. Then $f$ has a fixed point.*

*Proof.* Repeating the proof of Theorem 1, we construct an increasing Cauchy sequence $\{x_n\}$ with $x_n \neq x_m$ for all $m \neq n$ in $X$ such that $x_n \to z \in X$. Using the assumption on $X$ we have $x_n \preccurlyeq z$. Now, we show that $fz = z$. Suppose, on the contrary, that $fz \neq z$. By (1) and Lemma 2,

$$\frac{1}{s} d(z, fz) \leqslant \limsup_{n\to\infty} d(x_{n+1}, fz) \leqslant \limsup_{n\to\infty} \beta\bigl(d(x_n, z)\bigr) \limsup_{n\to\infty} M(x_n, z),$$

where

$$\begin{aligned}\lim_{n\to\infty} M(x_n, z) = \lim_{n\to\infty} \max\biggl\{ & d(x_n, z), \frac{d(x_n, fx_n)d(z, fz)}{1 + d(fx_n, fz)}, \frac{d(x_n, fx_n)d(z, fz)}{1 + d(x_n, z)}, \\ & \frac{d(x_n, fx_n)d(x_n, fz)}{1 + d(x_n, fz) + d(z, fx_n)} \biggr\} = 0\end{aligned}$$

(see (3)). Therefore, we deduce that $d(z, fz) \leqslant 0$, a contradiction. Hence, we have $z = fz$. □

We illustrate the usefulness of the obtained results by the following example.

*Example 3.* Let $X = \{a, b, c, \delta, e\}$ be equipped with the order $\preccurlyeq$ given by

$$\preccurlyeq = \bigl\{(a,a),(b,b),(c,c),(\delta,\delta),(e,e),(\delta,a),(\delta,b),(\delta,c),(\delta,e),(a,c),(b,c),(e,c)\bigr\},$$

and let $d : X \times X \to [0, +\infty)$ be given as

$$\begin{aligned}&d(x,x) = 0 \quad \text{for } x \in X,\\ &d(x,y) = d(y,x) \quad \text{for } x, y \in X,\\ &d(a,b) = 9t,\\ &d(a,c) = d(a,e) = d(b,c) = d(c,e) = t,\\ &d(a,\delta) = d(b,\delta) = d(b,e) = d(c,\delta) = d(\delta,e) = 4t,\end{aligned}$$

where $0 < t < -(1/4)\ln(3/4)$, i.e., $\mathrm{e}^{-4t} > 3/4$. Then it is easy to check that $(X, \preccurlyeq, d)$ is a (complete) ordered $b$-g.m.s. with parameter $s = 3$. Consider the mapping $f : X \to X$ defined as

$$f = \begin{pmatrix} a & b & c & \delta & e \\ c & c & c & a & c \end{pmatrix}.$$

It is easy to check that all the conditions of Theorem 1 are fulfilled with $\beta(u) = (1/3)\mathrm{e}^{-4u}$ for $u > 0$ and $\beta(0) \in [0, 1/3)$. In particular, the contractive condition (1) is nontrivial only in the case when $x \in \{a, b, c, e\}$, $y = \delta$ (or vice versa) when it reduces to

$$\begin{aligned}d(fx, fy) = d(c, a) = t = \frac{1}{3} \cdot \frac{3}{4} \cdot 4t < \frac{1}{3}\mathrm{e}^{-4t} \cdot 4t\\ = \beta(4t)d(x, y) \leqslant \beta\bigl(d(x, y)\bigr)M(x, y).\end{aligned}$$

It follows that $f$ has a (unique) fixed point (which is $z = c$).

Note that $(X,d)$ is, obviously, neither a metric space, nor a generalized metric space (for example, $d(a,b) = 9t > t+t+t = d(a,e)+d(e,c)+d(c,b)$). It is a $b$-metric space, though, but with parameter $\sigma = 9/2 > s$ (because, for example, $d(a,b) = 9t$ and $d(a,c)+d(c,b) = 2t$). However, the conclusion about the existence of fixed point cannot be obtained using, for example, [9, Thm. 3.8] (which is a $b$-metric version of our Theorem 1). Indeed, no matter how $t > 0$ and $\alpha > 0$ are chosen, the respective Geraghty-type function $\beta(u) = (2/9)\mathrm{e}^{-\alpha u}$, $u > 0$, cannot be used to satisfy the contractive condition

$$d(fx,fy) \leqslant \beta\bigl(d(x,y)\bigr)d(x,y).$$

Namely, if it were true, for $x \in \{a,b,c,e\}$ and $y = \delta$, we would get

$$d(fx,fy) = d(c,a) = t \leqslant \frac{2}{9}\mathrm{e}^{-4\alpha t} \cdot 4t = \beta(4t) \cdot 4t = \beta\bigl(d(x,y)\bigr)d(x,y),$$

which would imply that $\mathrm{e}^{-4\alpha t} \geqslant 9/8 > 1$, a contradiction.

If in the above theorems, we take $\beta(t) = r$, where $0 \leqslant r < 1/s$, then we have the following corollary.

**Corollary 1.** *Let $(X,\preccurlyeq,d)$ be a complete ordered b-g.m.s. with parameter $s$. Let $f: X \to X$ be an increasing mapping with respect to $\preccurlyeq$ such that there exists an element $x_0 \in X$ with $x_0 \preccurlyeq fx_0$. Suppose that for some $r$ with $0 \leqslant r < 1/s$,*

$$d(fx,fy) \leqslant rM(x,y)$$

*holds for all comparable elements $x,y \in X$, where $M(x,y)$ is as in Theorem 1. If $f$ is continuous, or, for any nondecreasing sequence $\{x_n\}$ in $X$ such that $x_n \to u \in X$, one has $x_n \preccurlyeq u$ for all $n \in \mathbb{N}$, then $f$ has a fixed point. Additionally, the set of fixed points of $f$ is well ordered if and only if $f$ has one and only one fixed point.*

**Corollary 2.** *Let $(X,\preccurlyeq,d)$ be a complete ordered b-g.m.s. with parameter $s$. Let $f: X \to X$ be an increasing mapping with respect to $\preccurlyeq$ such that there exists an element $x_0 \in X$ with $x_0 \preccurlyeq fx_0$. Suppose that*

$$\begin{aligned} d(fx,fy) \leqslant{}& \alpha d(x,y) + \beta\frac{d(x,fx)d(y,fy)}{1+d(fx,fy)} + \gamma\frac{d(x,fx)d(y,fy)}{1+d(x,y)} \\ &+ \delta\frac{d(x,fx)d(x,fy)}{1+d(x,fy)+d(y,fx)} \end{aligned}$$

*for all comparable elements $x,y \in X$, where $\alpha,\beta,\gamma,\delta \geqslant 0$ and $\alpha+\beta+\gamma+\delta < 1/s$. If $f$ is continuous, or, for any nondecreasing sequence $\{x_n\}$ in $X$ such that $x_n \to u \in X$, one has $x_n \preccurlyeq u$ for all $n \in \mathbb{N}$, then $f$ has a fixed point.*

*Proof.* Since

$$\begin{aligned}
&\alpha d(x,y)+\beta\frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}+\gamma\frac{d(x,fx)d(y,fy)}{1+d(x,y)}+\delta\frac{d(x,fx)d(x,fy)}{1+d(x,fy)+d(y,fx)}\\
&\quad\leqslant(\alpha+\beta+\gamma+\delta)\max\biggl\{d(x,y),\frac{d(x,fx)d(y,fy)}{1+d(fx,fy)},\frac{d(x,fx)d(y,fy)}{1+d(x,y)},\\
&\qquad\frac{d(x,fx)d(x,fy)}{1+d(x,fy)+d(y,fx)}\biggr\},
\end{aligned}$$

taking $r=\alpha+\beta+\gamma+\delta$, all the conditions of Corollary 1 hold, and hence, $f$ has a fixed point. □

**Corollary 3.** *Let $(X,\preccurlyeq,d)$ be a complete, totally ordered b-g.m.s. with parameter $s$, and let $f:X\to X$ be an increasing mapping with respect to $\preccurlyeq$ such that there exists an element $x_0\in X$ with $x_0\preccurlyeq fx_0$. Suppose that*

$$d\bigl(f^m x,f^m y\bigr)\leqslant\beta\bigl(M(x,y)\bigr)M(x,y)$$

*for some $\beta\in\mathcal{F}_s$ and a positive integer $m$, and for all elements $x,y\in X$, where*

$$\begin{aligned}
M(x,y)=\max\biggl\{&d(x,y),\frac{d(x,f^m x)d(y,f^m y)}{1+d(f^m x,f^m y)},\frac{d(x,f^m x)d(y,f^m y)}{1+d(x,y)},\\
&\frac{d(x,f^m x)d(x,f^m y)}{1+d(x,f^m y)+d(y,f^m x)}\biggr\}.
\end{aligned}$$

*If $f^m$ is continuous, or, for any nondecreasing sequence $\{x_n\}$ in $X$ such that $x_n\to u\in X$, one has $x_n\preccurlyeq u$ for all $n\in\mathbb{N}$, then $f$ has a fixed point.*

*Proof.* Since $f$ is an increasing mapping with respect to $\preccurlyeq$, so, $f^m$ is also an increasing mapping with respect to $\preccurlyeq$, and we have $x_0\preccurlyeq fx_0\preccurlyeq f^2x_0\preccurlyeq\cdots\preccurlyeq f^m x_0$. Thus, all conditions of Theorem 1 hold for $f^m$, and it has a fixed point $z\in X$, i.e., $f^m z=z$. Now we show that $fz=z$. If, on the contrary, $fz\neq z$, then, since the order $\preccurlyeq$ is total, we have $z\prec fz$ or $fz\prec z$. If $z\prec fz$, then we have $z\prec fz\preccurlyeq f^2z\preccurlyeq\cdots\preccurlyeq f^m z$, a contradiction. Similarly, for the case $fz\prec z$, we can get a contradiction. So, $f$ has a fixed point. □

### 3.2 Results using comparison functions

Let $\varPsi$ denote the family of all nondecreasing and continuous functions $\psi:[0,\infty)\to[0,\infty)$ such that $\lim_{n\to\infty}\psi^n(t)=0$ for all $t>0$, where $\psi^n$ denotes the $n$th iterate of $\psi$. It is easy to show that, for each $\psi\in\varPsi$, the following is satisfied:

1. $\psi(t)<t$ for all $t>0$;
2. $\psi(0)=0$.

**Theorem 3.** *Let $(X,\preccurlyeq,d)$ be a complete ordered b-g.m.s. with parameter $s$. Let $f:X\to X$ be an increasing mapping with respect to $\preccurlyeq$ such that there exists an element $x_0\in X$ with $x_0\preccurlyeq fx_0$. Suppose that*

$$sd(fx,fy)\leqslant\psi\bigl(M(x,y)\bigr),\tag{5}$$

*where*

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}\right\}$$

*for some $\psi \in \Psi$ and for all elements $x, y \in X$ with $x, y$ comparable. If $f$ is continuous, then $f$ has a fixed point. In addition, the set of fixed points of $f$ is well ordered if and only if $f$ has one and only one fixed point.*

*Proof.* We will deliver the proof for the case $s > 1$ (as noted before, the case when $s = 1$, i.e., when $(X, d)$ is a g.m.s., is easy and well known).

Since $x_0 \preccurlyeq fx_0$ and $f$ is an increasing function, we obtain by induction that

$$x_0 \preccurlyeq fx_0 \preccurlyeq f^2x_0 \preccurlyeq \cdots \preccurlyeq f^nx_0 \preccurlyeq f^{n+1}x_0 \preccurlyeq \cdots .$$

By letting $x_n = f^nx_0$, we have

$$x_0 \preccurlyeq x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \cdots .$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = fx_{n_0}$, and so, we have nothing for prove. Hence, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

*Step I.* We will prove that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Using condition (5), we obtain

$$sd(x_n, x_{n+1}) = sd(fx_{n-1}, fx_n) \leqslant \psi\big(M(x_{n-1}, x_n)\big),$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, fx_{n-1})d(x_n, fx_n)}{1+d(fx_{n-1}, fx_n)}\right\} \\ &= d(x_{n-1}, x_n). \end{aligned}$$

Hence,

$$d(x_n, x_{n+1}) \leqslant \frac{1}{s}\psi\big(d(x_{n-1}, x_n)\big) < \frac{1}{s}d(x_{n-1}, x_n).$$

By induction we get that

$$d(x_n, x_{n+1}) < \frac{1}{s^n}d(x_0, x_1) \to 0 \quad \text{as } n \to \infty.$$

It follows that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{6}$$

*Step II.* Suppose that $x_n = x_m$ for some $m, n \in \mathbb{N}$, $m < n$. Then

$$d(x_m, x_{m+1}) = d(x_n, x_{n+1}) \leqslant \psi^{n-m}\big(d(x_m, x_{m+1})\big) < d(x_m, x_{m+1}),$$

a contradiction. Hence, all elements of the Picard sequence $\{x_n\}$ are distinct. Now we will prove that $\{x_n\}$ is a $b$-g.m.s. Cauchy sequence. Suppose the contrary. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that $n_i$ is the smallest index for which

$$n_i > m_i > i \quad \text{and} \quad d(x_{m_i}, x_{n_i}) \geqslant \varepsilon. \tag{7}$$

This means that

$$d(x_{m_i}, x_{n_i-2}) < \varepsilon. \tag{8}$$

From (7) and using the $b$-rectangular inequality, we get

$$\varepsilon \leqslant d(x_{m_i}, x_{n_i}) \leqslant sd(x_{m_i}, x_{m_i+1}) + sd(x_{m_i+1}, x_{n_i-1}) + sd(x_{n_i-1}, x_{n_i}).$$

Taking the upper limit as $i \to \infty$, from (6) we get

$$\frac{\varepsilon}{s} \leqslant \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i-1}). \tag{9}$$

From the definition of $M(x, y)$ we have

$$\begin{aligned} M(x_{m_i}, x_{n_i-2}) &= \max\left\{d(x_{m_i}, x_{n_i-2}), \frac{d(x_{m_i}, fx_{m_i})d(x_{n_i-2}, fx_{n_i-2})}{1 + d(fx_{m_i}, fx_{n_i-2})}\right\} \\ &= \max\left\{d(x_{m_i}, x_{n_i-2}), \frac{d(x_{m_i}, x_{m_i+1})d(x_{n_i-2}, x_{n_i-1})}{1 + d(x_{m_i+1}, x_{n_i-1})}\right\}, \end{aligned}$$

and if $i \to \infty$, by (6) and (8) we have

$$\limsup_{i\to\infty} M(x_{m_i}, x_{n_i}) \leqslant \varepsilon.$$

Now, from (5) we have

$$sd(x_{m_i+1}, x_{n_i-1}) = sd(fx_{m_i}, fx_{n_i-2}) \leqslant \psi\bigl(M(x_{m_i}, x_{n_i-2})\bigr).$$

Again, if $i \to \infty$, by (9) we obtain

$$\varepsilon = s \cdot \frac{\varepsilon}{s} \leqslant s \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i-1}) \leqslant \psi(\varepsilon) < \varepsilon,$$

which is a contradiction. Consequently, $\{x_n\}$ is a $b$-g.m.s. Cauchy sequence in $X$. Therefore, the sequence $\{x_n\}$ $b$-g.m.s. converges to some $z \in X$, that is, $\lim_{n\to\infty} d(x_n, z) = 0$ for all $a \in X$.

*Step III.* Now, we show that $z$ is a fixed point of $f$. Suppose, on the contrary, that $fz \neq z$. Then, by Lemma 1, it follows that $x_n$ differs from both $fz$ and $z$ for $n$ sufficiently large. Using the $b$-rectangular inequality, we get

$$d(z, fz) \leqslant sd(z, fx_n) + sd(fx_n, fx_{n+1}) + sd(fx_{n+1}, fz).$$

Letting $n \to \infty$ and using the continuity of $f$, we get

$$d(z, fz) \leqslant 0.$$

Hence, we have $fz = z$. Thus, $z$ is a fixed point of $f$. □

**Theorem 4.** *Under the hypotheses of Theorem* 3*, without the continuity assumption on* $f$*, assume that whenever* $\{x_n\}$ *is a nondecreasing sequence in* $X$ *such that* $x_n \to u \in X$*, one has* $x_n \preccurlyeq u$ *for all* $n \in \mathbb{N}$*. Then* $f$ *has a fixed point.*

*Proof.* Following the proof of Theorem 3, we construct an increasing sequence $\{x_n\}$ in $X$ such that $x_n \to z \in X$. Using the given assumption on $X$, we have $x_n \preccurlyeq z$. Now, we show that $z = fz$. By (5) we have

$$sd(fz, x_n) = sd(fz, fx_{n-1}) \leqslant \psi\big(M(z, x_{n-1})\big), \tag{10}$$

where

$$M(z, x_{n-1}) = \max\left\{d(z, x_{n-1}), \frac{d(z, fz)d(x_{n-1}, fx_{n-1})}{1 + d(fz, fx_{n-1})}\right\}.$$

Letting $n \to \infty$ in the above relation, we get

$$\limsup_{n\to\infty} M(z, x_{n-1}) = 0. \tag{11}$$

Again, taking the upper limit as $n \to \infty$ in (10) and using Lemma 2 and (11), we get

$$s\left[\frac{1}{s}d(z, fz)\right] \leqslant s \limsup_{n\to\infty} d(x_n, fz) \leqslant \limsup_{n\to\infty} \psi\big(M(z, x_{n-1})\big) = 0.$$

So, we get $d(z, fz) = 0$, i.e., $fz = z$. □

### 3.3 Results for almost generalized weakly contractive mappings

Recall that Khan et al. introduced in [21] the concept of an altering distance function as follows.

**Definition 4.** (See [21].) A function $\varphi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties hold:

1. $\varphi$ is continuous and non-decreasing;
2. $\varphi(t) = 0$ if and only if $t = 0$.

So-called almost contractions were introduced by Berinde in [3] (see also [4]) and later generalized and used in a lot of papers in various situations. We will use here the following variant.

**Definition 5.** Let $(X, d)$ be a $b$-g.m.s. with parameter $s$, and let $f : X \to X$ be a mapping. For $x, y \in X$, set

$$M(x, y) = \max\big\{d(x, y), d(x, fx), d(y, fy)\big\}$$

and

$$N(x, y) = \min\big\{d(x, fx), d(x, fy), d(y, fx), d(y, fy)\big\}.$$

We say that $f$ is an almost generalized $(\psi,\varphi)_s$-contractive mapping if there exist $L \geqslant 0$ and two altering distance functions $\psi$ and $\varphi$ such that

$$\psi\bigl(sd(fx,fy)\bigr) \leqslant \psi\bigl(M(x,y)\bigr) - \varphi\bigl(M(x,y)\bigr) + L\psi\bigl(N(x,y)\bigr) \tag{12}$$

for all $x, y \in X$.

Now, let us prove our new result.

**Theorem 5.** *Let $(X, \preccurlyeq, d)$ be a complete ordered b-g.m.s. with parameter $s$. Let $f: X \to X$ be a continuous mapping, which is non-decreasing with respect to $\preccurlyeq$. Suppose that $f$ satisfies condition* (12) *for all elements $x, y \in X$ with $x, y$ comparable. If there exists $x_0 \in X$ such that $x_0 \preccurlyeq fx_0$, then $f$ has a fixed point. Moreover, the set of fixed points of $f$ is well ordered if and only if $f$ has one and only one fixed point.*

*Proof.* Starting with the given $x_0$, define a sequence $\{x_n\}$ in $X$ such that $x_{n+1} = fx_n$ for all $n \geqslant 0$. Since $x_0 \preccurlyeq fx_0 = x_1$ and $f$ is non-decreasing, we have $x_1 = fx_0 \preccurlyeq x_2 = fx_1$, and, by induction,

$$x_0 \preccurlyeq x_1 \preccurlyeq \cdots \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \cdots .$$

We will again assume that $s > 1$ and that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$. By (12) we have

$$\begin{aligned}\psi\bigl(sd(x_n,x_{n+1})\bigr) &= \psi\bigl(sd(fx_{n-1},fx_n)\bigr)\\ &\leqslant \psi\bigl(M(x_{n-1},x_n)\bigr) - \varphi\bigl(M(x_{n-1},x_n)\bigr)\\ &\quad + L\psi\bigl(N(x_{n-1},x_n)\bigr),\end{aligned} \tag{13}$$

where

$$\begin{aligned}M(x_{n-1},x_n) &= \max\bigl\{d(x_{n-1},x_n), d(x_{n-1},fx_{n-1}), d(x_n,fx_n)\bigr\}\\ &= \max\bigl\{d(x_{n-1},x_n), d(x_n,x_{n+1})\bigr\}\end{aligned} \tag{14}$$

and

$$\begin{aligned}N(x_{n-1},x_n) &= \min\bigl\{d(x_{n-1},fx_{n-1}), d(x_{n-1},fx_n), d(x_n,fx_{n-1}), d(x_n,fx_n)\bigr\}\\ &= \min\bigl\{d(x_{n-1},x_n), d(x_{n-1},x_{n+1}), 0, d(x_n,x_{n+1})\bigr\} = 0.\end{aligned} \tag{15}$$

From (13)–(15) and the properties of $\psi$ and $\varphi$ we get

$$\psi\bigl(sd(x_n,x_{n+1})\bigr) < \psi\bigl(\max\bigl\{d(x_{n-1},x_n), d(x_n,x_{n+1})\bigr\}\bigr),$$

i.e.,

$$d(x_n,x_{n+1}) < \frac{1}{s}\max\bigl\{d(x_{n-1},x_n), d(x_n,x_{n+1})\bigr\}. \tag{16}$$

If

$$\max\bigl\{d(x_{n-1},x_n), d(x_n,x_{n+1})\bigr\} = d(x_n,x_{n+1}),$$

then we get $d(x_n, x_{n+1}) < d(x_n, x_{n+1})/s$, a contradiction. Hence,

$$\max\bigl\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\bigr\} = d(x_{n-1}, x_n),$$

therefore, (16) becomes

$$d(x_n, x_{n+1}) < \frac{1}{s} d(x_{n-1}, x_n).$$

Now,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{17}$$

follows in the same way as in the proof of Theorem 1.

Also, again similarly as in the proof of Theorem 1, in what follows, we can assume that $x_n \neq x_m$ for $n \neq m$.

Next, we show that $\{x_n\}$ is a $b$-g.m.s. Cauchy sequence in $X$. Suppose the contrary, that is, $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that $n_i$ is the smallest index for which

$$n_i > m_i > i \quad \text{and} \quad (x_{m_i}, x_{n_i}) \geqslant \varepsilon. \tag{18}$$

This means that

$$d(x_{m_i}, x_{n_i-2}) < \varepsilon. \tag{19}$$

Using (19) and taking the upper limit as $i \to \infty$, we get

$$\limsup_{i\to\infty} d(x_{m_i}, x_{n_i-2}) \leqslant \varepsilon. \tag{20}$$

On the other hand, we have

$$d(x_{m_i}, x_{n_i}) \leqslant sd(x_{m_i}, x_{m_i+1}) + sd(x_{m_i+1}, x_{n_i-1}) + sd(x_{n_i-1}, x_{n_i}).$$

Using (17), (18) and taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leqslant \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i-1}). \tag{21}$$

Using the $b$-rectangular inequality once again, we have the following inequalities:

$$d(x_{m_i}, x_{n_i}) \leqslant sd(x_{m_i}, x_{n_i-2}) + sd(x_{n_i-2}, x_{n_i-1}) + sd(x_{n_i-1}, x_{n_i}).$$

Using (17), (18) and taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leqslant \limsup_{i\to\infty} d(x_{m_i}, x_{n_i-2}). \tag{22}$$

From (12) we have

$$\begin{aligned} \psi\bigl(sd(x_{m_i+1}, x_{n_i-1})\bigr) &= \psi\bigl(sd(fx_{m_i}, fx_{n_i-2})\bigr) \\ &\leqslant \psi(M(x_{m_i}, x_{n_i-2})) - \varphi\bigl(M(x_{m_i}, x_{n_i-2})\bigr) \\ &\quad + L\psi\bigl(N(x_{m_i}, x_{n_i-2})\bigr), \end{aligned} \tag{23}$$

where

$$M(x_{m_i}, x_{n_i-2}) = \max\bigl\{d(x_{m_i}, x_{n_i-2}), d(x_{m_i}, fx_{m_i}), d(x_{n_i-2}, fx_{n_i-2})\bigr\}$$
$$= \max\bigl\{d(x_{m_i}, x_{n_i-2}), d(x_{m_i}, x_{m_i+1}), d(x_{n_i-2}, x_{n_i-1})\bigr\}, \tag{24}$$

and

$$N(x_{m_i}, x_{n_i-2})$$
$$= \min\bigl\{d(x_{m_i}, fx_{m_i}), d(x_{m_i}, fx_{n_i-2}), d(x_{n_i-2}, fx_{m_i}), d(x_{n_i-2}, fx_{n_i-2})\bigr\}$$
$$= \min\bigl\{d(x_{m_i}, x_{m_i+1}), d(x_{m_i}, x_{n_i-1}), d(x_{n_i-2}, x_{m_i+1}), d(x_{n_i-2}, x_{n_i-1})\bigr\}. \tag{25}$$

Taking the upper limit as $i \to \infty$ in (24) and (25) and using (17), (20), we get

$$\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-2}) = \max\Bigl\{\limsup_{i\to\infty} d(x_{m_i}, x_{n_i-2}), 0, 0\Bigr\} \leqslant \varepsilon.$$

So, we have

$$\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-2}) \leqslant \varepsilon \tag{26}$$

and

$$\limsup_{i\to\infty} N(x_{m_i}, x_{n_i-2}) = 0. \tag{27}$$

Similarly, by taking the lower limit as $i \to \infty$ in (24) and using (17), (22), we get

$$\frac{\varepsilon}{s} \leqslant \liminf_{i\to\infty} M(x_{m_i}, x_{n_i-2}). \tag{28}$$

Now, taking the upper limit as $i \to \infty$ in (23) and using (21), (26) and (27), we have

$$\psi\left(s \cdot \frac{\varepsilon}{s}\right) \leqslant \psi\Bigl(s \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i-1})\Bigr)$$
$$\leqslant \psi\Bigl(\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-2})\Bigr) - \liminf_{i\to\infty} \varphi\bigl(M(x_{m_i}, x_{n_i-2})\bigr)$$
$$\leqslant \psi(\varepsilon) - \varphi\Bigl(\liminf_{i\to\infty} M(x_{m_i}, x_{n_i-2})\Bigr),$$

which further implies that

$$\varphi\Bigl(\liminf_{i\to\infty} M(x_{m_i}, x_{n_i-2})\Bigr) = 0,$$

so, $\liminf_{i\to\infty} M(x_{m_i}, x_{n_i-2}) = 0$, a contradiction with (28). Thus, $\{x_{n+1} = fx_n\}$ is a $b$-g.m.s.-Cauchy sequence in $X$.

As $X$ is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = u.$$

Now, suppose that $f$ is continuous. We show that $u$ is a fixed point of $f$. Suppose, on the contrary, that $fu \neq u$. Then, by Lemma 1, it follows that $x_n$ differs from both $fu$ and $u$ for $n$ sufficiently large. Using the $b$-rectangular inequality, we get

$$d(u, fu) \leqslant sd(u, fx_n) + sd(fx_n, fx_{n+1}) + sd(fx_{n+1}, fu).$$

Letting $n \to \infty$, we get

$$d(u, fu) \leqslant 0.$$

So, we have $fu = u$. Thus, $u$ is a fixed point of $f$. □

As in some earlier results, the continuity of $f$ can be replaced by another condition.

**Theorem 6.** *Under the hypotheses of Theorem* 5*, without the continuity assumption on $f$, assume that whenever $\{x_n\}$ is a non-decreasing sequence in $X$ such that $x_n \to x \in X$, one has $x_n \preccurlyeq x$, for all $n \in \mathbb{N}$. Then $f$ has a fixed point in $X$.*

*Proof.* Following similar arguments to those given in the proof of Theorem 5, we construct an increasing sequence $\{x_n\}$ in $X$ such that $x_n \to u$ for some $u \in X$. Using the assumption on $X$, we have that $x_n \preccurlyeq u$ for all $n \in \mathbb{N}$. Now, we show that $fu = u$. By (12) we have

$$\begin{aligned}\psi\big(sd(x_{n+1}, fu)\big) &= \psi\big(sd(fx_n, fu)\big) \\ &\leqslant \psi\big(M(x_n, u)\big) - \varphi\big(M(x_n, u)\big) + L\psi\big(N(x_n, u)\big),\end{aligned} \tag{29}$$

where

$$\begin{aligned}M(x_n, u) &= \max\big\{d(x_n, u), d(x_n, fx_n), d(u, fu)\big\} \\ &= \max\big\{d(x_n, u), d(x_n, x_{n+1}), d(u, fu)\big\}\end{aligned} \tag{30}$$

and

$$\begin{aligned}N(x_n, u) &= \min\big\{d(x_n, fx_n), d(x_n, fu), d(u, fx_n), d(u, fu)\big\} \\ &= \min\big\{d(x_n, x_{n+1}), d(x_n, fu), d(u, x_{n+1}), d(u, fu)\big\}.\end{aligned} \tag{31}$$

Letting $n \to \infty$ in (30) and (31), we get

$$M(x_n, u) \to d(u, fu) \tag{32}$$

and

$$N(x_n, u) \to 0.$$

Again, taking the upper limit as $i \to \infty$ in (29) and using Lemma 2 and (32), we get

$$\begin{aligned}\psi\big(d(u, fu)\big) = \psi\left(s \cdot \frac{1}{s} d(u, fu)\right) &\leqslant \psi\Big(s \limsup_{n\to\infty} d(x_{n+1}, fu)\Big) \\ &\leqslant \psi\Big(\limsup_{n\to\infty} M(x_n, u)\Big) - \liminf_{n\to\infty} \varphi\big(M(x_n, u)\big) \\ &\leqslant \psi\big(d(u, fu)\big) - \varphi\Big(\liminf_{n\to\infty} M(x_n, u)\Big).\end{aligned}$$

Therefore, $\varphi(\liminf_{n\to\infty} M(x_n, u)) \leqslant 0$, equivalently, $\liminf_{n\to\infty} M(x_n, u) = 0$. Thus, from (32) we get $u = fu$, and hence, $u$ is a fixed point of $f$. □

**Corollary 4.** *Let $(X, \preccurlyeq, d)$ be a complete ordered b-g.m.s. with parameter $s$. Let $f : X \to X$ be a non-decreasing continuous mapping with respect to $\preccurlyeq$. Suppose that there exist $k \in [0,1)$ and $L \geqslant 0$ such that*

$$d(fx, fy) \leqslant \frac{k}{s} \max\bigl\{d(x,y), d(x,fx), d(y,fy)\bigr\} + \frac{L}{s} \min\bigl\{d(x,fx), d(x,fy), d(y,fx), d(y,fy)\bigr\} \tag{33}$$

*for all elements $x, y \in X$ with $x, y$ comparable. If there exists $x_0 \in X$ such that $x_0 \preccurlyeq fx_0$, then $f$ has a fixed point provided that*

(a) *$f$ is continuous, or*
(b) *for any non-decreasing sequence $\{x_n\}$ in $X$ such that $x_n \to x \in X$, we have $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$.*

*Proof.* By choosing $\psi(t) = t$ and $\varphi(t) = (1-k)t$, the desired result can be obtained from Theorems 5 and 6. □

The following example, which demonstrates the usage of previous results, is inspired by [12, Ex. 3.5].

*Example 4.* Consider the set $X = A \cup [1,2]$, where $A = \{0, 1/2, 1/3, 1/4, 1/5, 1/6\}$ endowed with the order defined as follows:

$$t \preccurlyeq \frac{1}{3} \preccurlyeq \frac{1}{6} \preccurlyeq \frac{1}{5} \preccurlyeq \frac{1}{2} \preccurlyeq 0 \preccurlyeq \frac{1}{4} \quad \text{for all } t \in [1,2].$$

Define $d : X \times X \to [0, +\infty)$ as follows:

$$\begin{gathered}
d\left(0, \frac{1}{2}\right) = d\left(\frac{1}{3}, \frac{1}{4}\right) = d\left(\frac{1}{5}, \frac{1}{6}\right) = 0.09, \\
d\left(0, \frac{1}{3}\right) = d\left(\frac{1}{2}, \frac{1}{5}\right) = d\left(\frac{1}{4}, \frac{1}{5}\right) = 0.04, \\
d\left(0, \frac{1}{4}\right) = d\left(\frac{1}{2}, \frac{1}{3}\right) = d\left(\frac{1}{4}, \frac{1}{6}\right) = 0.16, \\
d\left(0, \frac{1}{5}\right) = d\left(\frac{1}{2}, \frac{1}{6}\right) = d\left(\frac{1}{3}, \frac{1}{6}\right) = 0.25, \\
d\left(0, \frac{1}{6}\right) = d\left(\frac{1}{2}, \frac{1}{4}\right) = d\left(\frac{1}{3}, \frac{1}{5}\right) = 0.36, \\
d(x,x) = 0 \quad \text{and} \quad d(x,y) = d(y,x) \quad \text{for } x, y \in X, \\
d(x,y) = (x-y)^2 \quad \text{if } \{x,y\} \cap [1,2] \neq \emptyset.
\end{gathered}$$

Obviously, $(X, d)$ is neither a metric, nor a generalized metric space. However, it is a $b$-g.m.s. with $s = 3$ (for example, $d(1,2) = 1 = 3(1/9 + 1/9 + 1/9) = 3(d(1, 4/3) + d(4/3, 5/3) + d(5/3, 2)))$.

Consider now the mapping $f : X \to X$ given as

$$fx = \begin{cases} 1/6 & \text{if } x \in [1,2], \\ 1/4 & \text{if } x \in A \setminus \{1/3\}, \\ 1/5 & \text{if } x = 1/3. \end{cases}$$

It is easy to check that $f$ is increasing w.r.t. $\preccurlyeq$ and that there exists $x_0 \in X$ such that $x_0 \preccurlyeq fx_0$. In order to show that the contractive condition (33) is fulfilled with $k = 16/23$, denote by $R$ the right-hand side of this condition and consider the following possibilities:

1. $x \in [1,2]$, $y \in A \setminus \{1/3\}$. Then $fx = 1/6$, $fy = 1/4$, $M(x,y) \geqslant d(x,fx) \geqslant (5/6)^2 > 0.69$ and
$$d(fx,fy) = 0.16 = \frac{1}{3} \cdot \frac{16}{23} \cdot 0.69 < R.$$
2. $x \in [1,2]$, $y = 1/3$. Then $fx = 1/6$, $fy = 1/5$, $M(x,y) > 0.69$ and
$$d(fx,fy) = 0.09 < \frac{1}{3} \cdot \frac{16}{23} \cdot 0.69 < R.$$
3. $x \in A \setminus \{1/3\}$, $y = 1/3$, $fx = 1/4$, $fy = 1/5$, $M(x,y) = 0.36$ and
$$d(fx,fy) = 0.04 < \frac{1}{3} \cdot \frac{16}{23} \cdot 0.36 \leqslant R.$$

Hence, all the conditions of Corollary 4 are satisfied and $f$ has a unique fixed point (which is $u = 1/4$).

**Remark.** Added in proof: During the revision process we have learned that $b$-g.m.s. were also introduced and some fixed point results (under different conditions than here) were obtained in the (now published) papers [8, 10, 11].

**Acknowledgment.** The authors are highly indebted to the referees of this paper, who helped us to improve our presentation.

The fourth author acknowledges with thanks DSR, KAU for financial support. The third author is thankful to the Ministry of Education, Science and Technological Development of Serbia.

## References

1. T.V. An, N.V. Dung, Z. Kadelburg, S. Radenović, Various generalizations of metric spaces and fixed point theorems, *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM*, **109**:175–198, 2015.
2. I.A. Bakhtin, The contraction principle in quasimetric spaces, *Funkts. Anal.*, **30**:26–37, 1989 (in Russian).

3. V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, *Nonlinear Anal. Forum*, **9**:43–53, 2004.
4. V. Berinde, General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces, *Carpathian J. Math.*, **24**:10–19, 2008.
5. A. Branciari, A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces, *Publ. Math.*, **57**:31–37, 2000.
6. S. Czerwik, Contraction mappings in $b$-metric spaces, *Acta Math. Inform. Univ. Ostrav.*, **1**: 5–11, 1993.
7. S. Czerwik, Nonlinear set-valued contraction mappings in $b$-metric spaces, *Atti Semin. Mat. Fis. Univ. Modena*, **46**:263–276, 1998.
8. H.-S. Ding, V. Ozturk, S. Radenović, On some new fixed point results in $b$-rectangular metric spaces, *J. Nonlinear Sci. Appl.*, **8**:378–386, 2015.
9. D. Đukić, Z. Kadelburg, S. Radenović, Fixed points of Geraghty-type mappings in various generalized metric spaces, *Abstract Appl. Anal.*, **2011**:561245, 2011.
10. R. George, H.A. Nabweg, K.P. Reshma, R. Rajagopalan, Generalized cone $b$-metric spaces and contraction principles, *Mat. Vesn.*, **67**:246–257, 2015.
11. R. George, S. Radenović, K.P. Reshma, S. Shukla, Rectangular $b$-metric spaces and contraction principle, *J. Nonlinear Sci. Appl.*, **8**:1005–1013, 2015.
12. R. George, R. Rajagopalan, Common fixed point results for $\psi$-$\varphi$ contractions in rectangular metric spaces, *Bull. Math. Anal. Appl.*, **5**(1):44–52, 2013.
13. M. Geraghty, On contractive mappings, *Proc. Am. Math. Soc.*, **40**:604–608, 1973.
14. N. Hussain, D. Đorić, Z. Kadelburg, S. Radenović, Suzuki-type fixed point results in metric type spaces, *Fixed Point Theory Appl.*, **2012**:126, 2012.
15. N. Hussain, V. Parvaneh, J.R. Roshan, Z. Kadelburg, Fixed points of cyclic $(\psi, \varphi, L, A, B)$-contractive mappings in ordered $b$-metric spaces with applications, *Fixed Point Theory Appl.*, **2013**:256, 2013.
16. N. Hussain, J.R. Roshan, V. Parvaneh, M. Abbas, Common fixed point results for weak contractive mappings in ordered $b$-dislocated metric spaces with applications, *J. Inequal. Appl.*, **2013**:486, 2013.
17. N. Hussain, R. Saadati, R.P. Agarwal, On the topology and $wt$-distance on metric type spaces, *Fixed Point Appl.*, **2014**:88, 2014.
18. M. Jleli, B. Samet, The Kannan fixed point theorem in a cone rectangular metric space, *J. Nonlinear Sci. Appl.*, **2**:161–167, 2009.
19. Z. Kadelburg, S. Radenović, Fixed point results in generalized metric spaces without Hausdorff property, *Math. Sci., Springer*, **8**:125, 2014.
20. Z. Kadelburg, S. Radenović, On generalized metric spaces: a survey, *TWMS J. Pure Appl. Math.*, **5**(1):3–13, 2014.
21. M.S. Khan, M. Swaleh, S. Sessa, Fixed point theorems by altering distances between the points, *Bull. Aust. Math. Soc.*, **30**:1–9, 1984.
22. W. Kirk, N. Shahzad, *Fixed Point Theory in Distance Spaces*, Springer, 2014.

23. P. Kumam, W. Sintunavarat, The existence of fixed point theorems for partial $q$-set-valued quasi-contractions in $b$-metric spaces and related results, *Fixed Point Theory Appl.*, **2014**:226, 2014.
24. P. Kumam, W. Sintunavarat, S. Sedghi, N. Shobkolaei, Common fixed point of two $r$-weakly commuting mappings in $b$-metric spaces, *J. Funct. Spaces*, **2015**:350840, 2015.
25. S. Phiangsungnoen, W. Sintunavarat, P. Kumam, Generalized Ulam–Hyers stability for fixed point equation in $b$-metric space via $\alpha$-admissible, *Fixed Point Theory Appl.*, **2014**(188), 2014.
26. B. Samet, Discussion on "A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces" by A. Branciari, *Publ. Math.*, **76**:493–494, 2010.
27. I.R. Sarma, J.M. Rao, S.S. Rao, Contractions over generalized metric spaces, *J. Nonlinear Sci. Appl.*, **2**(3):180–182, 2009.
28. N. Shobe, S. Sedghi, J.R. Roshan, N. Hussain, Suzuki-type fixed point results in metric-like spaces, *J. Funct. Spaces*, **2013**:143686, 2013.
29. W. Sintunavarat, S. Plubtieng, P. Katchang, Fixed point result and applications on $b$-metric space endowed with an arbitrary binary relation, *Fixed Point Theory Appl*, **2013**:296, 2013.
30. T. Suzuki, Generalized metric spaces do not have the compatible topology, *Abstract Appl. Anal.*, **2014**:458098, 2014.