LMD mokslo darbai, 96–100
1998, Vilnius

The Euler approximation of stochastic differential equations driven by a fractional Brownian motion

K. Kubilius (MII, VGTU)

In this note, we examine the strong approximation of stochastic differential equation (SDE) of the form

$$dX_t = f(t, X_t)\,dt + g(t)\,dB_t^H, \tag{1}$$

or equivalently

$$X_t = X_0 + \int_0^t f(s, X_s)\,ds + \int_0^t g(s)\,dB_s^H,$$

where B^H is a fractional Brownian motion (fBm) with the Hurst index $1/2 \leqslant H < 1$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The case $H = 1/2$ corresponds to the ordinary Brownian motion. Results of this type are known only in the case $H = 1/2$ (see [3]).

As is well-known (see, for example, [4]), a centered Gaussian process $(X_t)_{t\geqslant 0}$ with $X_0 = 0$ is a fBm if

$$\mathrm{Cov}\big(X_t, X_s\big) = \frac{1}{2}\mathrm{Var}(X_1)\big(t^{2H} + s^{2H} - |t-s|^{2H}\big),$$

for all $t, s \geqslant 0$. If $\mathrm{Var}(X_1) = 1$, we write $X = B^H$ and call it a standard fBm.

Equation (1) differs from ordinary SDE by its second term on the right side. The fBm B^H is not a semimartingale (see [4], [5]). There are some ways of defining stochastic integral with respect to fBm. For example, Lin [4] defined the stochastic integral with respect to B^H in the case, where the integrands are either deterministic bounded functions or the compositions of deterministic bounded functions and B^H (see also [1]). Lin found existence and uniqueness conditions of the solution of equation (1).

In this paper, we use another definition of the integral $\int_0^t g(s)\,dB_s^H$. We define it as the Riemann–Stieltjes integral using Young [6] results.

Let $\{t_k,\ 0 \leqslant k \leqslant n\}$ be a partition of the interval $[0, T]$, i.e., $0 = t_0 < t_1 < \cdots < t_n = T$, and $\delta_n = \max_k (t_k - t_{k-1})$. For a given time discretization (t_k), we define Euler approximation

$$Y^n(0) = X(0),$$
$$Y^n(t) = Y^n(t_k) + f\big(t_k, Y^n(t_k)\big)(t - t_k) + g(t_k)\big(B^H(t) - B^H(t_k)\big), \quad t \in [t_k, t_{k+1}),$$

or, equivalently,

$$Y^n(t) = X_0 + \int_0^t f_n(s)\,ds + \int_0^t g_n(s)\,dB_s^H,$$

where $f_n(s) := f(t_k, Y^n(t_k))$ and $g_n(s) := g(t_k)$ for $s \in [t_k, t_{k+1})$, $0 \leqslant k \leqslant n-1$.

Our main result is the following:

THEOREM 1. *Let K, T be positive numbers, $\delta_n \leqslant T/n$, and let $f(s,x)$, $g(s)$ be Borel functions such that*

1. $|g(t) - g(s)| \leqslant K\,|t-s|$ *for all* $s, t \in [0,T]$;
2. $|f(s,x)| \leqslant K(1+|x|)$ *for every fixed* $s \in [0,T]$;
3. $|f(t,x) - f(s,y)| \leqslant K(|x-y| + |t-s|)$ *for all* $s,t \in [0,T]$ *and* $x, y \in \mathbf{R}$.
4. $\mathbf{E}|X_0|^p < \infty$.

Then there exists a constant C such that

$$\mathbf{E}\sup_{t\leqslant T}|X_t - Y_t^n|^p \leqslant C\delta_n^{Hp}, \qquad p > 1.$$

THEOREM 2. *Let conditions* 2 *and* 3 *of Theorem* 1 *be fulfilled. If moreover* $g \in \mathcal{W}_q([0,T])$ *and* $q^{-1} + \lambda > 1$, *where* $\lambda < H$, *then, for almost all* ω, *there exists a* $C(\omega) = C(X_0(\omega), p, K, T)$ *such that*

$$\sup_{t\leqslant T}|X_t(\omega) - Y_t^n(\omega)|^p \leqslant C(\omega)\delta_n^{\lambda p}, \qquad p > 1.$$

Preliminaries

All facts mentioned bellow are taken from [2] and [6].

Let f be a real-valued function defined on a closed interval $[a,b]$. The p-variation, $0 < p < \infty$, of f is defined by

$$v_p(f) = v_p(f;[a,b]) = \sup_{\varkappa}\sum_{i=1}^n |f(x_i) - f(x_{i-1})|^p,$$

where the supremum is taken over all subdivisions $\varkappa$ of $[a,b]$: $\varkappa$: $a = x_0 < \cdots < x_n = b$, $n \geqslant 1$. If $v_p(f) < \infty$, f is said to have a bounded p-variation on $[a,b]$. If f is a Hölder function with $0 < \alpha \leqslant 1$, then it has bounded a $1/\alpha$-variation.

Denote by $\mathcal{W}_p([a,b])$ the class of functions defined on $[a,b]$ with a bounded p-variation, that is

$$\mathcal{W}_p([a,b]) := \{f\colon\ [a,b] \to \mathbf{R}\colon\ v_p(f;[a,b]) < \infty\}.$$

Let $a < c < b$ and let $f \in \mathcal{W}_p([a,b])$ with $0 < p < \infty$. Then

$$v_p(f;[a,c]) + v_p(f;[c,b]) \leqslant v_p(f;[a,b]). \tag{2}$$

Young [6] proved that, if $f \in \mathcal{W}_p([a,b])$ and $h \in \mathcal{W}_p([a,b])$ with $p, q > 0$, $1/p + 1/q > 1$, have no common discontinuities, then the Riemann–Stieltjes integral $\int_a^b f\,dh$ exists and, for any $\xi \in [a,b]$, the following inequalities hold:

$$\left| \int_a^b f\,dh - f(\xi)\big[h(b) - h(a)\big] \right| \leqslant \Big(1 + \zeta(p^{-1} + q^{-1})\Big) V_p(f; [a,b]) V_q(h; [a,b]), \quad (3)$$

where $\zeta(s)$ denotes the zeta function, i.e., $\zeta(s) = \sum_{n \geqslant 1} n^{-s}$, $V_p(f) = V_p(f; [a,b]) = v_p^{1/p}(f)$.

If the function h is continuous, then the indefinite integral $\int_a^y f\,dh$, $y \in [a,b]$, is a continuous function ([2], Lemma 3.23).

1. Proofs

It is known that, with probability 1, sample functions of fBm B^H satisfy the Hölder condition of exponent λ for each $\lambda < H$. So fBm B^H, $1/2 \leqslant H < 1$, has a bounded $1/\lambda$-variation with probability 1 and $v_{1/\lambda}(B^H; [0,T]) \leqslant T$.

Now we prove Theorem 1.

From Hölder's inequality and Gronwall's lemma it is evident that

$$\begin{aligned} \sup_{0 \leqslant t \leqslant T} |X_t - Y_t^n|^p \leqslant 2^{p-1} e^{pKT} \Bigg(& T^{p-1} \int_0^T \big| f(s, Y_s^n) - f_n(s) \big|^p ds \\ & + \sup_{0 \leqslant t \leqslant T} \bigg| \int_0^t \big[g(s) - g_n(s)\big] dB_s^H \bigg|^p \Bigg) \end{aligned} \quad (4)$$

We further have

$$\int_0^T \big| f(s, Y_s^n) - f_n(s) \big|^p ds \leqslant K^p 2^{p-1} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \Big[|s - t_{k-1}|^p + |Y^n(s) - Y^n(t_{k-1})|^p \Big] ds. \quad (5)$$

For $s \in [t_{k-1}, t_k)$, we have

$$\begin{aligned} \big| Y^n(s) - Y^n(t_{k-1}) \big|^p &\leqslant | f(t_{k-1}, Y^n(t_{k-1}))(s - t_{k-1}) \\ &\quad + g(t_{k-1})\big(B^H(s) - B^H(t_{k-1})\big) \big|^p \\ &\leqslant 4^{p-1} K^p \big(1 + |Y^n(t_{k-1})|^p\big)(s - t_{k-1})^p \\ &\quad + 2^{p-1} |g|_\infty^p \big| B^H(s) - B^H(t_{k-1}) \big|^p. \end{aligned} \quad (6)$$

By Gronwall's inequality we get

$$\max_{1 \leqslant k \leqslant n} |Y^n(t_k)| \leqslant e^{KT} \left(|X_0| + KT + \sup_{0 \leqslant t \leqslant T} \left| \int_0^t g_n(s)\, dB_s^H \right| \right). \quad (7)$$

In [4], Lin showed that there exists a constant $C_{H,p}$, $0 < p < \infty$, depending only on H and p such that, for any bounded measurable function h on $[0, T]$,

$$\mathbf{E} \sup_{0\leqslant t\leqslant T} \left| \int_0^t h(s)\, dB_s^H \right|^p \leqslant C_{p,H} |h|_\infty^p. \tag{8}$$

So $\max_{1\leqslant k\leqslant n} |Y^n(t_k)|^p$ is integrable.

It is known that that for each $p > 1$ there is a $C_p < \infty$ such that

$$\mathbf{E}\left|B_t^H - B_s^H\right|^p \leqslant C_p |t-s|^{pH}. \tag{9}$$

Now from (6)–(9) we get

$$\mathbf{E}\left|Y^n(s) - Y^n(t_{k-1})\right|^p \leqslant C_1 \delta_n^p + C_2 |g|_\infty^p \delta_n^{pH}, \qquad s \in [t_{k-1}, t_k]. \tag{10}$$

The statement of the theorem follows from (4), (5), (10) and the inequality

$$\mathbf{E} \sup_{0\leqslant t\leqslant T} \left| \int_0^t [g(s) - g_n(s)] dB_s^H \right|^p \leqslant C_{p,H} |g - g_n|_\infty^p \leqslant C_{p,H} \delta_n^p. \qquad \square$$

Now we prove Theorem 2.

First note that $g_n \in \mathcal{W}_q([0, T])$, $q > 0$, and $V_q(g_n; [0, T]) \leqslant V_q(g; [0, T])$. Then from (3) we have

$$\begin{aligned} \sup_{0\leqslant t\leqslant T} \left| \int_0^t g_n(s)\, dB_s^H \right| &\leqslant C_{q,\lambda}\big[V_q(g_n; [0, T]) + |g(0)|\big] V_{1/\lambda}\big(B^H; [0, T]\big) \\ &\leqslant C_{q,\lambda}\big[V_q(g; [0, T]) + |g(0)|\big] T^\lambda, \end{aligned} \tag{11}$$

where $C_{q,\lambda} = 1 + \zeta(q^{-1} + \lambda)$.

From Hölder continuity of the sample paths of B^H, inequalities (6), (7), and (11), it follows that, for almost all ω, there is a $C(\omega) = C(X_0(\omega), p, K, T)$ such that

$$\left|Y^n(s, \omega) - Y^n(t_{k-1}, \omega)\right| \leqslant C(\omega) \delta_n^\lambda \qquad s \in [t_{k-1}, t_k]. \tag{12}$$

Let $0 < \varepsilon < 1$ be such that $\lambda + \varepsilon < H$. Then from (2), (3) and Hölder inequality we have, for $t \in [t_m, t_{m+1}]$,

$$\left| \int_0^t [g(s) - g_n(s)] dB_s^H \right| \leqslant \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} [g(s) - g(t_{k-1})] dB_s^H \right| + \left| \int_{t_m}^t [g(s) - g(t_m)] dB_s^H \right|$$

$$
\begin{aligned}
&\leqslant C_{q,\lambda,\varepsilon} \sum_{k=1}^{m} V_q\big(g, [t_{k-1}, t_k]\big) V_{1/(\lambda+\varepsilon)}\big(B^H, [t_{k-1}, t_k]\big) \\
&\quad + C_{q,\lambda,\varepsilon} V_q\big(g, [t_m, t]\big) V_{1/(\lambda+\varepsilon)}\big(B^H, [t_m, t]\big) \\
&\leqslant C_{q,\lambda,\varepsilon} \Big(\sum_{k=1}^{m} v_q\big(g, [t_{k-1}, t_k]\big) \Big)^{1/q} \\
&\quad \times \Big(\sum_{k=1}^{m} V_{1/(\lambda+\varepsilon)}^{1/\varepsilon}\big(B^H, [t_{k-1}, t_k]\big) + V_{1/(\lambda+\varepsilon)}^{1/\varepsilon}\big(B^H, [t_m, t]\big) \Big)^{\varepsilon} \\
&\leqslant C_{q,\lambda,\varepsilon} K\, V_q\big(g, [0, T]\big) \Big(\sum_{k=1}^{m+1} (t_k - t_{k-1})^{1+\lambda/\varepsilon} \Big)^{\varepsilon} \\
&\leqslant C_{q,\lambda,\varepsilon} K\, V_q\big(g, [0, T]\big) \delta_n^{\lambda} T^{\varepsilon},
\end{aligned}
$$

where $C_{q,\lambda,\varepsilon} = 1 + \zeta(q^{-1} + \lambda + \varepsilon)$. Since $\varepsilon > 0$ is arbitrary then

$$
\sup_{t \leqslant T} \left| \int_0^t [g(s) - g_n(s)] dB_s^H \right| \leqslant \big(1 + \zeta(q^{-1} + \lambda)\big) K\, V_q\big(g, [0, T]\big) \delta_n^{\lambda}. \tag{13}
$$

The statement of the theorem now follows from inequalities (4)–(7) and (11)–(13).

REFERENCES

[1] Dai, W. and Heyde, C. C., Itô's formula with respect to fractional Brownian motion and its application, *J. Appl. Math. Stochastic Anal.*, **9** (1996), 439-448.

[2] Dudley R. M. and Norvaiša R., Product integrals, Young integrals and p-variation. Preprint (1997)

[3] S. Kanagawa, The rate of convergence for approximate solutions of stochastic differencial equations, *Tokyo J. Math.*, **12** (1986), 33-48.

[4] S. J. Lin, Stochastic analysis of fractional Brownian motions, *Stochastics and Stochastics Reports*, **55** (1995), 121-140.

[5] Liptser, R.S., Shiryayev, A.N., *Theory of Martingales*, Nauka, Moscow, 1986.

[6] Young, L.C., An inequality of the Hölder type, connected with Stieltjes integration, *Acta Math.*, **67** (1936), 251-282.

Stochastinių diferencialinių lygčių, generuotų trupmeninio brauno judesio, Eulerio aproksimacija

K. Kubilius (MII, VGTU)

Darbe nagrinėjama stochastinė diferencialinė lygtis, kurioje integralas, atžvilgiu trupmeninio Brauno judesio, apibrežiamas dviem skirtingais būdais. Gauti du skirtingi įverčiai.