LMD konf. darbai, 83–88
1997, Vilnius

On m-sided rational surface patches

K. Karčiauskas (VU)

1. INTRODUCTION

The m-sided parametric surface patches were investigated by many authors from different points of view (cf. [1]–[4]). The aim of our approach is to overcome some shortcomings of Ch. Loop & T. DeRose [2] and J. Warren [4]: in [2] the functions defining m-sided patches are linearly dependent in general; in [4] functions are linearly independent, but corresponding control nets have a natural structure only for $m = 6$. Here we present some new schemes of m-sided rational parametric patches with a natural combinatorial structure of a control net, constructed using linearly independent functions.

The paper is organized as follows. In Section 2 we describe the combinatorial structure of control net of m-sided patches and corresponding labeling of basic functions, control points and weights. In Section 3 we present the first general approach, when m-sided parametric patch is defined over m-sided convex polygonal domain. The second general approach, when m-sided patch is created via blowing up base points of the parameterization, is described in Section 4. Both general schemes can be improved for $m = 5$, as presented in Section 5.

In this paper we do not give proofs. We also do not consider smooth joining of m-sided patches. Proofs, problems of smooth joining and some other results can be found in [5].

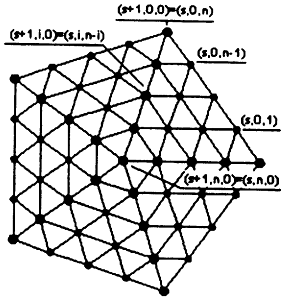
2. A COMBINATORIAL STRUCTURE OF THE CONTROL NET

All control nets of m-sided patches we define in this article have combinatorial structure of a triangulation of a convex m-gon, constructed as described bellow.

Here and later on for an integer a, $0 \leqslant a \leqslant N-1$, we simply write $a+1$ and $a-1$ instead of $a+1 \bmod N$ and $a-1 \bmod N$ respectively (it means N is considered as 0 and -1 as $N-1$). Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{m-1}$ are vertices of a convex polygon, $\mathbf{v}$ — its fixed inner point and n — fixed natural number. For each triangle with vertices $\mathbf{v}, \mathbf{v}_{s+1}, \mathbf{v}_s$, $0 \leqslant s \leqslant m-1$, the points

$$\mathbf{T}^s_{ij} = (i/n)\mathbf{v} + (j/n)\mathbf{v}_{s+1} + ((n-i-j)/n)\mathbf{v}_s, \quad i, j \geqslant 0,\ i+j \leqslant n$$

linked together, form its standard triangulation. All together they form the triangulation of m-gon (in a figure common points of triangulations of triangles are shown bigger as others). Its labeling is convenient to organize in a following manner.



Let $\mathcal{I}$ be a set of all triples (s, i, j), $0 \leqslant s \leqslant m-1$, $0 \leqslant i \leqslant n$, $0 \leqslant j \leqslant n-i$, when triples $(s, i, n-i)$ and $(s+1, i, 0)$ are identified. It means that in $\mathcal{I}$ we have $(s, i, n-i) = (s+1, i, 0)$. The basic functions, control points, weights will be labeled by $\mathcal{I}$. It consists of $mn(n+1)/2+1$ elements.

3. *M*-SIDED PATCHES OVER THE CONVEX *M*-GON

For a convex polygon P with vertices $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{m-1}$ and $r \in \{0, 1, \dots, m-1\}$ let l_r denote fixed linear function, which is zero on a line $\overline{\mathbf{v}_r\mathbf{v}_{r+1}}$ and takes positive values on other points of P. For $q = (s, i, j) \in \mathcal{I}$ and $r \in \{0, 1, \dots, m-1\}$ we define

$$d(q, r) = \begin{cases} i, & \text{if } r = s, \\ i + j, & \text{if } r = s - 1, \\ n - j, & \text{if } r = s + 1, \\ n, & \text{otherwise.} \end{cases}$$

The $d(q, r)$ can be considered as a "combinatorial distance" from q to the line $\overline{\mathbf{v}_r\mathbf{v}_{r+1}}$. And now for $q \in \mathcal{I}$ we define functions $f_q = \prod_{r=0}^{m-1} l_r^{d(q,r)}$. Finally for the fixed points $\mathbf{V}_q$ of $\mathbb{R}^3$ and weights w_q, $q \in \mathcal{I}$, a parametric m-sided patch $F\colon P \to \mathbb{R}^3$ is defined by

$$F(p) = \frac{\sum_{q\in\mathcal{I}} w_q \mathbf{V}_q f_q(p)}{\sum_{q\in\mathcal{I}} w_q f_q(p)}\,.$$

The boundary curves of this patch are rational Bézier curves of degree n. A general line is mapped by F to a rational Bézier curve of degree mn, since $\deg f_q \leqslant mn$. If all weights are positive then the patch lies in a convex hull of control points $\mathbf{V}_q$.

Remark. The functions f_q can be changed by multiplying them with positive numbers. It is shown in [5] how to do this in order to get most natural set of basic functions. There are also some specific features described for the patches, defined over triangle and parallelogram.

4. BLOWING UP BASE POINTS

In this section we assume $m \geqslant 5$ and by k denote an integer part of $(m+1)/2$. Let K be convex polygon with vertices $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{k-1}$. If m is even ($m = 2k$), then all vertices of K are blown up; if m is odd ($m = 2k-1$), then vertices $\mathbf{p}_1, \ldots, \mathbf{p}_{k-1}$ are blown up. By l_r, $0 \leqslant r \leqslant k-1$, we denote linear function, which is zero on a line $\overline{\mathbf{p}_r\mathbf{p}_{r+1}}$ and takes positive values on other points of K. And as in the previous section, we first define "combinatorial distance" $d(q,r)$ for all $q = (s,i,j) \in \mathcal{I}$ and $r \in \{0, 1, \ldots, k-1\}$:

(*) $m = 2k$

(1) $s = 2\tilde{s}$, $0 \leqslant \tilde{s} \leqslant k-1$

$$d(q,r) = \begin{cases} i, & \text{if } r = \tilde{s}, \\ n+j, & \text{if } r = \tilde{s}-1, \\ 2n-i-j, & \text{if } r = \tilde{s}+1, \\ 2n-i, & \text{otherwise.} \end{cases}$$

(2) $s = 2\tilde{s}+1$, $0 \leqslant \tilde{s} \leqslant k-1$

$$d(q,r) = \begin{cases} i+j, & \text{if } r = \tilde{s}, \\ n-j, & \text{if } r = \tilde{s}+1, \\ 2n-i, & \text{otherwise.} \end{cases}$$

(**) $m = 2k-1$

(1) $s = 0$

$$d(q,r) = \begin{cases} i, & \text{if } r = 0, \\ 2n-i-j, & \text{if } r = 1, \\ i+j, & \text{if } r = k-1, \\ 2n-i, & \text{otherwise.} \end{cases}$$

(2) $s = 2\tilde{s}$, $1 \leqslant \tilde{s} \leqslant k-2$

$$d(q,r) = \begin{cases} i, & \text{if } r = \tilde{s}, \\ n+j, & \text{if } r = \tilde{s}-1, \\ 2n-i-j, & \text{if } r = \tilde{s}+1, \\ 2n-i, & \text{otherwise.} \end{cases}$$

(3) $s = 2k-2$

$$d(q,r) = \begin{cases} n-j, & \text{if } r = 0, \\ n+j, & \text{if } r = k-2, \\ i, & \text{if } r = k-1, \\ 2n-i, & \text{otherwise.} \end{cases}$$

(4) $s = 1$

$$d(q,r) = \begin{cases} i+j, & \text{if } r = 0, \\ n-j, & \text{if } r = 1, \\ n+j, & \text{if } r = k-1, \\ 2n-i, & \text{otherwise.} \end{cases}$$

(5) $s = 2\tilde{s} + 1$, $1 \leqslant \tilde{s} \leqslant k - 3$ (if $k > 3$)

$$d(q,r) = \begin{cases} i + j, & \text{if } r = \tilde{s}, \\ n - j, & \text{if } r = \tilde{s} + 1, \\ 2n - i, & \text{otherwise.} \end{cases}$$

(6) $s = 2k - 3$

$$d(q,r) = \begin{cases} 2n - i - j, & \text{if } r = 0, \\ i + j, & \text{if } r = k - 2, \\ n - j, & \text{if } r = k - 1, \\ 2n - i, & \text{otherwise.} \end{cases}$$

Now for $q \in \mathcal{I}$ the functions f_q are defined by $f_q = \prod_{r=0}^{k-1} l_r^{d(q,r)}$. For fixed points $\mathbf{V}_q$ of $\mathbb{R}^3$ and weights w_q a map F: $K \to \mathbb{R}^3$ is defined by

$$F(p) = \frac{\sum_{q \in \mathcal{I}} w_q \mathbf{V}_q f_q(p)}{\sum_{q \in \mathcal{I}} w_q f_q(p)} . \tag{1}$$

We get m-sided patch via blowing up base points of F. The boundary curves are rational curves of degree n and the convex hull property holds too. A general line is mapped by F to a rational Bézier curve of degree $n(2k-3)$, since $\deg f_q \leqslant n(2k-3)$.

5. TWO SPECIAL SCHEMES FOR THE 5-SIDED PATCH

In this section we set $m = 5$. Let K be triangle with vertices $\mathbf{p}_0 = (0; 0)$, $\mathbf{p}_1 = (1; 0)$, $\mathbf{p}_2 = (0; 1)$ and $l_0 = y$, $l_1 = 1 - x - y$, $l_2 = x$. For $q = (s, i, j) \in \mathcal{I}$ functions f_q are defined by the following formulas

$$s = 0: \; f_q = l_0^i l_1^{2n-i-j} l_2^{i+j} (l_1 + l_2)^j,$$
$$s = 1: \; f_q = l_0^{i+j} l_1^{n-j} l_2^{n+j} (l_1 + l_2)^{n-i-j},$$
$$s = 2: \; f_q = l_0^{n+j} l_1^i l_2^{2n-i-j},$$
$$s = 3: \; f_q = l_0^{2n-i-j} l_1^{i+j} l_2^{n-j} (l_1 + l_0)^j,$$
$$s = 4: \; f_q = l_0^{n-j} l_1^{n+j} l_2^i (l_1 + l_0)^{n-i-j}.$$

A map F: $K \to \mathbb{R}^3$ is now defined by the formula (1).

This 5-sided patch has a combinatorial symmetry of a pentagon (the construction does not have this symmetry!): if we set $\widetilde{\mathbf{V}}_{(s,i,j)} := \mathbf{V}_{(s+1,i,j)}$, $\widetilde{w}_{(s,i,j)} := w_{(s+1,i,j)}$, then two sets $\{\mathbf{V}_q, w_q\}$, $\{\widetilde{\mathbf{V}}_q, \widetilde{w}_q\}$, $q \in \mathcal{I}$ of control points and weights define the same patch. More precisely it means following. Let g be a birational transformation of the triangle K, given by the formula

$$g(x, y) = \left(\frac{1 - x - y}{1 - y} ; \frac{xy}{(1 - x)(1 - y)} \right) .$$

If F and $\widetilde{F}$ are the maps $K \to \mathbb{R}^3$, definied by $\{\mathbf{V}_q, w_q\}$ and $\{\widetilde{\mathbf{V}}_q, \widetilde{w}_q\}$ respectively, then $\widetilde{F} = F \circ g$.

This symmetry property still remains if we redefine the basic functions f_g setting $f_{(s,i,j)} := c^n_{i,j} f_{(s,i,j)}$. The integers $c^n_{i,j}$, $0 \leqslant i \leqslant n$, $0 \leqslant j \leqslant n-i$, are definied recursively (for $i < 0$ we set $c^n_{i,j} = 0$):

(*) $c^1_{0,0} = 1$, $c^1_{0,1} = 1$, $c^1_{1,0} = 3$;

(**) $n \geqslant 1$

(1) $i = n+1$, $j = 0$:

$$c^{n+1}_{n+1,0} = 10c^n_{n-1,0} + 5c^n_{n-2,1} + 3c^n_{n,0};$$

(2) $0 \leqslant i \leqslant n$, $j = 0$:

$$c^{n+1}_{i,0} = c^n_{i,0} + 3c^n_{i-1,0} + 2c^n_{i-1,1} + 2c^n_{i-2,0} + 4c^n_{i-2,1} + 2c^n_{i-3,1};$$

(3) $0 \leqslant i \leqslant n$, $j = n+1-i$: $c^{n+1}_{i,n+1-i} = c^{n+1}_{i,0}$;

(4) $0 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n-i$:

$$c^{n+1}_{i,j} = c^n_{i,j-1} + c^n_{i,j} + c^n_{i-1,j-1} + 3c^n_{i-1,j} + c^n_{i-1,j+1} + 2c^n_{i-2,j} + 2c^n_{i-2,j+1} + c^n_{i-3,j+1}.$$

For example, we get
$c^2_{0,0} = 1$, $c^2_{0,1} = 2$, $c^2_{1,0} = 8$, $c^2_{2,0} = 19$;
$c^3_{0,0} = 1$, $c^3_{0,1} = 3$, $c^3_{1,0} = 15$, $c^3_{1,1} = 24$, $c^3_{2,0} = 69$, $c^3_{3,0} = 147$;
It is a full list for $c^2_{i,j}$ and $c^3_{i,j}$ since $c^n_{i,j} = c^n_{i,n-i-j}$. Using so redefined basic functions we get 5-sided patches with additional properties. Here are some of them:

- if $w_{(s,0,j)} = 1$, $0 \leqslant s \leqslant 4$, $0 \leqslant j \leqslant n$, then the boundary Bézier curves are integral;
- there exists degree elevation;
- implicit degree of the patches are $\leqslant 5n^2$;
- a shape of the patch more adequately reacts to a change of the weights w_q.

Remark 1. The functions form a basis for the polynomials of degree $\leqslant 3n$, having singularities of multiplicity $\geqslant n$ at $(0;1)$, $(1;0)$ and the infinite points of coordinate axes. This simple algebraic description of basic functions is a reason, why we get better results as using general blowing up scheme.

Remark 2. The 6-sided patch created via blowing up all vertices of triangle T has an analogous hexagonal symmetry. A reparametrization g is given by the formula

$$g(x,y) = \left(\frac{x(1-x-y)}{y(1-y)+x(1-x-y)}; \frac{xy}{y(1-y)+x(1-x-y)} \right).$$

In this case we also redefine basic functions by multiplying them with positive integers. The symmetry property still holds and we get the 6-sided patch from [4].

Now we describe another special scheme for the 5-sided patch. Let P be a pentagon with vertices $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. As in Section 3 by l_r, $0 \leqslant r \leqslant 4$, we denote fixed linear function, wich is zero on a line $\overline{\mathbf{v}_r \mathbf{v}_{r+1}}$ and takes positive values on the other points of P. Let $\mathbf{e}_r$, $0 \leqslant r \leqslant 4$, denote an intersection of the lines $\overline{\mathbf{v}_{r-1} \mathbf{v}_r}$ and $\overline{\mathbf{v}_{r+1} \mathbf{v}_{r+2}}$. And by h_r, $0 \leqslant r \leqslant 4$, we denote fixed linear function, wich is zero on a line $\overline{\mathbf{e}_{r-1} \mathbf{e}_r}$ and takes positive values on P. For $q = (s, i, j) \in \mathcal{I}$ we define the functions f_q by the formula

$$f_q = l_s^i l_{s+1}^{n-j} l_{s+2}^{2n-i-j} l_{s-2}^{n+j} l_{s-1}^{i+j} h_s^{n-i-j} h_{s+1}^j.$$

A map F: $P \to \mathbb{R}^3$ is defined by the same formula as in Section 3.

Remark 3. This construction was the first we have found looking for a new scheme of 5-sided patch. The functions f_q form a basis for the polynomials of degree $\leqslant 5n$, having at the points $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ singularities of multiplicity $\geqslant 2n$. In fact we get the same class of surfaces as via blowing up two vertices of the triangle.

REFERENCES

[1] J. A. Gregory, N-sided surface patches, in: *The Mathematics of Surfaces*, J. Gregory (Ed.), Clarendon Press, Oxford, England, 1986, 217–232.

[2] Ch. Loop and T. deRose, A multisided generalization Bézier surfaces, *ACM Transactions on Graphics*, **8** (3) (1989), 204–234.

[3] T. Varady, Survey and new results in n-sided patch generation, in: *The Mathematics of Surfaces* II, R. R. Martin (Ed.), Clarendon Press, Oxford, England, 1987, 203–236.

[4] J. Warren, Creating multisided rational Bézier surfaces using base points, *ACM Transactions on Graphics*, **11** (2) (1992), 127–139.

[5] K. Karčiauskas, New rational schemes for m-sided surface patches, Preprint, Vilnius University, Department of Mathematics, 1997.

Racionalios m-kampės paviršių skiautės

K. Karčiauskas

Straipsnyje pateikiamos dvi naujos daugiakampių skiaučių schemos. Jų privalumai, lyginant su jau žinomomis schemomis, yra šie: natūrali kombinatorinė kontrolinio tinklo struktūra; bazinės funkcijos tiesiškai nepriklausomos. Penkiakampėms skiautėms pateiktos dvi papildomos schemos.