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Existence of positive solutions for a semipositone discrete boundary value problem

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Abstract. We investigate the existence of positive solutions for a nonlinear second-order difference equation with a linear term and a sign-changing nonlinearity, supplemented with multi-point boundary conditions. In the proof of our main results, we use the Guo–Krasnosel'skii fixed point theorem.

Keywords: difference equations, multi-point boundary conditions, positive solutions, sign-changing nonlinearity.

1 Introduction

We consider the nonlinear difference equation

$$\Delta^2 u_{n-1} - L u_n + f(n, u_n) = 0, \quad n = \overline{1, N-1}, \tag{E}$$

with the multi-point boundary conditions

$$u_0 = \sum_{i=1}^{p} a_i u_{\xi_i}, \qquad u_N = \sum_{i=1}^{q} b_i u_{\eta_i}, \tag{BC}$$

where $N \in \mathbb{N}$, $N > 2$, $p, q \in \mathbb{N}$, Δ is the forward difference operator with stepsize 1, $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_{n-1} = u_{n+1} - 2u_n + u_{n-1}$, and $n = \overline{k, m}$ means that $n = k, k+1, \dots, m$ for $k, m \in \mathbb{N}$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1, p}$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1, q}$, $1 \leqslant \xi_1 < \cdots < \xi_p \leqslant N-1$, $1 \leqslant \eta_1 < \cdots < \eta_q \leqslant N-1$, L is a positive constant, and f is a sign-changing nonlinearity.

Under some assumptions on the function f, we will investigate the existence of at least one or two positive solutions for problem (E)–(BC). Problem (E)–(BC) with $L = 0$ and a positive parameter in (E), and $a_i = 0$ for all $i = \overline{1, p}$ in (BC) was recently studied in the paper [20]. Equation (E) with $L = 0$, where the nonlinearity f may be unbounded below or nonpositive, subject to the boundary conditions $u_0 = u_1$ and $u_N = u_{N-1}$, which is a resonant problem, has been investigated in the paper [7] by transforming it into a nonresonant problem. The existence, nonexistence and multiplicity of positive solutions for difference equations and systems of difference equations with parameters or without parameters, with nonnegative or sign-changing nonlinearities, supplemented with various boundary conditions were investigated in the papers [1, 3–6, 8–10, 12, 14–17, 21–23] and the monograph [13]. For various applications of the nonlinear difference equations in many domains, we recommend the readers the monographs [2], [18] and [19].

2 Preliminary results

We study in this section the second-order difference equation

$$\Delta^2 u_{n-1} - Lu_n + y_n = 0, \quad n = \overline{1, N-1}, \tag{1}$$

with the multi-point boundary conditions (BC), where $y_n \in \mathbb{R}$ for all $n = \overline{1, N-1}$.

We denote by $A = (L + 2 + \sqrt{L^2 + 4L})/2$ the biggest solution of the characteric equation $r^2 - (L+2)r + 1 = 0$ associated to equation (1). The other solution is $1/A$. We also denote by

$$\Delta_1 = \left(\sum_{i=1}^{p} a_i A^{\xi_i} - 1\right)\left(\frac{1}{A^N} - \sum_{i=1}^{q} b_i \frac{1}{A^{\eta_i}}\right) + \left(1 - \sum_{i=1}^{p} a_i \frac{1}{A^{\xi_i}}\right)\left(A^N - \sum_{i=1}^{q} b_i A^{\eta_i}\right).$$

Lemma 1. *If $\Delta_1 \neq 0$, then the unique solution of problem* (1)–(BC) *is given by*

$$\begin{aligned}
u_n = {} & -\frac{A}{A^2 - 1}\sum_{j=1}^{n-1}\left(A^{n-j} - A^{j-n}\right) y_j \\
& + \frac{A^{n+1}}{\Delta_1 (A^2 - 1)}\left[\left(\frac{1}{A^N} - \sum_{i=1}^{q} b_i \frac{1}{A^{\eta_i}}\right)\sum_{i=1}^{p} a_i \sum_{j=1}^{\xi_i - 1}\left(A^{\xi_i - j} - A^{j - \xi_i}\right) y_j\right. \\
& + \left(1 - \sum_{i=1}^{p} a_i \frac{1}{A^{\xi_i}}\right)\sum_{j=1}^{N-1}\left(A^{N-j} - A^{j-N}\right) y_j \\
& \left. - \left(1 - \sum_{i=1}^{p} a_i \frac{1}{A^{\xi_i}}\right)\sum_{i=1}^{q} b_i \sum_{j=1}^{\eta_i - 1}\left(A^{\eta_i - j} - A^{j-\eta_i}\right) y_j\right] \\
& + \frac{1}{A^{n-1}\Delta_1 (A^2 - 1)}\left[\left(\sum_{i=1}^{p} a_i A^{\xi_i} - 1\right)\sum_{j=1}^{N-1}\left(A^{N-j} - A^{j-N}\right) y_j\right. \\
& - \left(\sum_{i=1}^{p} a_i A^{\xi_i} - 1\right)\sum_{i=1}^{q} b_i \sum_{j=1}^{\eta_i - 1}\left(A^{\eta_i - j} - A^{j-\eta_i}\right) y_j \\
& \left. - \left(A^N - \sum_{i=1}^{q} b_i A^{\eta_i}\right)\sum_{i=1}^{p} a_i \sum_{j=1}^{\xi_i - 1}\left(A^{\xi_i - j} - A^{j-\xi_i}\right) y_j\right], \quad n = \overline{0, N}.
\end{aligned} \tag{2}$$

We consider here that $\sum_{j=1}^{-1} z_j = 0$ and $\sum_{j=1}^{0} z_j = 0$.

Proof. The general solution for the homogeneous equation associated to (1), that is, the equation $\Delta^2 u_{n-1} - Lu_n = 0$, $n = \overline{1, N-1}$, is

$$u_n^h = C_1 A^n + C_2 \frac{1}{A^n}, \quad n = \overline{0, N},$$

with $C_1, C_2 \in \mathbb{R}$.

We will determine a particular solution for the nonhomogeneous equation (1) by using the variation of constants method. Namely, we will look for a solution of (1) of the form

$$\widetilde{u}_n = P_n A^n + Q_n \frac{1}{A^n}, \quad n = \overline{0, N},$$

where $P_n, Q_n \in \mathbb{R}$ for all $n = \overline{0, N}$. For the sequences $(P_n)_{n=\overline{0,N}}$ and $(Q_n)_{n=\overline{0,N}}$, we obtain the system

$$\begin{aligned}
&(P_n - P_{n-1})A^n + (Q_n - Q_{n-1})\frac{1}{A^n} = 0, \\
&(P_n - P_{n-1})\left(A^n - A^{n-1}\right) + (Q_n - Q_{n-1})\left(\frac{1}{A^n} - \frac{1}{A^{n-1}}\right) = -y_n
\end{aligned}$$

(we consider that $y_0 = 0$ and $y_N = 0$). By solving the above system in the unknowns $P_n - P_{n-1}$ and $Q_n - Q_{n-1}$ we deduce

$$P_n - P_{n-1} = -\frac{1}{A^{n-1}(A^2 - 1)} y_n, \qquad Q_n - Q_{n-1} = \frac{A^{n+1}}{A^2 - 1} y_n,$$

from which we conclude

$$P_n = -\sum_{j=1}^{n} \frac{1}{A^{j-1}(A^2 - 1)} y_j, \quad Q_n = \sum_{j=1}^{n} \frac{A^{j+1}}{A^2 - 1} y_j, \quad n = \overline{0, N}.$$

Then we obtain for $\widetilde{u}_n$ the expression

$$\widetilde{u}_n = -\frac{A}{A^2 - 1} \sum_{j=1}^{n-1} \left(A^{n-j} - A^{j-n}\right) y_j, \quad n = \overline{0, N}.$$

Therefore the general solution of equation (1) is

$$\begin{aligned}
u_n &= u_n^h + \widetilde{u}_n \\
&= C_1 A^n + C_2 \frac{1}{A^n} - \frac{A}{A^2 - 1} \sum_{j=1}^{n-1} \left(A^{n-j} - A^{j-n}\right) y_j, \quad n = \overline{0, N}. \qquad (3)
\end{aligned}$$

Now we impose to sequence $(u_n)_{n=\overline{0,N}}$ (given by (3)) the boundary conditions (BC), namely $u_0 = \sum_{i=1}^{p} a_i u_{\xi_i}$ and $u_N = \sum_{i=1}^{q} b_i u_{\eta_i}$. We obtain for the constants C_1 and C_2 the system

$$
\begin{aligned}
C_1 + C_2 &= \sum_{i=1}^{p} a_i \left[C_1 A^{\xi_i} + C_2 \frac{1}{A^{\xi_i}} - \frac{A}{A^2-1} \sum_{j=1}^{\xi_i - 1} \bigl(A^{\xi_i - j} - A^{j-\xi_i}\bigr) y_j \right], \\
C_1 A^N + C_2 \frac{1}{A^N} &- \frac{A}{A^2-1} \sum_{j=1}^{N-1} \bigl(A^{N-j} - A^{j-N}\bigr) y_j \\
&= \sum_{i=1}^{q} b_i \left[C_1 A^{\eta_i} + C_2 \frac{1}{A^{\eta_i}} - \frac{A}{A^2-1} \sum_{j=1}^{\eta_i - 1} \bigl(A^{\eta_i - j} - A^{j-\eta_i}\bigr) y_j \right]
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
&C_1 \left(\sum_{i=1}^{p} a_i A^{\xi_i} - 1 \right) - C_2 \left(1 - \sum_{i=1}^{p} a_i \frac{1}{A^{\xi_i}} \right) \\
&\quad = \frac{A}{A^2-1} \sum_{i=1}^{p} a_i \sum_{j=1}^{\xi_i - 1} \bigl(A^{\xi_i - j} - A^{j-\xi_i}\bigr) y_j, \\
&C_1 \left(A^N - \sum_{i=1}^{q} b_i A^{\eta_i} \right) + C_2 \left(\frac{1}{A^N} - \sum_{i=1}^{q} b_i \frac{1}{A^{\eta_i}} \right) \\
&\quad = \frac{A}{A^2-1} \sum_{j=1}^{N-1} \bigl(A^{N-j} - A^{j-N}\bigr) y_j \\
&\qquad - \frac{A}{A^2-1} \sum_{i=1}^{q} b_i \sum_{j=1}^{\eta_i - 1} \bigl(A^{\eta_i - j} - A^{j-\eta_i}\bigr) y_j.
\end{aligned}
$$

The determinant of the above system in the unknowns C_1 and C_2 is Δ_1, which by assumption of this lemma is different from zero. So, the above system has a unique solution given by

$$
\begin{aligned}
C_1 = \frac{A}{\Delta_1(A^2-1)} \Biggl\{ &\left(\frac{1}{A^N} - \sum_{i=1}^{q} b_i \frac{1}{A^{\eta_i}} \right) \sum_{i=1}^{p} a_i \sum_{j=1}^{\xi_i - 1} \bigl(A^{\xi_i - j} - A^{j-\xi_i}\bigr) y_j \\
&+ \left(1 - \sum_{i=1}^{p} a_i \frac{1}{A^{\xi_i}} \right) \\
&\times \left[\sum_{j=1}^{N-1} \bigl(A^{N-j} - A^{j-N}\bigr) y_j - \sum_{i=1}^{q} b_i \sum_{j=1}^{\eta_i - 1} \bigl(A^{\eta_i - j} - A^{j-\eta_i}\bigr) y_j \right] \Biggr\}, \qquad (4_1)
\end{aligned}
$$

$$
\begin{aligned}
C_2 = \frac{A}{\Delta_1(A^2-1)} \Biggl\{ & \Biggl(\sum_{i=1}^{p} a_i A^{\xi_i} - 1 \Biggr) \\
& \times \Biggl[\sum_{j=1}^{N-1} \bigl(A^{N-j} - A^{j-N}\bigr) y_j - \sum_{i=1}^{q} b_i \sum_{j=1}^{\eta_i - 1} \bigl(A^{\eta_i - j} - A^{j-\eta_i}\bigr) y_j \Biggr] \\
& - \Biggl(A^N - \sum_{i=1}^{q} b_i A^{\eta_i} \Biggr) \sum_{i=1}^{p} a_i \sum_{j=1}^{\xi_i - 1} \bigl(A^{\xi_i - j} - A^{j-\xi_i}\bigr) y_j \Biggr\}.
\end{aligned} \tag{4_2}
$$

By replacing the expressions for C_1 and C_2 from (4) in relation (3) we obtain the solution of problem (1)–(BC) given by (2). □

To express the solution of problem (1)–(BC) by using the associated Green function, we will firstly investigate the solution of problem (1)–(BC) with $a_i = 0$ for all $i = \overline{1,p}$ and $b_j = 0$ for all $j = \overline{1,q}$, and discover the corresponding Green function.

Lemma 2. *The unique solution of equation* (1) *with the boundary conditions* $u_0 = 0$ *and* $u_N = 0$ *is* $u_n^0 = \sum_{i=1}^{N-1} g(n,i) y_i$, $n = \overline{0,N}$, *where the Green function* g *is given by*

$$
\begin{aligned}
g(n,i) = {} & \frac{A}{(A^2-1)(A^N - A^{-N})} \\
& \times \begin{cases} (A^i - A^{-i})(A^{N-n} - A^{n-N}), & 1 \leqslant i < n \leqslant N, \\ (A^n - A^{-n})(A^{N-i} - A^{i-N}), & 0 \leqslant n \leqslant i \leqslant N-1. \end{cases}
\end{aligned} \tag{5}
$$

Proof. By using Lemma 1 and (2) we deduce that the solution of equation (1) with the boundary conditions $u_0 = 0$ and $u_N = 0$ is

$$
\begin{aligned}
u_n^0 = {} & -\frac{A}{A^2-1} \sum_{i=1}^{n-1} \bigl(A^{n-i} - A^{i-n}\bigr) y_i \\
& + \frac{A^{N+n+1}}{(A^2-1)(A^{2N}-1)} \sum_{i=1}^{N-1} \bigl(A^{N-i} - A^{i-N}\bigr) y_i \\
& - \frac{A^{N-n+1}}{(A^2-1)(A^{2N}-1)} \sum_{i=1}^{N-1} \bigl(A^{N-i} - A^{i-N}\bigr) y_i, \quad n = \overline{0,N}.
\end{aligned} \tag{6}
$$

Therefore by (6) we obtain

$$
\begin{aligned}
u_n^0 = {} & -\frac{A}{A^2-1} \sum_{i=1}^{n-1} \bigl(A^{n-i} - A^{i-n}\bigr) y_i \\
& + \Biggl(\frac{A^{N+n+1}}{(A^2-1)(A^{2N}-1)} - \frac{A^{N-n+1}}{(A^2-1)(A^{2N}-1)} \Biggr) \sum_{i=1}^{N-1} \bigl(A^{N-i} - A^{i-N}\bigr) y_i
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-1}\left[\frac{A^{N+n+1}-A^{N-n+1}}{(A^2-1)(A^{2N}-1)}\left(A^{N-i}-A^{i-N}\right)-\frac{A}{A^2-1}\left(A^{n-i}-A^{i-n}\right)\right]y_i\\
&\quad+\sum_{i=n}^{N-1}\frac{A^{N+1}(A^n-A^{-n})}{(A^2-1)(A^{2N}-1)}\left(A^{N-i}-A^{i-N}\right)y_i\\
&= \sum_{i=1}^{n-1}\frac{A^{N+1}(A^n-A^{-n})(A^{N-i}-A^{i-N})-A(A^{2N}-1)(A^{n-i}-A^{i-n})}{(A^2-1)(A^{2N}-1)}y_i\\
&\quad+\frac{A^{N+1}}{(A^2-1)(A^{2N}-1)}\sum_{i=n}^{N-1}\left(A^n-A^{-n}\right)\left(A^{N-i}-A^{i-N}\right)y_i\\
&= \frac{A^{N+1}}{(A^2-1)(A^{2N}-1)}\sum_{i=1}^{n-1}\left[\left(A^n-A^{-n}\right)\left(A^{N-i}-A^{i-N}\right)\right.\\
&\quad\left.-\left(A^N-A^{-N}\right)\left(A^{n-i}-A^{i-n}\right)\right]y_i\\
&\quad+\frac{A^{N+1}}{(A^2-1)(A^{2N}-1)}\sum_{i=n}^{N-1}\left(A^n-A^{-n}\right)\left(A^{N-i}-A^{i-N}\right)y_i\\
&= \frac{A^{N+1}}{(A^2-1)(A^{2N}-1)}\sum_{i=1}^{n-1}\left(A^i-A^{-i}\right)\left(A^{N-n}-A^{n-N}\right)y_i\\
&\quad+\frac{A^{N+1}}{(A^2-1)(A^{2N}-1)}\sum_{i=n}^{N-1}\left(A^n-A^{-n}\right)\left(A^{N-i}-A^{i-N}\right)y_i,\quad n=\overline{0,N}.
\end{aligned}
$$

Then we deduce $u_n^0=\sum_{i=1}^{N-1}g(n,i)y_i$ for all $n=\overline{0,N}$, where g is given by (5). □

Lemma 3. *If $\Delta_1\neq 0$, then the solution of problem* (1)–(BC) *given by* (2) *can be expressed as $u_n=\sum_{j=1}^{N-1}G(n,j)y_j$, $n=\overline{0,N}$, where the Green function G is given by*

$$
\begin{aligned}
G(n,j) &= g(n,j)\\
&\quad+\frac{1}{\Delta_1}\left[A^n\left(\sum_{i=1}^{q}b_i\frac{1}{A^{\eta_i}}-\frac{1}{A^N}\right)+\frac{1}{A^n}\left(A^N-\sum_{i=1}^{q}b_iA^{\eta_i}\right)\right]\sum_{i=1}^{p}a_ig(\xi_i,j)\\
&\quad+\frac{1}{\Delta_1}\left[A^n\left(1-\sum_{i=1}^{p}a_i\frac{1}{A^{\xi_i}}\right)+\frac{1}{A^n}\left(\sum_{i=1}^{p}a_iA^{\xi_i}-1\right)\right]\sum_{i=1}^{q}b_ig(\eta_i,j)
\end{aligned}\tag{7}
$$

for all $n=\overline{0,N}$ and $j=\overline{1,N-1}$ with g given by (5).

Proof. By (2) and using the function g given by (5), we obtain

$$
u_n=-\frac{A}{A^2-1}\sum_{j=1}^{n-1}\frac{(A^{n-j}-A^{j-n})(A^N-A^{-N})}{A^N-A^{-N}}y_j+C_1A^n+C_2\frac{1}{A^n}
$$

$$
\begin{aligned}
&= \frac{A}{(A^2-1)\big(A^N-A^{-N}\big)} \sum_{j=1}^{n-1} \big[-\big(A^n-A^{-n}\big)\big(A^{N-j}-A^{j-N}\big) \\
&\quad + \big(A^j-A^{-j}\big)\big(A^{N-n}-A^{n-N}\big)\big] y_j + C_1 A^n + C_2 \frac{1}{A^n} \\
&= \frac{A}{(A^2-1)\big(A^N-A^{-N}\big)} \left[\sum_{j=1}^{n-1} \big(A^j-A^{-j}\big)\big(A^{N-n}-A^{n-N}\big) y_j \right. \\
&\quad \left. + \sum_{j=n}^{N-1} \big(A^n-A^{-n}\big)\big(A^{N-j}-A^{j-N}\big) y_j - \sum_{j=1}^{N-1} \big(A^n-A^{-n}\big)\big(A^{N-j}-A^{j-N}\big) y_j \right] \\
&\quad + C_1 A^n + C_2 \frac{1}{A^n} \\
&= \sum_{j=1}^{n-1} g(n,j) y_j - \frac{A}{(A^2-1)\big(A^N-A^{-N}\big)} \sum_{j=1}^{N-1} \big(A^n-A^{-n}\big)\big(A^{N-j}-A^{j-N}\big) y_j \\
&\quad + C_1 A^n + C_2 \frac{1}{A^n}, \quad n = \overline{0,N},
\end{aligned}
$$

where C_1 and C_2 are defined by (4). Then we deduce

$$
\begin{aligned}
u_n &= \sum_{j=1}^{n-1} g(n,j) y_j \\
&\quad + \frac{A}{(A^2-1)\big(A^N-A^{-N}\big)\Delta_1} \left\{ \left[\left(\sum_{i=1}^{p} a_i A^{\xi_i} - 1 \right) \left(\frac{1}{A^N} - \sum_{i=1}^{q} b_i \frac{1}{A^{\eta_i}} \right) \right. \right. \\
&\quad \left. + \left(1 - \sum_{i=1}^{p} a_i \frac{1}{A^{\xi_i}} \right) \left(A^N - \sum_{i=1}^{q} b_i A^{\eta_i} \right) \right] \\
&\quad \times \left[- \sum_{j=1}^{N-1} \big(A^n-A^{-n}\big)\big(A^{N-j}-A^{j-N}\big) y_j \right] \\
&\quad + A^n \left(\frac{1}{A^N} - \sum_{i=1}^{q} b_i \frac{1}{A^{\eta_i}} \right) \big(A^N-A^{-N}\big) \sum_{i=1}^{p} a_i \sum_{j=1}^{\xi_i-1} \big(A^{\xi_i-j}-A^{j-\xi_i}\big) y_j \\
&\quad + A^n \left(1 - \sum_{i=1}^{p} a_i \frac{1}{A^{\xi_i}} \right) \big(A^N-A^{-N}\big) \sum_{j=1}^{N-1} \big(A^{N-j}-A^{j-N}\big) y_j \\
&\quad - A^n \left(1 - \sum_{i=1}^{p} a_i \frac{1}{A^{\xi_i}} \right) \big(A^N-A^{-N}\big) \sum_{i=1}^{q} b_i \sum_{j=1}^{\eta_i-1} \big(A^{\eta_i-j}-A^{j-\eta_i}\big) y_j \\
&\quad + \frac{1}{A^n} \left(\sum_{i=1}^{p} a_i A^{\xi_i} - 1 \right) \big(A^N-A^{-N}\big) \sum_{j=1}^{N-1} \big(A^{N-j}-A^{j-N}\big) y_j
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{A^n}\left(\sum_{i=1}^{p} a_i A^{\xi_i} - 1\right)\left(A^N - A^{-N}\right)\sum_{i=1}^{q} b_i \sum_{j=1}^{\eta_i - 1}\left(A^{\eta_i - j} - A^{j-\eta_i}\right)y_j \\
&- \frac{1}{A^n}\left(A^N - \sum_{i=1}^{q} b_i A^{\eta_i}\right)\left(A^N - A^{-N}\right)\sum_{i=1}^{p} a_i \sum_{j=1}^{\xi_i - 1}\left(A^{\xi_i - j} - A^{j-\xi_i}\right)y_j \Bigg\} \\
= &\sum_{j=1}^{n-1} g(n,j)y_j + \frac{A}{(A^2-1)\left(A^N - A^{-N}\right)\Delta_1}\Bigg\{-\left(\sum_{i=1}^{p} a_i A^{\xi_i} - 1\right) \\
&\times \left(\frac{1}{A^N} - \sum_{i=1}^{q} b_i \frac{1}{A^{\eta_i}}\right)\left(A^n - A^{-n}\right)\sum_{j=1}^{N-1}\left(A^{N-j} - A^{j-N}\right)y_j \\
&- \left(1 - \sum_{i=1}^{p} a_i \frac{1}{A^{\xi_i}}\right)\left(A^N - \sum_{i=1}^{q} b_i A^{\eta_i}\right)\left(A^n - A^{-n}\right)\sum_{j=1}^{N-1}\left(A^{N-j} - A^{j-N}\right)y_j \\
&+ A^n\left(1 - \sum_{i=1}^{p} a_i \frac{1}{A^{\xi_i}}\right)\left(A^N - A^{-N}\right)\sum_{j=1}^{N-1}\left(A^{N-j} - A^{j-N}\right)y_j \\
&+ \frac{1}{A^n}\left(\sum_{i=1}^{p} a_i A^{\xi_i} - 1\right)\left(A^N - A^{-N}\right)\sum_{j=1}^{N-1}\left(A^{N-j} - A^{j-N}\right)y_j \\
&+ \left[A^n\left(\frac{1}{A^N} - \sum_{i=1}^{q} b_i \frac{1}{A^{\eta_i}}\right) - \frac{1}{A^n}\left(A^N - \sum_{i=1}^{q} b_i A^{\eta_i}\right)\right] \\
&\times \sum_{i=1}^{p} a_i \sum_{j=1}^{\xi_i - 1}\left(A^N - A^{-N}\right)\left(A^{\xi_i - j} - A^{j-\xi_i}\right)y_j \\
&+ \left[-A^n\left(1 - \sum_{i=1}^{p} a_i \frac{1}{A^{\xi_i}}\right) - \frac{1}{A^n}\left(\sum_{i=1}^{p} a_i A^{\xi_i} - 1\right)\right] \\
&\times \sum_{i=1}^{q} b_i \sum_{j=1}^{\eta_i - 1}\left(A^N - A^{-N}\right)\left(A^{\eta_i - j} - A^{j-\eta_i}\right)y_j \Bigg\} \\
= &\sum_{j=1}^{n-1} g(n,j)y_j + \frac{A}{(A^2-1)\left(A^N - A^{-N}\right)\Delta_1}\Bigg\{-\left(\sum_{i=1}^{p} a_i A^{\xi_i} - 1\right) \\
&\times \left(\frac{1}{A^N} - \sum_{i=1}^{q} b_i \frac{1}{A^{\eta_i}}\right)\left(A^n - A^{-n}\right)\sum_{j=1}^{N-1}\left(A^{N-j} - A^{j-N}\right)y_j \\
&- \left(1 - \sum_{i=1}^{p} a_i \frac{1}{A^{\xi_i}}\right)\left(A^N - \sum_{i=1}^{q} b_i A^{\eta_i}\right)\left(A^n - A^{-n}\right)\sum_{j=1}^{N-1}\left(A^{N-j} - A^{j-N}\right)y_j
\end{aligned}
$$

$$
\begin{aligned}
&+ A^n\left(1-\sum_{i=1}^{p}a_i\frac{1}{A^{\xi_i}}\right)\left(A^N-A^{-N}\right)\sum_{j=1}^{N-1}\left(A^{N-j}-A^{j-N}\right)y_j\\
&+ \frac{1}{A^n}\left(\sum_{i=1}^{p}a_iA^{\xi_i}-1\right)\left(A^N-A^{-N}\right)\sum_{j=1}^{N-1}\left(A^{N-j}-A^{j-N}\right)y_j\\
&+ \left[A^n\left(\frac{1}{A^N}-\sum_{i=1}^{q}b_i\frac{1}{A^{\eta_i}}\right)-\frac{1}{A^n}\left(A^N-\sum_{i=1}^{q}[2pt]b_iA^{\eta_i}\right)\right]\\
&\times\sum_{i=1}^{p}a_i\sum_{j=1}^{\xi_i-1}\left[\left(A^{\xi_i}-A^{-\xi_i}\right)\left(A^{N-j}-A^{j-N}\right)\right.\\
&\left.-\left(A^j-A^{-j}\right)\left(A^{N-\xi_i}-A^{\xi_i-N}\right)\right]y_j\\
&+ \left[-A^n\left(1-\sum_{i=1}^{p}a_i\frac{1}{A^{\xi_i}}\right)-\frac{1}{A^n}\left(\sum_{i=1}^{p}a_iA^{\xi_i}-1\right)\right]\\
&\times\sum_{i=1}^{q}b_i\sum_{j=1}^{\eta_i-1}\left[\left(A^{\eta_i}-A^{-\eta_i}\right)\left(A^{N-j}-A^{j-N}\right)\right.\\
&\left.\left.-\left(A^j-A^{-j}\right)\left(A^{N-\eta_i}-A^{\eta_i-N}\right)\right]y_j\right\},\quad n=\overline{0,N}.
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
u_n=&\sum_{j=1}^{n-1}g(n,j)y_j+\frac{A}{(A^2-1)\left(A^N-A^{-N}\right)\varDelta_1}\left\{-\left(\sum_{i=1}^{p}a_iA^{\xi_i}-1\right)\right.\\
&\times\left(\frac{1}{A^N}-\sum_{i=1}^{q}b_i\frac{1}{A^{\eta_i}}\right)\left(A^n-A^{-n}\right)\sum_{j=1}^{N-1}\left(A^{N-j}-A^{j-N}\right)y_j\\
&-\left(1-\sum_{i=1}^{p}a_i\frac{1}{A^{\xi_i}}\right)\left(A^N-\sum_{i=1}^{q}b_iA^{\eta_i}\right)\left(A^n-A^{-n}\right)\sum_{j=1}^{N-1}\left(A^{N-j}-A^{j-N}\right)y_j\\
&+A^n\left(1-\sum_{i=1}^{p}a_i\frac{1}{A^{\xi_i}}\right)\left(A^N-A^{-N}\right)\sum_{j=1}^{N-1}\left(A^{N-j}-A^{j-N}\right)y_j\\
&+\frac{1}{A^n}\left(\sum_{i=1}^{p}a_iA^{\xi_i}-1\right)\left(A^N-A^{-N}\right)\sum_{j=1}^{N-1}\left(A^{N-j}-A^{j-N}\right)y_j\\
&+\left[A^n\left(\frac{1}{A^N}-\sum_{i=1}^{q}b_i\frac{1}{A^{\eta_i}}\right)-\frac{1}{A^n}\left(A^N-\sum_{i=1}^{q}b_iA^{\eta_i}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\times \Biggl\{ \sum_{i=1}^{p} a_i \Biggl[\sum_{j=1}^{\xi_i-1} \bigl(-\bigl(A^j - A^{-j}\bigr)\bigl(A^{N-\xi_i} - A^{\xi_i-N}\bigr)\bigr) y_j \\
&- \sum_{j=\xi_i}^{N-1} \bigl(A^{\xi_i} - A^{-\xi_i}\bigr)\bigl(A^{N-j} - A^{j-N}\bigr) y_j \Biggr] \\
&+ \sum_{i=1}^{P} a_i \sum_{j=1}^{N-1} \bigl(A^{\xi_i} - A^{-\xi_i}\bigr)\bigl(A^{N-j} - A^{j-N}\bigr) y_j \Biggr\} \\
&+ \Biggl[-A^n \Biggl(1 - \sum_{i=1}^{p} a_i \frac{1}{A^{\xi_i}}\Biggr) - \frac{1}{A^n} \Biggl(\sum_{i=1}^{p} a_i A^{\xi_i} - 1\Biggr) \Biggr] \\
&\times \Biggl\{ \sum_{i=1}^{q} b_i \Biggl[\sum_{j=1}^{\eta_i-1} \bigl(-\bigl(A^j - A^{-j}\bigr)\bigl(A^{N-\eta_i} - A^{\eta_i-N}\bigr)\bigr) y_j \\
&- \sum_{j=\eta_i}^{N-1} \bigl(A^{\eta_i} - A^{-\eta_i}\bigr)\bigl(A^{N-j} - A^{j-N}\bigr) y_j \Biggr] \\
&+ \sum_{i=1}^{q} b_i \sum_{j=1}^{N-1} \bigl(A^{\eta_i} - A^{-\eta_i}\bigr)\bigl(A^{N-j} - A^{j-N}\bigr) y_j \Biggr\} \Biggr\} \\
= &\sum_{j=1}^{N-1} g(n,j) y_j + \frac{1}{\Delta_1} \Biggl[-A^n \Biggl(\frac{1}{A^N} - \sum_{i=1}^{q} b_i \frac{1}{A^{\eta_i}}\Biggr) + \frac{1}{A^n} \Biggl(A^N - \sum_{i=1}^{q} b_i A^{\eta_i}\Biggr) \Biggr] \\
&\times \sum_{j=1}^{N-1} \sum_{i=1}^{p} a_i g(\xi_i, j) y_j \\
&+ \frac{1}{\Delta_1} \Biggl[A^n \Biggl(1 - \sum_{i=1}^{p} a_i \frac{1}{A^{\xi_i}}\Biggr) + \frac{1}{A^n} \Biggl(\sum_{i=1}^{p} a_i A^{\xi_i} - 1\Biggr) \Biggr] \sum_{j=1}^{N-1} \sum_{i=1}^{q} b_i g(\eta_i, j) y_j \\
&+ \frac{A}{(A^2-1)(A^N - A^{-N})\Delta_1} \Biggl\{ -\Biggl(\sum_{i=1}^{p} a_i A^{\xi_i} - 1\Biggr) \\
&\times \Biggl(\frac{1}{A^N} - \sum_{i=1}^{q} b_i \frac{1}{A^{\eta_i}}\Biggr) \bigl(A^n - A^{-n}\bigr) \sum_{j=1}^{N-1} \bigl(A^{N-j} - A^{j-N}\bigr) y_j \\
&- \Biggl(1 - \sum_{i=1}^{p} a_i \frac{1}{A^{\xi_i}}\Biggr) \Biggl(A^N - \sum_{i=1}^{q} b_i A^{\eta_i}\Biggr) \bigl(A^n - A^{-n}\bigr) \sum_{j=1}^{N-1} \bigl(A^{N-j} - A^{j-N}\bigr) y_j \\
&+ A^n \Biggl(1 - \sum_{i=1}^{p} a_i \frac{1}{A^{\xi_i}}\Biggr) \bigl(A^N - A^{-N}\bigr) \sum_{j=1}^{N-1} \bigl(A^{N-j} - A^{j-N}\bigr) y_j
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{A^n}\left(\sum_{i=1}^{p} a_i A^{\xi_i}-1\right)\left(A^N-A^{-N}\right)\sum_{j=1}^{N-1}\left(A^{N-j}-A^{j-N}\right)y_j\\
&+A^n\left(\frac{1}{A^N}-\sum_{i=1}^{q} b_i\frac{1}{A^{\eta_i}}\right)\sum_{i=1}^{p} a_i\left(A^{\xi_i}-A^{-\xi_i}\right)\sum_{j=1}^{N-1}\left(A^{N-j}-A^{j-N}\right)y_j\\
&-\frac{1}{A^n}\left(A^N-\sum_{i=1}^{q} b_i A^{\eta_i}\right)\sum_{i=1}^{p} a_i\left(A^{\xi_i}-A^{-\xi_i}\right)\sum_{j=1}^{N-1}\left(A^{N-j}-A^{j-N}\right)y_j\\
&-A^n\left(1-\sum_{i=1}^{p} a_i\frac{1}{A^{\xi_i}}\right)\sum_{i=1}^{q} b_i\left(A^{\eta_i}-A^{-\eta_i}\right)\sum_{j=1}^{N-1}\left(A^{N-j}-A^{j-N}\right)y_j\\
&-\frac{1}{A^n}\left(\sum_{i=1}^{p} a_i A^{\xi_i}-1\right)\sum_{i=1}^{q} b_i\left(A^{\eta_i}-A^{-\eta_i}\right)\sum_{j=1}^{N-1}\left(A^{N-j}-A^{j-N}\right)y_j\Bigg\}\\
&=\sum_{j=1}^{N-1}\Bigg\{g(n,j)+\frac{1}{\Delta_1}\left[A^n\left(\sum_{i=1}^{q} b_i\frac{1}{A^{\eta_i}}-\frac{1}{A^N}\right)+\frac{1}{A^n}\left(A^N-\sum_{i=1}^{q} b_i A^{\eta_i}\right)\right]\\
&\quad\times\sum_{i=1}^{p} a_i g(\xi_i,j)\\
&\quad+\frac{1}{\Delta_1}\left[A^n\left(1-\sum_{i=1}^{p} a_i\frac{1}{A^{\xi_i}}\right)+\frac{1}{A^n}\left(\sum_{i=1}^{p} a_i A^{\xi_i}-1\right)\right]\sum_{i=1}^{q} b_i g(\eta_i,j)\Bigg\}y_j\\
&=\sum_{j=1}^{N-1} G(n,j)y_j,\quad n=\overline{0,N},
\end{aligned}
$$

where the Green function G is given by (7). □

Lemma 4. *The Green function g given by* (5) *satisfies the inequalities*

$$k(n)h(j)\leqslant g(n,j)\leqslant h(j)\quad n=\overline{0,N},\ j=\overline{1,N-1},$$

where

$$
\begin{aligned}
h(j)&=g(j,j)\\
&=\frac{A}{(A^2-1)\left(A^N-A^{-N}\right)}\left(A^j-A^{-j}\right)\left(A^{N-j}-A^{j-N}\right),\quad j=\overline{1,N-1},
\end{aligned}
$$

and

$$k(n)=\frac{1}{A^{N-1}-A^{1-N}}\min\left\{A^n-A^{-n},\ A^{N-n}-A^{n-N}\right\},\quad n=\overline{0,N}.$$

Proof. Because the function $\varphi(x)=A^x-A^{-x}$, $x\geqslant 0$, is strictly increasing, with $\varphi(0)=0$, we deduce that $g(n,j)\leqslant g(j,j)=h(j)$ for all $n=\overline{0,N}$, $j=\overline{1,N-1}$.

For $1 \leqslant j < n \leqslant N$, we have

$$\frac{g(n,j)}{g(j,j)} = \frac{A^{N-n} - A^{n-N}}{A^{N-j} - A^{j-N}} \geqslant \frac{A^{N-n} - A^{n-N}}{A^{N-1} - A^{1-N}},$$

and for $0 \leqslant n \leqslant j \leqslant N-1$, we have

$$\frac{g(n,j)}{g(j,j)} = \frac{A^{n} - A^{-n}}{A^{j} - A^{-j}} \geqslant \frac{A^{n} - A^{-n}}{A^{N-1} - A^{1-N}},$$

and then we obtain $g(n,j) \geqslant k(n)h(j)$ for all $n = \overline{0,N}$, $j = \overline{1,N-1}$, where $k(n) = (1/(A^{N-1} - A^{1-N}))\min\{A^n - A^{-n},\ A^{N-n} - A^{n-N}\}$ for all $n = \overline{0,N}$. □

Remark 1. Because $N > 2$, we have $k(n) < 1$ for all $n = \overline{0,N}$, and then $\min_{n=\overline{1,N-1}} k(n) \in (0,1)$.

Lemma 5. *We assume that $a_i \geqslant 0$ for all $i = \overline{1,p}$, $b_j \geqslant 0$ for all $j = \overline{1,q}$, $\sum_{i=1}^{p} a_i A^{\xi_i} \geqslant 1$, $\sum_{i=1}^{p} a_i/A^{\xi_i} \leqslant 1$, $\sum_{i=1}^{q} b_i/A^{\eta_i} \geqslant 1/A^N$, $\sum_{i=1}^{q} b_i A^{\eta_i} \leqslant A^N$, and $\Delta_1 > 0$. Then the Green function G given by* (7) *satisfies the inequalities*

$$k(n)h(j) \leqslant G(n,j) \leqslant \Lambda h(j), \quad n = \overline{0,N},\ j = \overline{1,N-1},$$

where

$$\begin{aligned}\Lambda = 1 &+ \frac{1}{\Delta_1}\left[A^N\left(\sum_{i=1}^{q} b_i \frac{1}{A^{\eta_i}} - \frac{1}{A^N}\right) + A^N - \sum_{i=1}^{q} b_i A^{\eta_i}\right]\sum_{i=1}^{p} a_i \\ &+ \frac{1}{\Delta_1}\left[A^N\left(1 - \sum_{i=1}^{p} a_i \frac{1}{A^{\xi_i}}\right) + \sum_{i=1}^{p} a_i A^{\xi_i} - 1\right]\sum_{i=1}^{q} b_i.\end{aligned}$$

Proof. Under the assumptions of this lemma, by using Lemma 4 we have $G(n,j) \geqslant g(n,j) \geqslant k(n)h(j)$ for all $n = \overline{0,N}$, $j = \overline{1,N-1}$. By using again Lemma 4 we obtain

$$\begin{aligned}G(n,j) \leqslant h(j) &+ \frac{1}{\Delta_1}\left[A^N\left(\sum_{i=1}^{q} b_i \frac{1}{A^{\eta_i}} - \frac{1}{A^N}\right) + A^N - \sum_{i=1}^{q} b_i A^{\eta_i}\right]\sum_{i=1}^{p} a_i h(j) \\ &+ \frac{1}{\Delta_1}\left[A^N\left(1 - \sum_{i=1}^{p} a_i \frac{1}{A^{\xi_i}}\right) + \sum_{i=1}^{p} a_i A^{\xi_i} - 1\right]\sum_{i=1}^{q} b_i h(j) \\ &= \Lambda h(j), \quad n = \overline{0,N},\ j = \overline{1,N-1}.\end{aligned}$$

□

Lemma 6. *Under the assumptions of Lemma* 5, *the solution u_n, $n = \overline{0,N}$, of problem* (1)–(BC) *satisfies the inequality $u_n \geqslant (1/\Lambda)k(n)u_m$ for all $n,m = \overline{0,N}$.*

Proof. By using Lemma 3 and Lemma 5 we obtain

$$\begin{aligned}u_n &= \sum_{j=1}^{N-1} G(n,j)y_j \geqslant \sum_{j=1}^{N-1} k(n)h(j)y_j = k(n)\sum_{j=1}^{N-1} h(j)y_j \\ &\geqslant \frac{k(n)}{\Lambda}\sum_{j=1}^{N-1} G(m,j)y_j = \frac{k(n)}{\Lambda}u_m, \quad n,m = \overline{0,N}.\end{aligned}$$

□

In the proof of our main results, we will use the Guo–Krasnosel'skii fixed point theorem presented below (see [11]).

Theorem 1. *Let X be a Banach space and let $C \subset X$ be a cone in X. Assume Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $\mathcal{A} : C \cap (\overline{\Omega}_2 \setminus \Omega_1) \to C$ be a completely continuous operator such that either*

(i) $\|\mathcal{A}u\| \leqslant \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \geqslant \|u\|$, $u \in C \cap \partial\Omega_2$, *or*
(ii) $\|\mathcal{A}u\| \geqslant \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \leqslant \|u\|$, $u \in C \cap \partial\Omega_2$.

Then $\mathcal{A}$ has a fixed point in $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Existence of positive solutions

In this section, we will investigate the existence of at least one or two positive solutions for problem (E)–(BC). We present now the basic assumptions that we will use in the sequel.

(H1) $a_i \geqslant 0$ for all $i = \overline{1,p}$, $b_j \geqslant 0$ for all $j = \overline{1,q}$, $\sum_{i=1}^p a_i A^{\xi_i} \geqslant 1$, $\sum_{i=1}^p a_i/A^{\xi_i} \leqslant 1$, $\sum_{i=1}^q b_i/A^{\eta_i} \geqslant 1/A^N$, $\sum_{i=1}^q b_i A^{\eta_i} \leqslant A^N$, $\Delta_1 > 0$ and $L > 0$.
(H2) The function $f : \{1,\dots,N-1\} \times \mathbb{R}_+ \to \mathbb{R}$ is continuous, and there exist $c_n \geqslant 0$, $n = \overline{1,N-1}$, with $\sum_{i=1}^{N-1} c_i > 0$ such that $f(n,u) \geqslant -c_n$ for all $n = \overline{1,N-1}$, $u \in \mathbb{R}_+$ ($\mathbb{R}_+ = [0,\infty)$).

We remark that, under assumption (H2), the nonlinearity in equation (E), namely $-Lu + f(n,u)$, may be unbounded below.

We denote by $(r_n)_{n=\overline{0,N}}$ the solution of problem (1)–(BC) with $y_n = c_n$ for all $n = \overline{1,N-1}$, namely the solution of problem

$$\Delta^2 u_{n-1} - Lu_n + c_n = 0, \quad n = \overline{1,N-1},$$
$$u_0 = \sum_{i=1}^p a_i u_{\xi_i}, \qquad u_N = \sum_{i=1}^q b_i u_{\eta_i},$$

where c_n, $n = \overline{0,N}$, are given in (H2). So, by using the Green function G and Lemma 3 we have $r_n = \sum_{j=1}^{N-1} G(n,j)c_j$ for all $n = \overline{0,N}$.

We consider now the difference equation

$$\Delta^2 v_{n-1} - Lv_n + f\big(n, (v_n - r_n)^*\big) + c_n = 0, \quad n = \overline{1,N-1}, \tag{8}$$

with the multi-point boundary conditions

$$v_0 = \sum_{i=1}^p a_i v_{\xi_i}, \qquad v_N = \sum_{i=1}^q b_i v_{\eta_i}, \tag{9}$$

where $z^* = z$ if $z \geqslant 0$ and $z^* = 0$ if $z < 0$.

We obtain easily the following lemma.

Lemma 7. *The sequence* $(u_n)_{n=\overline{0,N}}$ *is a positive solution of problem* (E)–(BC) ($u_n > 0$ *for all* $n = \overline{0,N}$) *if and only if* $(v_n)_{n=\overline{0,N}}$, $v_n = u_n + r_n$, $n = \overline{0,N}$, *is a solution of the boundary value problem* (8)–(9) *with* $v_n > r_n$ *for all* $n = \overline{0,N}$.

By using Lemma 3 we also obtain the following result.

Lemma 8. *The sequence* $(v_n)_{n=\overline{0,N}}$ *is a solution of problem* (8)–(9) *if and only if* $(v_n)_{n=\overline{0,N}}$ *is a solution of the problem*

$$v_n = \sum_{j=1}^{N-1} G(n,j)\bigl(f\bigl(j,(v_j - r_j)^*\bigr) + c_j\bigr), \quad n = \overline{0,N}. \tag{10}$$

We consider the Banach space $X = \mathbb{R}^{N+1} = \{v = (v_n)_{n=\overline{0,N}},\ v_n \in \mathbb{R},\ n = \overline{0,N}\}$ endowed with the maximum norm $\|v\| = \max_{n=\overline{0,N}} |v_n|$, and we define the operator $Q : X \to X$, $Q(v) = (Q_n(v))_{n=\overline{0,N}}$, where

$$Q_n(v) = \sum_{j=1}^{N-1} G(n,j)\bigl(f\bigl(j,(v_j - r_j)^*\bigr) + c_j\bigr), \quad n = \overline{0,N},$$
$$v = (v_n)_{n=\overline{0,N}}.$$

By (H2) the operator Q is completely continuous. We also define the cone

$$P = \left\{v \in X\colon\ v = (v_n)_{n=\overline{0,N}},\ v_n \geqslant \frac{k(n)}{\varLambda}\|v\|\ \ \forall n = \overline{0,N}\right\}.$$

By using Lemma 6 we deduce that $Q(P) \subset P$. In addition, we have the following lemma.

Lemma 9. *The sequence* $(v_n)_{n=\overline{0,N}}$ *is a solution of problem* (10) *if and only if* $(v_n)_{n=\overline{0,N}}$ *is a fixed point of operator* Q.

So, the existence of positive solutions of problem (E)–(BC) is reduced in three steps (Lemmas 7–9) to the fixed point problem of operator Q in the cone P.

Let $k_0 = \min\{k(n),\ n = \overline{1,N-1}\}$. By Remark 1 we have $k_0 \in (0,1)$. We define the functions

$$\varPhi(r) = \max\left\{f(n,u) + c_n,\ n = \overline{1,N-1},\ u \in \left[\frac{k_0}{\varLambda}r - \varLambda\sum_{j=1}^{N-1} h(j)c_j,\ r\right]\right\},$$
$$\varPsi(r) = \min\left\{f(n,u) + c_n,\ n = \overline{1,N-1},\ u \in \left[\frac{k_0}{\varLambda}r - \varLambda\sum_{j=1}^{N-1} h(j)c_j,\ r\right]\right\}$$

for $r > (\varLambda^2/k_0)\sum_{j=1}^{N-1} h(j)c_j$.

Theorem 2. *We assume that* (H1), (H2) *and*

(H3) *There exist* $r, R > 0$ *such that* $(\varLambda^2/k_0)\sum_{j=1}^{N-1} h(j)c_j < r < R$ *and*

$$\begin{aligned} \varPhi(r) &\leqslant \frac{r}{\max_{n=\overline{0,N}} \sum_{j=1}^{N-1} G(n,j)}, \\ \varPsi(R) &> \frac{R}{\max_{n=\overline{1,N-1}} \sum_{j=1}^{N-1} G(n,j)}, \end{aligned} \tag{11}$$

hold. Then problem (E)–(BC) *has at least one positive solution.*

In addition, if

(H4) $f(n,u) \leqslant 0$ *for all* $n = \overline{1, N-1}$ *and* $u > 0$ *sufficiently large,*

holds, then problem (E)–(BC) *has at least two positive solutions.*

Proof. We assume that (H1)–(H3) hold. We define the sets $\varOmega_1 = \{v \in X,\ \|v\| < r\}$ and $\varOmega_2 = \{v \in X,\ \|v\| < R\}$, where r and R are given in assumption (H3).

For $v \in P \cap \partial\varOmega_1$, by Lemma 5 we obtain

$$\begin{aligned} r &\geqslant v_j - r_j \geqslant \frac{k(j)}{\varLambda}\|v\| - r_j \geqslant \frac{k(j)}{\varLambda}\|v\| - \varLambda\sum_{i=1}^{N-1} h(i)c_i \\ &\geqslant \frac{k_0 r}{\varLambda} - \varLambda\sum_{i=1}^{N-1} h(i)c_i > 0, \quad j = \overline{1, N-1}, \end{aligned}$$

and then

$$f(j, v_j - r_j) + c_j \leqslant \varPhi(r), \quad j = \overline{1, N-1}.$$

Therefore by Lemma 5 we deduce

$$Q_n(v) \leqslant \sum_{j=1}^{N-1} G(n,j)\varPhi(r), \quad n = \overline{0, N},$$

and then

$$\bigl\|Q(v)\bigr\| = \max_{n=\overline{0,N}} \bigl|Q_n(v)\bigr| \leqslant r = \|v\|, \quad v \in P \cap \partial\varOmega_1. \tag{12}$$

For $v \in P \cap \partial\varOmega_2$, by Lemma 5, for $j = \overline{1, N-1}$, we obtain

$$R \geqslant v_j - r_j \geqslant \frac{k(j)}{\varLambda}\|v\| - \varLambda\sum_{i=1}^{N-1} h(i)c_i \geqslant \frac{k_0}{\varLambda}R - \varLambda\sum_{i=1}^{N-1} h(i)c_i,$$

and then

$$f(j, v_j - r_j) + c_j \geqslant \varPsi(R), \quad j = \overline{1, N-1}.$$

Then by (H3) we have for all $v \in P \cap \partial\Omega_2$ and $n = \overline{1, N-1}$,

$$Q_n(v) \geqslant \sum_{j=1}^{N-1} G(n,j)\Psi(R),$$

and so

$$\begin{aligned} \|Q(v)\| &= \max_{n=\overline{0,N}} |Q_n(v)| \geqslant \max_{n=\overline{1,N-1}} |Q_n(v)| \\ &> R = \|v\|, \quad v \in P \cap \partial\Omega_2. \end{aligned} \tag{13}$$

Therefore by (12), (13) and Theorem 1 we deduce that operator Q has a fixed point $v^1 = (v_n^1)_{n=\overline{0,N}} \in P$ satisfying $r \leqslant \|v^1\| < R$. By Lemma 5 we obtain

$$\begin{aligned} &u_n^1 = v_n^1 - r_n \geqslant \frac{k(n)}{\Lambda}\|v^1\| - \Lambda \sum_{j=1}^{N-1} h(j)c_j \geqslant \frac{k_0 r}{\Lambda} - \Lambda \sum_{j=1}^{N-1} h(j)c_j > 0, \\ &\quad n = \overline{1, N-1}, \\ &u_0^1 = v_0^1 - r_0 = \sum_{i=1}^{p} a_i u_{\xi_i}^1 > 0, \qquad u_N^1 = v_N^1 - r_N = \sum_{i=1}^{q} b_i u_{\eta_i}^1 > 0. \end{aligned}$$

Hence, by using Lemmas 7–9 we conclude that $u^1 = (u_n^1)_{n=\overline{0,N}}$ is a positive solution of problem (E)–(BC).

Now we assume in addition that (H4) holds. We prove that problem (E)–(BC) has a distinct second positive solution u^2. By (H4) we deduce that there exists $M > 0$ such that $f(n,u) + c_n \leqslant c_n$ for all $n = \overline{1, N-1}$ and $u > M$. We choose

$$R_1 > \max\left\{R, \frac{\Lambda}{k_0}\left(M + \Lambda \sum_{j=1}^{N-1} h(j)c_j\right)\right\}, \qquad R_1 \geqslant c\Lambda \sum_{j=1}^{N-1} h(j),$$

where $c = \max\{c_n, n = \overline{1, N-1}\} > 0$. Let $\Omega_3 = \{v \in X, \|v\| < R_1\}$. For $v \in P \cap \partial\Omega_3$, we obtain

$$v_n - r_n \geqslant \frac{k_0}{\Lambda} R_1 - \Lambda \sum_{j=1}^{N-1} h(j)c_j > M, \quad n = \overline{1, N-1},$$

which implies that $f(n, (v_n - r_n)^*) + c_n \leqslant c_n \leqslant c$ for all $n = \overline{1, N-1}$.

Then for all $v \in P \cap \partial\Omega_3$, we conclude

$$Q_n(v) \leqslant \Lambda \sum_{j=1}^{N-1} h(j)\bigl(f\bigl(j, (v_j - r_j)^*\bigr) + c_j\bigr) \leqslant c\Lambda \sum_{j=1}^{N-1} h(j) \leqslant R_1,$$

and so

$$\|Q(v)\| \leqslant \|v\|, \quad v \in P \cap \partial\Omega_3. \tag{14}$$

Therefore by (13), (14) and Theorem 1, the operator Q has a fixed point $v^2 = (v_n^2)_{n=\overline{0,N}} \in P$ such that $R < \|v^2\| \leqslant R_1$. By similar arguments used for v^1 we have that $v_n^2 > r_n$ for all $n = \overline{0,N}$, and so by Lemmas 7–9 we obtain that $u^2 = v^2 - \widetilde{r}$, where $\widetilde{r} = (r_n)_{n=\overline{0,N}}$, is a second positive solution of problem (E)–(BC). $\square$

Theorem 3. *We assume that* (H1), (H2) *and* (H3) *hold. If* $\lim_{u\to\infty} f(n,u)/u = 0$ *uniformly for* $n = \overline{1,N-1}$*, then problem* (E)–(BC) *has at least two positive solutions.*

Proof. We consider the sets Ω_1 and Ω_2 defined at the beginning of the proof of Theorem 2. By the proof of Theorem 2 we know that operator Q has a fixed point $v^1 = (v_n^1)_{n=\overline{0,N}} \in P$ such that $r \leqslant \|v^1\| < R$ with $v_n^1 > r_n$ for all $n = \overline{0,N}$. Then $u^1 = (u_n^1)_{n=\overline{0,N}}$, $u_n^1 = v_n^1 - r_n$, $n = \overline{0,N}$, is a positive solution of problem (E)–(BC). We will prove that Q has a second fixed point $\widetilde{v} \in P$.

From the condition $\lim_{u\to\infty} f(n,u)/u = 0$, uniformly for $n = \overline{1,N-1}$, we deduce that $\lim_{u\to\infty}(f(n,u)+c_n)/u = 0$ uniformly with respect to $n = \overline{1,N-1}$. Then for $\varepsilon > 0$, $\varepsilon \leqslant (\varLambda\sum_{j=1}^{N-1} h(j))^{-1}$, there exists $M_1 > 0$ such that $f(n,u) + c_n \leqslant \varepsilon u$ for all $u > M_1$ and $n = \overline{1,N-1}$. We choose $R_2 > \max\{R, (\varLambda/k_0)(M_1 + \varLambda\sum_{j=1}^{N-1} h(j)c_j)\}$ and $\Omega_3 = \{v \in X,\ \|v\| < R_2\}$. For $v \in P \cap \partial\Omega_3$, we obtain

$$v_n - r_n \geqslant \frac{k_0}{\varLambda}R_2 - \varLambda\sum_{j=1}^{N-1} h(j)c_j > M_1, \quad n = \overline{1,N-1},$$

and hence

$$f(n,\, v_n - r_n) + c_n \leqslant \varepsilon(v_n - r_n) \leqslant \varepsilon v_n \leqslant \varepsilon R_2, \quad n = \overline{1,\, N-1}.$$

Thus, for $v \in P \cap \partial\Omega_3$, we deduce

$$Q_n(v) \leqslant \varepsilon\varLambda R_2\sum_{j=1}^{N-1} h(j) \leqslant R_2, \quad n = \overline{0,N},$$

and then

$$\big\|Q(v)\big\| \leqslant \|v\|, \quad v \in P \cap \partial\Omega_3.$$

Therefore by Theorem 1 we conclude that operator Q has a fixed point $\widetilde{v} = (\widetilde{v}_n)_{n=\overline{0,N}} \in P$ such that $R < \|\widetilde{v}\| \leqslant R_2$ and hence $\widetilde{u} = (\widetilde{u}_n)_{n=\overline{0,N}}$, where $\widetilde{u}_n = \widetilde{v}_n - \widetilde{r}_n$, $n = \overline{0,N}$, is a second positive solution of problem (E)–(BC). $\square$

Theorem 4. *We assume that* (H1), (H2) *and*

(H5) *There exist* $r, R > 0$ *such that* $(\varLambda^2/k_0)\sum_{j=1}^{N-1} h(j)c_j < r < R$ *and*

$$\varPhi(R) < \frac{R}{\max_{n=\overline{0,N}}\sum_{j=1}^{N-1} G(n,j)},$$
$$\varPsi(r) \geqslant \frac{r}{\max_{n=\overline{1,N-1}}\sum_{j=1}^{N-1} G(n,j)},$$

hold. Then problem (E)–(BC) *has at least one positive solution.*

In addition, if

(H6) *There exists* $C_0 > \Lambda(k_0^2 \sum_{j=1}^{N-1} h(j))^{-1}$ *such that* $f(n,u) \geqslant C_0 u$ *for all* $n = \overline{1, N-1}$ *and* $u > 0$ *sufficiently large,*

holds, then problem (E)–(BC) *has at least two positive solutions.*

Proof. We suppose that assumptions (H1), (H2) and (H5) hold. We consider the sets Ω_1 and Ω_2 defined at the beginning of the proof of Theorem 2. In a similar manner as that used in the proof of Theorem 2, we obtain $\|Q(v)\| \geqslant \|v\|$ for all $v \in P \cap \partial\Omega_1$ and $\|Q(v)\| < \|v\|$ for all $v \in P \cap \partial\Omega_2$. Thus by Theorem 1 operator Q has a fixed point $v^1 = (v_n^1)_{n=\overline{0,N}} \in P$, with $r \leqslant \|v^1\| < R$ and satisfies $v_n^1 - r_n > 0$ for all $n = \overline{0, N}$. Therefore by Lemmas 7–9, $(u_n^1)_{n=\overline{0,N}}$, where $u_n^1 = v_n^1 - r_n$, $n = \overline{0, N}$, is a positive solution of problem (E)–(BC).

Now, we assume assumption (H6) true. Then there exists $M_2 > 0$ such that, for $u > M_2$, we have

$$f(n,u) + c_n \geqslant C_0 u + c_n \geqslant C_0 u, \quad n = \overline{1, N-1}, \qquad u > M_2.$$

We choose

$$R_3 > \max\left\{R,\ \frac{\Lambda}{k_0}\left(M_2 + \Lambda \sum_{j=1}^{N-1} h(j)c_j\right)\right\},$$

$$R_3 \geqslant k_0 C_0 \Lambda \sum_{j=1}^{N-1} h(j) \sum_{j=1}^{N-1} h(j)c_j \left(\frac{k_0^2 C_0}{\Lambda} \sum_{j=1}^{N-1} h(j) - 1\right)^{-1}.$$

Let $\Omega_3 = \{v \in X,\ \|v\| < R_3\}$. Then for $v \in P \cap \partial\Omega_3$, we obtain

$$v_n - r_n \geqslant \frac{k_0}{\Lambda} R_3 - \Lambda \sum_{j=1}^{n-1} h(j)c_j > M_2, \quad n = \overline{1, N-1},$$

which implies

$$f\bigl(n, (v_n - r_n)^*\bigr) + c_n \geqslant C_0(v_n - r_n), \quad n = \overline{1, N-1}.$$

Thus, for all $v \in P \cap \partial\Omega_3$ and $n = \overline{1, N-1}$, we deduce

$$\begin{aligned} Q_n(v) &\geqslant C_0 k_0 \sum_{j=1}^{N-1} h(j)(v_j - r_j) \\ &\geqslant C_0 k_0 \sum_{j=1}^{N-1} h_j \left(\frac{k_0}{\Lambda} R_3 - \Lambda \sum_{i=1}^{N-1} h(i)c_i\right) \\ &\geqslant R_3, \end{aligned}$$

and then $\|Q(v)\| \geqslant \|v\|$, $v \in P \cap \partial\Omega_3$. Therefore operator Q has another fixed point $v^2 \in P$ such that $R < \|v^2\| \leqslant R_3$, and so $u^2 = v^2 - \widetilde{r}$ is a second positive solution of problem (E)–(BC). □

The following theorem results from Theorem 4.

Theorem 5. *We assume that* (H1), (H2) *and* (H5) *hold. If* $\lim_{u\to\infty} f(n,u)/u = \infty$ *uniformly for* $n = \overline{1, N-1}$*, then problem* (E)–(BC) *has at least two positive solutions.*

We also obtain existence of at least one positive solution of problem (E)–(BC) if we combine the first condition of (11) with condition $\lim_{u\to\infty} f(n,u)/u = \infty$ uniformly for $n = \overline{1, N-1}$. So, we have the following theorem.

Theorem 6. *We assume that* (H1), (H2)*,*

(H3′) *There exist* $r > (\Lambda^2/k_0)\sum_{j=1}^{N-1} h(j)c_j$ *such that*

$$\Phi(r) \leqslant \frac{r}{\max_{n=\overline{0,N}} \sum_{j=1}^{N-1} G(n,j)},$$

hold, and $\lim_{u\to\infty} f(n,u)/u = \infty$ *uniformly for* $n = \overline{1, N-1}$*. Then problem* (E)–(BC) *has at least one positive solution.*

Proof. The proof of this theorem follows from the first part of the proof of Theorem 2 (with the set Ω_1, for which $\|Q(v)\| \leqslant \|v\|$ for all $v \in P \cap \partial\Omega_1$) and the second part of the proof of Theorem 4 (with a set Ω_3, for which $\|Q(v)\| \geqslant \|v\|$ for all $v \in P \cap \partial\Omega_3$). □

4 An example

Let $N = 20$, $L = 2$, $p = 2$, $q = 1$, $\xi_1 = 5$, $\xi_2 = 15$, $a_1 = 2$, $a_2 = 1/3$, $\eta_1 = 10$, $b_1 = 1/2$. We consider the difference equation

$$\Delta^2 u_{n-1} - 2u_n + f(n, u_n) = 0, \quad n = \overline{1, 19}, \tag{E$_0$}$$

with the multi-point boundary conditions

$$u_0 = 2u_5 + \frac{1}{3}u_{15}, \qquad u_{20} = \frac{1}{2}u_{10}. \tag{BC$_0$}$$

We obtain $A = 2 + \sqrt{3}$, $\Delta_1 \approx 2.73999 \cdot 10^{11} > 0$, $\sum_{i=1}^{p} a_i A^{\xi_i} \approx 1.26502 \cdot 10^8 > 1$, $\sum_{i=1}^{p} a_i/A^{\xi_i} \approx 0.00276244 < 1$, $\sum_{i=1}^{q} b_i/A^{\eta_i} \approx 9.53882 \cdot 10^{-7} > 1/A^{20} \approx 3.63956 \times 10^{-12}$, $\sum_{i=1}^{q} b_i A^{\eta_i} \approx 262087 < A^{20} \approx 2.74758 \cdot 10^{11}$. Therefore assumption (H1) is satisfied. In addition, we have

$$g(n,j) = \frac{A}{(A^2-1)(A^{20}-A^{-20})} \times \begin{cases} (A^j - A^{-j})(A^{20-n} - A^{n-20}), & 1 \leqslant j < n \leqslant 20, \\ (A^n - A^{-n})(A^{20-j} - A^{j-20}), & 0 \leqslant n \leqslant j \leqslant 19, \end{cases}$$

$$
\begin{aligned}
G(n,j) = g(n,j)&\\
&+ \frac{1}{\Delta_1}\left[A^n\left(\frac{1}{2}\frac{1}{A^{10}} - \frac{1}{A^{20}}\right) + \frac{1}{A^n}\left(A^{20} - \frac{1}{2}A^{10}\right)\right]\left(2g(5,j) + \frac{1}{3}g(15,j)\right)\\
&+ \frac{1}{\Delta_1}\left[A^n\left(1 - 2\frac{1}{A^5} - \frac{1}{3}\frac{1}{A^{15}}\right) + \frac{1}{A^n}\left(2A^5 + \frac{1}{3}A^{15} - 1\right)\right]\frac{1}{2}g(10,j),
\end{aligned}
$$

$n = \overline{0,20}$, $j = \overline{1,19}$,

$$
h(j) = \frac{A}{(A^2-1)(A^{20}-A^{-20})}\left(A^j - A^{-j}\right)\left(A^{20-j} - A^{j-20}\right), \quad j = \overline{1,19},
$$

$$
k(n) = \frac{1}{A^{19} - A^{-19}}\min\left\{A^n - A^{-n},\, A^{20-n} - A^{n-20}\right\}, \quad n = \overline{0,20}.
$$

We deduce $\Lambda \approx 3.84003043$ and $k_0 \approx 4.7053 \cdot 10^{-11}$. We consider $c_n = \ln((n+3)/n) > 0$ for all $n = \overline{1,19}$, and then we have $a := (\Lambda^2/k_0)\sum_{j=1}^{N-1} h(j)c_j \approx 6.535016 \cdot 10^{11}$. We denote by $b = a+1$ and, for $n = \overline{1,19}$, define the function

$$
f(n,u) = \begin{cases}
\frac{(u+1)^{1/2}}{n(n+4)} + \ln\frac{n}{n+3}, & u \in [0,b],\\
u - b + \frac{(b+1)^{1/2}}{n(n+4)} + \ln\frac{n}{n+3}, & u \in (b,2b],\\
\frac{(u+3)^2}{n(n+2)} - \frac{(2b+3)^2}{n(n+2)} + \frac{(b+1)^{1/2}}{n(n+4)} + b + \ln\frac{n}{n+3}, & u \in (2b,\infty).
\end{cases}
$$

The function f is a continuous one, and it satisfies the inequality $f(n,u) \geqslant -c_n$ for all $u \in \mathbb{R}_+$ and $n = \overline{1,19}$. Then assumption (H2) is also satisfied. After some computations, we obtain $S := \max_{n=\overline{0,20}}(\sum_{j=1}^{19} G(n,j)) \approx 0.499998$. Therefore for $r = b$, we deduce $\Phi(r) \approx 161679 < r/S \approx 1.30701 \cdot 10^{12}$, so assumption (H3′) is satisfied. Because $\lim_{u\to\infty} f(n,u)/u = \infty$ uniformly for $n = \overline{1,19}$, by Theorem 6 we conclude that problem (E_0)–(BC_0) has at least one positive solution $u = (u_n)_{n=\overline{0,20}}$.

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