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Bergström expansion for mixtures of lattice distributions

V. Čekanavičius (VU)

Let $\mathcal{F}$ be the set of all probability measures, $\mathcal{M}$ be the set of all measures of bounded variation on $\mathbb{R}$. If $W \in \mathcal{M}$ then, due to the Jordan-Hahn decomposition, $W = W^+ - W^-$.

We denote by $\| W \|$ the total variation norm of W, i.e., $\| W \| = W^+\{\mathbb{R}\} + W^-\{\mathbb{R}\}$. Let E_a be the distribution concentrated at a point a (i.e. $E_a\{a\} = 1$), $E \equiv E_0$. The notation $C(\cdot)$ will be used for different positive constants depending on the indicated argument only. Products and powers of measures will be understood in the convolution sense: $FG = F * G$, $W^n = W^{*n}$, $W^0 = E$. For $W \in \mathcal{M}$ we shall denote its Fourier-Stieltjes transform by $\widehat{W}(t) = \int_{\mathbb{R}} \exp\{itx\}\, W\{dx\}$, $t \in \mathbb{R}$ and the analogue of the uniform distance by

$$|W| = \sup_{x\in\mathbb{R}} |W\{(-\infty, x)\}| = \sup_{x\in\mathbb{R}} |W(x)|\,.$$

Let $\mathbb{N}$ be the set of all natural numbers, Z be the set of all integer numbers.

H. Bergström [1] used asymptotic expansions based on the following identity

$$F^n = \sum_{j=0}^{s} \binom{n}{j} G^{n-j}(F-G)^j + r_n^{(s+1)},$$

with

$$r_n^{(s+1)} = \sum_{\mu=s+1}^{n} \binom{\mu-1}{s} F^{n-\mu}(F-G)^{s+1}G^{\mu-s-1}, \tag{1}$$

Bergström expansion was applied in [1]–[11]. In [3], [4], two generalizations of (1) for the convolutions of non-identical distributions were given. We shall use the generalization of (1) from [8] (see also [6]).

Let $F_1, F_2, \dots, F_n \in \mathcal{M}$, $G_1, \dots, G_n \in \mathcal{M}$, $0 \leqslant s \leqslant n-1$. Analogously to (1) we have

$$\prod_{j=1}^{n} F_j = \sum_{\nu=0}^{s} \Delta_\nu + \sum_{\nu=s+1}^{n} \Delta_\nu = \sum_{\nu=0}^{s} \Delta_\nu + R_n^{(s+1)}, \tag{2}$$

$$\Delta_\nu = \sum\nolimits_{n,\mu}^{\nu} \prod_{m=1}^{n} G_m^{1-\mu_m}(F_m - G_m)^{\mu_m}. \tag{3}$$

$$R_n^{(s+1)} = \sum_{j=s+1}^{n} (F_j - G_j) \prod_{i=j+1}^{n} \sum\nolimits_{j-1,\mu}^{s} \prod_{m=1}^{j-1} G_m^{1-\mu_m}(F_m - G_m)^{\mu_m}. \tag{4}$$

Here $\sum_{n,\mu}^{\nu}$ means summation over all possible $\mu_1, \mu_2, \ldots, \mu_n \in \{0, 1\}$ such that $\mu_1 + \ldots + \mu_n = \nu$, i.e.

$$\sum\nolimits_{n,\mu}^{\nu} = \sum\{\mu_1 + \ldots + \mu_n = \nu,\ \mu_m \in \{0, 1\},\ m = 1, \ldots, n\}.$$

Let $F \in \mathcal{F},\ i = 1, \ldots, n$,

$$\varphi_i(F) = \sum_{j=0}^{\infty} p_{ij} F^j, \qquad \psi_i(F) = \sum_{j=0}^{\infty} q_{ij} F^j, \tag{5}$$

$$\sum_{j=0}^{\infty} p_{ij} = \sum_{j=0}^{\infty} q_{ij} = 1, \quad \sum_{j=0}^{\infty} |p_{ij}| < \infty, \quad \sum_{j=0}^{\infty} |q_{ij}| < \infty. \tag{6}$$

Note that if $p_{ij}, q_{ij} \geqslant 0$,then $\varphi_i(F),\ \psi_i(F)$ are distributions of the sums of a random number of i.i.d.r.v. In general, we deal with signed measures. We shall say that, $\varphi_i(F)$ and $\psi_i(F)$ satisfy condition (λ_i), if there exists $\lambda_i < C$ such that

$$\max\{|\varphi_i(\widehat{F}(t))|,\ |\psi_i(\widehat{F}(t))|\} \leqslant \exp\{\lambda_i(Re\widehat{F}(t) - 1)\}. \tag{7}$$

Here $Re\widehat{F}(t)$ denotes the real part of $\widehat{F}(t)$ and

$$\varphi_i(\widehat{F}(t)) = \sum_{j=0}^{\infty} p_{ij}(\widehat{F}(t))^j = \widehat{\varphi_i(F)}(t).$$

The following Lemma asserts that the class of measures satisfying condition (λ) is large enough.

LEMMA 1. *Let* $F \in \mathcal{F}$,

$$\varphi(F) = \sum_{j=0}^{\infty} p_j F^{j'}, \quad \sum_{j=0}^{\infty} p_j = 1, \quad \sum_{j=0}^{\infty} |p_j| < \infty, \quad \beta_2(\varphi(E_1)) < \infty.$$

Then, for all $t \in \mathbb{R}$,

$$|\varphi(\widehat{F}(t))| \leqslant \exp\{\big(\alpha_1(\varphi(E_1)) - \alpha_1^2(\varphi(E_1)) - \beta_2(\varphi(E_1))\big)(Re\widehat{F}(t) - 1)\}.$$

Proof. By the Bergström identity and definition of $\varphi(E_1)$ we get

$$\begin{aligned} &|\varphi(\widehat{F}(t)) - 1 - \alpha_1(\varphi(E_1))(\widehat{F}(t) - 1)| \\ &= \Big|\sum_{j=0}^{\infty} p_j(\widehat{F}^j(t) - 1 - j(\widehat{F}(t) - 1))\Big| = \Big|\sum_{j=2}^{\infty} p_j \sum_{\mu=2}^{j} (\mu - 1)\widehat{F}^{j-\mu}(t)(\widehat{F}(t) - 1)^2\Big| \\ &\leqslant \sum_{j=2}^{\infty} |p_j| \binom{j}{2} |\widehat{F}(t) - 1|^2 \leqslant \beta_2(\varphi(E_1))|\widehat{F}(t) - 1|^2/2. \end{aligned} \tag{8}$$

Therefore

$$|\varphi(\widehat{F}(t))| \leqslant |1+\alpha_1(\varphi(E_1))(\widehat{F}(t)-1)| + \beta_2(\varphi(E_1))|\widehat{F}(t)-1|^2/2. \tag{9}$$

Taking into account that

$$\widehat{F}(t) = Re\widehat{F}(t) + iIm\widehat{F}(t), \qquad (Im\widehat{F}(t))^2 \leqslant 1 - \big(Re\widehat{F}(t)\big)^2,$$

we get from (9)

$$\begin{aligned}|\varphi(\widehat{F}(t))| &\leqslant |1+2(Re\widehat{F}(t)-1)\big(\alpha_1(\varphi(E_1)) - \alpha_1^2(\varphi(E_1))\big)|^{1/2} + \beta_2(\varphi(E_1))(1-Re\widehat{F}(t))\\ &\leqslant \exp\{\big(\alpha_1(\varphi(E_1)) - \alpha_1^2(\varphi(E_1)) - \beta_2(\varphi(E_1))\big)(Re\widehat{F}(t)-1)\}. \qquad \square\end{aligned}$$

Now we shall formulate the main result of this note. Let us denote a summand of the Bergström expansion by

$$\Delta_\nu(F) = \sum\nolimits_{n,\mu}^{\nu} \prod_{j=1}^{n} \psi_j^{1-\mu_j}(F)\big(\varphi_j(F) - \psi_j(F)\big)^{\mu_j}.$$

THEOREM 1. *Let* $F \in \mathcal{F}$, $F\{\mathrm{Z}\} = 1$ *and let, for* $m \geqslant 2$, $i = 1, 2\dots, n$, $k = 1, 2, \dots, m-1$, *the following conditions be satisfied*

$$\alpha_k(\varphi_i(E_1) - \psi_i(E_1)) = 0, \qquad \beta_m(\varphi_i(E_1) - \psi_i(E_1)) < \infty,$$

$$\max\{\beta_2(\varphi(E_1)),\ \beta(\psi(E_1))\} < \infty, \qquad \lambda_i \geqslant 0.$$

Then, for all $s \leqslant n-1$, $\nu = 1, 2, \dots, s$, *the following inequalities hold*

$$\begin{aligned}&\sup_{x\in\mathrm{Z}} |\Delta_\nu(F)\{x\}|\\ &\leqslant C(m,\nu)h^{-\nu/2}\Big(\sum\nolimits_{n,\mu}^{\nu}\prod_{i=1}^{n}\beta_m(\varphi_i(E_1)-\psi_i(E_1))^{\mu_i}\Big)\min\big\{1, (h(1-F\{0\}))^{1/2}\big\} \\ &\leqslant C(m,\nu)h^{-\nu/2}\min\big\{1, (h(1-F\{0\}))^{1/2}\big\}\Big(\sum_{i=1}^{n}\beta_m(\varphi_i(E_1)-\psi_i(E_1)\Big)^{\nu},\end{aligned} \tag{10}$$

$$\begin{aligned}\sup_{x\in\mathrm{Z}}\left|\prod_{i=1}^{n}\varphi_i(F)\{x\} - \sum_{\nu=0}^{s}\Delta_\nu(F)\{x\}\right| &\leqslant C(m,s)\left(\sum_{i=1}^{n}\beta_m(\varphi_i(E_1)-\psi_i(E_1)\right)^{s+1}\\ &\times h^{-(s+1)/2}\min\big\{1, (h(1-F\{0\}))^{1/2}\big\}.\end{aligned} \tag{11}$$

Here $h = \max\big\{1, \sum_{i=1}^{n}\lambda_i\big\}$.

Proof. Analogously to the proof of (8) we get

$$|\varphi_i(\widehat{F}(t)) - \psi_i(\widehat{F}(t))| \leqslant C(m)\beta_m(\varphi_i(E_1) - \psi_i(E_1))|\widehat{F}(t) - 1|^m \\ \leqslant C(m)\beta_m(\varphi_i(E_1) - \psi_i(E_1))(1 - Re\widehat{F}(t))^{m/2}. \tag{12}$$

Noting that, if $\lambda_i > 0$ then $\lambda_i \leqslant 1$, we get

$$\left|\sum\nolimits_{n,\mu}^{\nu} \prod_{i=1}^{n} \psi_i^{1-\mu_i}(\widehat{F}(t))\big(\varphi_i(\widehat{F}(t)) - \psi_i(\widehat{F}(t))\big)^{\mu_i}\right|$$

$$\leqslant C(m,\nu)\big(1 - Re\widehat{F}(t)\big)^{\nu/2} \sum\nolimits_{n,\mu}^{\nu} \exp\left\{\sum_{i=1}^{n}(1-\mu_i)\lambda_i(Re\widehat{F}(t) - 1)\right\}$$

$$\times \prod_{i=1}^{n} \beta_m^{\mu_i}(\varphi_i(E_1) - \psi_i(E_1))$$

$$\leqslant C(m,\nu)\exp\left\{\sum_{l=1}^{n}\lambda_l(Re\widehat{F}(t) - 1)/2\right\} h^{-\nu/2} \sum\nolimits_{n,\mu}^{\nu} \prod_{i=1}^{n} \beta_m^{\mu_i}(\varphi_i(E_1) - \psi_i(E_1)).$$

By the formula of inversion

$$|\Delta_\nu(F)| \leqslant C(m,\nu)h^{-\nu/2} \sum\nolimits_{n,\mu}^{\nu} \prod_{i=1}^{n} \beta_m^{\mu_i}(\varphi_i(E_1) - \psi_i(E_1)) \\ \times \int_{-\pi}^{\pi} \exp\left\{\sum_{l=1}^{n}\lambda_l(Re\widehat{F}(t) - 1)/2\right\} dt. \tag{13}$$

Note that

$$Re\widehat{F}(t) = \widehat{F}(t)/2 + \widehat{F}(-t)/2,$$

i.e., $Re\widehat{F}(t)$ is a characteristic function. To end the proof of (10) one should apply the following inequality:

$$\int_{-\pi}^{\pi} \exp\{a(\widehat{F}(t) - 1)\}\,dt \leqslant C(1 - F\{0\})^{-1/2}a^{-1/2}.$$

The proof of (11) is similar. *Q.E.D.*

Example. Let $0 \leqslant p \leqslant 1/2$, $F \in \mathcal{F}$, $F\{\mathbb{Z}\} = 1$, $n \in \mathbb{N}$. Then

$$\sup_{x\in\mathbb{Z}} |\big((1-p)E + pF\big)^n\{x\} - \exp\{np(F-E) - np^2(F-E)^2/2\}\{x\}| \\ \leqslant Cn^{-1}(1 - F\{0\})^{-1/2}.$$

To prove this inequality one must note that

$$\beta_3\big(((1-p)E + pF)^n - \exp\{np(F-E) - np^2(F-E)^2/2\}\big) \leqslant Cp^3,$$

and

$$\max\{|1 + p(e^{it} - 1)|, |\exp\{p(e^{it} - 1) - p^2(e^{it} - 1)^2/2\}|\} \leqslant \exp\{-Cp\sin^2(t/2)\}$$

and apply Theorem 1.

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Bergstremo skleidiniai gardelinių skirstinių mišiniams

V. Čekanavičius (VU)

Lyginame dviejų atsitiktinių dydžių sumų pasiskirstymų artumą lokalioje metrikoje. Kiekvienos sumos dėmenys savo ruožtu yra atsitiktinių dydžių su atsitiktiniais rėžiais sumos. Sąlygos keliamos atsitiktiniams rėžiams.