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On large deviations for the negative binomial law

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1. Preliminaries

Let E_a denote the distribution concentrated at a point a, $E \equiv E_0$. Products and powers of measures are defined in the convolution sense: $FG = F * G$, $F^n = F^{*n}$, $F^0 = E$. For any signed measure of bounded variation W we denote by $\exp\{W\} = \sum_{k=0}^{\infty} W^k/k!$ its exponential measure, by $|W| = \sup_x |W\{(-\infty, x)\}|$ the analogue of the uniform distance, $\widehat{W}(t) = \int_{-\infty}^{\infty} \exp\{itx\}\, dW$ its Fourier–Stieltjes transform.

Let F be a distribution concentrated on $0, 1, 2, \ldots$. We denote its factorial cumulants by Γ_k. Note that, for F having s finite absolute moments, we have

$$\ln \widehat{F}(t) = \sum_{k=1}^{s-1} \frac{\Gamma_k}{k!}(e^{it} - 1)^k + o(|t|^s), \qquad \text{as } t \to 0.$$

We use notation C for absolute positive constants. The symbol θ is used for all quantities satisfying $|\theta| \leqslant 1$.

Let ξ be a lattice random variable concentrated on non-negative integers, having distribution F and $\mathbf{E}\xi = \lambda > 0$. We say that ξ satisfies $(\tilde{S})$ condition if, for some $\Delta \geqslant 1$,

$$|\Gamma_k| \leqslant \frac{k!\lambda}{\Delta^{k-1}}, \qquad k = 2, 3, \ldots \tag{$\tilde{S}$}$$

Obviously, $(\tilde{S})$ condition is a lattice analogue of the Statulevičius (S) condition for cumulants – see, for example, [12, 13]. It was introduced in [1].

We note that, in 1976, Bikelis and Žemaitis [4] formulated the following analogue of (S):

$$|\Gamma_k| \leqslant \Pi(\Delta)\frac{k!\lambda^k}{\Delta^{k-1}}, \qquad k = 2, 3, \ldots \tag{P}$$

Here $\Pi(\Delta)$, for $0 < \Delta < \infty$, is some non-negative and bounded function.

Estimates under (P) condition were considered in [5, 8–11] and under $(\tilde{S})$ in [1, 2, 6]. Statulevičius (S) condition for other infinitely divisible approximations was considered in [3].

The aim of this note is a demonstration of the fact that, under $(\tilde{S})$ condition, the negative binomial approximation can be used as well. Note that, in [11], the negative binomial distribution was also considered, but only as an approximated law.

2. Results

Let $S_n = \xi_1 + \xi_2 + \ldots + \xi_n$ be a sum of independent identically distributed random variables. Let ξ_1 be concentrated on $0, 1, \ldots$ and have a distribution satisfying condition $(\tilde{S})$. Let

$$\mathbf{E}\xi_1 = \mu, \quad \lambda = n\mu, \qquad y = (x - \lambda)/(n\mathbf{D}\xi_1). \tag{1}$$

Further on we assume that, for all x such that $n\mu \leqslant x \leqslant n\mu\Delta/(100\pi\sqrt{6})$, the following relations hold

$$n\mu \to \infty, \; x = o(n), \; \mu = o(1), \; 1/\Delta = o(1), \; \mu\Delta \to \infty, \tag{2}$$

as $n \to \infty$.

Let G be a negative binomial distribution with the Fourier–Stieltjes transform:

$$\widehat{G}^n(t) = (1 - \mu(e^{it} - 1))^{-n}. \tag{3}$$

In theorems 1, 2 we assume that x is an integer number.

THEOREM 1. *The following relation holds*

$$\frac{1 - F^n(x)}{1 - G^n(x)} = e^{L(x)}\left(1 + O\left(\mu\sqrt{x} + \frac{\sqrt{x}}{\Delta} + y\sqrt{x} + \frac{x\sqrt{x}}{n}\right)\right). \tag{4}$$

Here $L(x)$ denotes Cramer series.

Due to the lattice structure of F and G we can obtain a local estimate.

THEOREM 2. *The following relation holds*

$$\frac{F^n\{x\}}{G^n\{x\}} = e^{L(x)}\big(1 + O(x/n)\big). \tag{5}$$

Remark. Conditions (1) and (2) can be weakened. In principle, it suffices to take small μ and Δ, not necessarily vanishing. However, then the proof becomes longer.

Example. Let $F = (1 - p)E + pE_1$, i.e. let F^n be a binomial distribution. Then $\Delta = 1/p$, $\mu = p$. Let $np^2 \to \infty$, $np^3 \to 0$, $x = np + o(\sqrt{n})$ and $x \geqslant np$. Then it can be established that $1 - F^n(x) = (1 - G^n(x))(1 + o(1))$. In this situation the same relation holds for the standard Poisson approximation, see [7].

3. Proofs

As usual we employ conjugate distributions. Set

$$F_h\{k\} = e^{hk}F\{k\}/\sum_{j=0}^{\infty} e^{hj}F\{j\}, \quad Q_z\{k\} = e^{zk}G\{k\}/\sum_{j=-\infty}^{\infty} e^{zj}G\{j\}.$$

We have

$$1 - F^n(x) = e^{nK(h)-hx}\sum_{k\geq x} e^{-h(k-x)}F_h^n\{k\}, \tag{6}$$

$$1 - G^n(x) = e^{nM(z)-zx}\sum_{k\geq x} e^{-z(k-x)}Q_z^n\{k\}. \tag{7}$$

Here $K(h) = \ln \mathrm{E}\exp\{h\xi\}$, $M(z) = -\ln(1-\mu(e^z-1))$. Quantities h and z are chosen from the saddle point equations: $x = n(K(h))'$ and $x = n(M(z))'$. Note that

$$x = \frac{n\mu e^z}{1-\mu(e^z-1)}, \quad \widehat{Q}_z^n(t) = (1-(x/n)(e^{it}-1))^{-n}. \tag{8}$$

From (6) and (7) we have

$$\begin{aligned} \frac{1-F^n(x)}{1-G^n(x)} &= e^{nK(h)-hx-nM(z)+zx} \\ &\times\left(1+\frac{\sum_{k\geqslant x} e^{-h(k-x)}(F_h^n\{k\}-Q_z^n\{k\}) + \sum_{k\geqslant x}(e^{-h(k-x)}-e^{-z(k-x)})Q_z^n\{k\}}{\sum_{k\geqslant x} e^{-z(k-x)}Q_z^n\{k\}}\right) \\ &= e^{L(x)}\left(1+\frac{A_1(x)+A_2(x)}{A_3(x)}\right). \end{aligned} \tag{9}$$

By Abel's summation formula we obtain

$$|A_1(x)| \leqslant 2|F_h^n - Q_z^n|. \tag{10}$$

Further on we assume that $y \leqslant \Delta/(50e)$, $\lambda \geqslant 1$, $\Delta \geqslant 12e$. Then we can use some auxiliary estimates, which can be derivered from the estimates in [1].

LEMMA 1. *The following inequalities hold*

$$e^h < 3x/(2\lambda), \quad |e^h - 1| \leqslant 7y, \tag{11}$$

$$e^h - 1 = y + \sum_{k=2}^{\infty} d_k y^k, \quad d_k = 3\theta(6e/\Delta)^{k-1}, \tag{12}$$

$$n\ln\widehat{F}_h(t) = x(e^{it}-1) + \theta\frac{11x^2}{\lambda\Delta}\sin^2(t/2), \tag{13}$$

$$|\widehat{F}_h(t)| \leqslant \exp\{-(3x/2n)\sin^2(t/2)\}. \tag{14}$$

From the definition of Q, for all sufficiently large n, we get

$$\begin{aligned} |\widehat{Q}_z(t)| &\leqslant \left|\exp\left\{\frac{x}{n}(e^{it}-1)+\sum_{j=2}^{n}\left(\frac{x}{n}\right)^j\frac{1}{j}(e^{it}-1)^j\right\}\right| \\ &\leqslant \exp\left\{-2\frac{x}{n}\sin^2(t/2)+2\sin^2(t/2)\left(\frac{x}{n}\right)^2\sum_{j=2}^{n}\left(\frac{2x}{n}\right)^{j-2}\right\} \\ &\leqslant \exp\{-(3x/2n)\sin^2(t/2)\}. \end{aligned} \tag{15}$$

Consequently, for all large n,

$$\begin{aligned} |\widehat{F}_h^n(t)-\widehat{Q}_z^n(t)| &\leqslant \exp\{-(3x/2n)\sin^2(t/2)\}n|\ln\widehat{F}_h(t)-\ln\widehat{Q}_z(t)| \\ &\leqslant C\exp\{-(3x/2n)\sin^2(t/2)\}\left(\frac{x^2}{\lambda\Delta}+\frac{x^2}{n}\right)\sin^2(t/2). \end{aligned} \tag{16}$$

By the formula of inversion and Tsaregradskii inequality we have

$$|F_h^n - Q_z^n| = O\left(\frac{x}{n}\left(\frac{1}{\mu\Delta}+1\right)\right), \tag{17}$$

$$\sup_k |F_h^n\{k\} - Q_z^n\{k\}| = O\left(\frac{\sqrt{x}}{n}\left(\frac{1}{\mu\Delta}+1\right)\right). \tag{18}$$

By conditioning and the formula of inversion

$$I = \sup_k |Q_z^n\{k\} - \exp\{x(E_1 - E)\}| = o(\sqrt{x}). \tag{19}$$

Consequently,

$$A_3(x) \geqslant Q_z^n\{x\} \geqslant \exp\{x(E_1 - E)\}\{x\} - I = C\sqrt{x}(1+o(1)). \tag{20}$$

From (8) it follows that

$$e^z - 1 = y\left(1 + O\left(\frac{1}{\Delta}+\mu\right)\right). \tag{21}$$

Hence

$$A_2(x) \leqslant C|e^h - e^z|(\min(h,z))^{-1}\sum_{k\geq x} Q_z^n\{k\} = O(1/\Delta + \mu + y). \tag{22}$$

Combining (9), (10), (20) and (22) we get the assertion of theorem 1. Theorem 2 can be proved similarly. Note that $L(x) = nK(h) - hx + zy - nM(z)$. Moreover, $nK(h) - hx$ can be expanded in the powers of y (see [1]).

REFERENCES

[1] A. Aleškevičienė and V. Statulevičius, Large deviations in approximation by Poisson law, In: *Probab. Theor. and Math. Stat.*, B. Grigelionis et. al. (Eds), VSP/TEV, Utrecht/Vilnius, 1994, 1–18.

[2] A. Aleškevičienė and V. Statulevičius, Large deviations in power zones in the approximation by the Poisson law, *Uspekhi Mat. Nauk*, **50**(5) (1995), 64–82.

[3] A. Aleškevičienė and V. Statulevičius, Probabilities of large deviations in the approximation by χ^2 law, *Lith. Math. J.*, **37**(4) (1997), 301–309.

[4] A. Bikelis and A. Žemaitis, Asymptotic expansions for probabilities of large deviations. Normal approximation, *Liet. Matem. Rink.*, **16**(3) (1976), 31–50.

[5] A. Bikelis and A. Žemaitis, Asymptotic expansions for probabilities of large deviations, In: *Application of Probab. Theory and Math. Statist.*, Vol. 3, Vilnius, 1980, 9–39.

[6] V. Čekanavičius and P. Vaitkus, Large deviations for signed compound Poisson approximations, *Statistics and Decisions*, **17** (1997), 379–396.

[7] G. I. Ivchenko, On comparison of the Binomial and Poisson laws, *Teor. Veroyatn. Primen.*, **19** (1974), 612–615.

[8] Š. Jakševičius, Asymptotic expansions for distributions. Poisson approximation, In: *Application of Probab. Theory and Math. Statist.*, Vol. 3, Vilnius, 1980, 41–86.

[9] Š. Jakševičius, Asymptotic expansions for distributions, I, *Liet. matem. rink.*, **23**(3) (1983), 196–213.

[10] J. Milaševičius, On convergence of the Grigelionis–Franken asymptotic expansion, In: *Application of Probab. Theory and Math. Statist.*, Vol. 3, Vilnius, 1980, 87–110.

[11] J. Milaševičius, On asymptotic expansions for probabilities of large deviations. Poisson approximation, *Optimization of Control Systems*, (1981), 232–239.

[12] L. Saulis and V. Statulevičius, *Limit Theorems on Large Deviations*, Kluver Acad. Publ., Dordrecht, Boston, London, 1991.

[13] V. Statulevičius, On large deviations, *Z. Wahrscheinlichkeitstheor. verw. Geb.*, **6**(2) (1966), 133–144.

Apie didelius nuokrypius neigiamam binominiam dėsniui

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Parodyta, kad esant reguliariam faktorialinių kumuliantų nykimui (Satatulevičiaus (S) sąlygos analogui) didelių nuokrypių rezultatus galima gauti ne tik Puasono dėsniui, bet ir neigiamam binominiam skirstiniui.