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Local limit theorems for multiplicative functions on semigroups

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1. Introduction

Denote by G a free commutative semigroup, generated by a subset P of prime elements p. In G it is defined a completely additive *degree function* $\delta: \mathbf{G} \to \mathbf{N} \cup \{0\}$ such that $\delta(p) \geq 1$ for each $p \in$ P. More precise definition of G and other traditional notations see in papers [1], [3], [5].

Let, further, denote by M(G) the class of multiplicative functions g: $\mathbf{G} \to \mathbf{R}$ satysfying the conditions

$$\sum_{\substack{p\in P,\ \delta(p)=l \\ g(p)=v}} 1 =: (\lambda_v + \rho_v(l)) \sum_{p\in P, \delta(p)=l} 1 \ , \ v \in R, l \geq 1, \tag{1}$$

where $\lambda_v \in [0,1]$ are constants, and the *remainder terms* $\rho_v(l)$ satisfy the conditions $\rho_v(l) := C_v(l) r^{-1}(l)$ with some function $r(l)$ such that

$$\int_2^\infty \frac{du}{ur(u)} < \infty \ .$$

For simplicity only, we let $r(u) = u^\alpha$ with $\alpha > 0$, but nontrivialy result can be obtained with $r(u) = (\ln u)^{2+\varepsilon}$. Besides,

$$\sum_v |C_v(l)| < \infty$$

uniformly in $l \geq 1$.

In [2] it was proved some local limit theorems for the *multiplicative* real-valued functions defined on N. The purpose of present paper is to prove analogical local theorems for arithmetic functions from the class M(G).

Local limit theorems of such kind for *additive* functions f: $\mathbf{G} \to \mathbf{Z}$ can be found in [1], [4], [5].

2. Notations

We consider only the principal value of logarithms and powers. Let $0^z = 0$ for every $z \in$ C. Further: $k = 0,1$;

$$\chi_k := \chi_k(t) = \sum_{v, v\neq 0} \lambda_v \, |v|^{it} \operatorname{sgn}^k v \; ; \quad E_k = \sum_{v, v\neq 0} \lambda_v \operatorname{sgn}^k v \ln |v| \; ;$$

$$\sigma_k^2 = \sum_{v, v\neq 0} \lambda_v \operatorname{sgn}^k v \ln^2 |v| \; ; \; \varphi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \; ; \; \gamma_k = \sum_{v, v\neq 0} \lambda_v \operatorname{sgn}^k v \; ;$$

$$\lambda = \sqrt{\log n} \; ; \quad y_k = \frac{\ln |m| - E_k \lambda^2}{\lambda} \; ; \; \eta_k(t) = \sum_{v, v\neq 0} \lambda_v \operatorname{sgn}^k v \cos(t \ln |v|).$$

Let t_0 and τ_0 denote arbitrary solutions of the equations $\eta_0(t) = \gamma_0$ and $\eta_0(\tau) = -\gamma_0$ respectively , belonging to the interval $(-\pi, \pi]$.

$$f_k(a,t) := |g(a)|^{it} \operatorname{sgn}^k g(a) \; ; \; \|a\| := q^{\delta(a)};$$

$$H_k(g,G) := A^{-\lambda_0} H_1(f_k) + (-1)^n \frac{I(G)}{A} A_1^{-\gamma_0} H_2(f_k);$$

$$A_1 = \frac{1}{A} \prod_{p\in P} \left(1 - \frac{1}{\|p\|}\right)^{-1} \left(1 - \frac{(-1)^{\delta(p)}}{\|p\|}\right)^{-1};$$

$$H_1(f_k) = \frac{1}{\Gamma(\gamma_0)} \sum_{t_0} e^{-it_0 \ln|m|} \prod_{p\in P} \left(1 - \frac{1}{\|p\|}\right)^{\gamma_0} \sum_{j\geq 0, g(p^j)\neq 0} \frac{f_k(p^j, t_0)}{\|p\|^j},$$

$$H_2(f_k) = \frac{1}{\Gamma(\gamma_0)} \sum_{\tau_0} e^{-i\tau_0 \ln|m|} \prod_{p\in P} \left(1 - \frac{(-1)^{\delta(p)}}{\|p\|}\right)^{-\gamma_0} \sum_{j\geq 0, g(p^j)\neq 0} \frac{(-1)^{j\delta(p)} f_k(p^j, \tau_o)}{\|p\|^j}.$$

3. Results

THEOREM 1. *Let $g \in M(G)$, $\sigma_0^2 > 0$, and for every $a \in G$ such, that $g(a) \neq 0$, $\ln|g(a)|$ assume only integer values. Let, further, the series*

$$\sum_{v, v\neq 0} |\ln |v||^3 \lambda_v \, , \quad \sum_{p, j\geq 2, g(p^j)\neq 0} \left|\ln |g(p^j)|\right| q^{-j\delta(p)}, \quad \sum_{v, v\neq 0} |\ln |v|| |C_v(l)| \qquad (2)$$

converge (the last one uniformly in $l \geq 1$). Then, for every $m \neq 0$, we have

$$\nu_n(m) := \frac{1}{Aq^n}\#\{a \in G; \delta(a) = n, g(a) = m\}$$

$$= \sum_{k=0}^{1} \frac{\operatorname{sgn}^k m}{2n^{1-\gamma_0}}\left(H_k(g,G)\frac{\varphi(y_0/\sigma_0)}{\lambda\sigma_0} + O\left(\frac{1}{\lambda^2}\right)\right) + O(n^{-\alpha}\ln n).$$

as $n\to\infty$.

THEOREM 2. *If* $g \in M(G)$ *and* $m = 0$, *then*

$$\nu_n(0) = \left(1 - \frac{h_1(\gamma_0)}{\Gamma(\gamma_0)}(An)^{-\lambda_0} - \frac{(-1)^n I(G)}{\Gamma(-\gamma_0)(An)^{\gamma_0+1}} h_2(\gamma_0)\right) + O(n^{-\alpha}\ln n),$$

where

$$h_1(\gamma_0) = \prod_p \left(1 - \frac{1}{\|p\|}\right)^{\gamma_0} \sum_{j\geq 0} \frac{\varepsilon(p^j)}{\|p\|^j}, \quad h_2(\gamma_0) = \prod_p \left(1 - \frac{1}{\|p\|}\right)^{-\gamma_0} \sum_{j\geq 0} \frac{(-1)^{j\delta(p)}\varepsilon(p^j)}{\|p\|^j}$$

and $\varepsilon(m) = \operatorname{sgn}^2 g(m)$.

4. Proof of Theorem 1

As usual we use the main result from paper [3] concerning the mean values of multiplicative functions defined on G. If multiplicative function $g(a) \in$ M(G), then functions $f_k(a,t)$ depend to the class M(G) defined in [3], and we obtain ([3], Theorem 1):

$$\frac{1}{Aq^n}\sum_{\delta(a)=n} f_k(a,t) = \frac{(An)^{\chi_k - 1}}{\Gamma(\chi_k)} \prod_{p\in P}\left(1 - \frac{1}{\|p\|}\right)^{\chi_k} \sum_{j=0}^{\infty} \frac{f_k(p^j,t)}{\|p\|^j}$$

$$+ \mathrm{I}(G)\frac{(-1)^n A_1^{\chi_k} n^{-\chi_k - 1}}{A\Gamma(-\chi_k)} \prod_{p\in P}\left(1 - \frac{(-1)^{\delta(p)}}{\|p\|}\right)^{\chi_k} \sum_{j=0}^{\infty} \frac{(-1)^{j\delta(p)} f_k(p^j,t)}{\|p\|^j} + O(n^{-\alpha}\ln n)$$

$$:= \frac{(An)^{\chi_k - 1}}{\Gamma(\chi_k)} h_{k1}(t) + \mathrm{I}(G)\frac{(-1)^n A_1^{\chi_k} n^{-\chi_k - 1}}{A\Gamma(-\chi_k)} h_{k2}(t) + O(n^{-\alpha}\ln n) \qquad (3)$$

uniformly for any $t \in R$. Here Γ denotes the Euler gamma-function.

Using the formula (3) and the equality

$$\nu_n(m) = \frac{1}{4\pi A q^n} \sum_k \operatorname{sgn}^k m \int_{-\pi}^{\pi} e^{-it\ln|m|} \sum_{\delta(a)=n} f_k(a,t)\; dt\,,$$

we obtain

$$\nu_n(m) = \sum_k \operatorname{sgn}^k m \;\cdot J_{kj} + O(n^{-\alpha}\ln n)\,,\; j=1;2\,, \tag{4}$$

where

$$J_{k1} = \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{A^{\chi_k-1} h_{k1}(t)}{\Gamma(\chi_k)} e^{-it\ln|m|} n^{\chi_k-1} dt = \frac{1}{4\pi\; n^{\lambda_0}} \int_{-\pi}^{\pi} L_{k1}(t)\exp\left\{\lambda^2 \mu_{k1}(t) - ity_k\lambda\right\} dt$$

and

$$J_{k2} = I(G)\frac{(-1)^n}{4\pi\;A}\int_{-\pi}^{\pi} \frac{A_1^{\chi_k} h_{k2}(\tau)}{\Gamma(-\chi_k)} e^{-i\tau\ln|m|} n^{-\chi_k-1} d\tau$$

$$= I(G)\frac{(-1)^n}{4\pi\;A n^{\lambda_0}}\int_{-\pi}^{\pi} L_{k2}(\tau)\exp\left\{\lambda^2 \mu_{k2}(\tau) - i\tau y_k\lambda\right\} d\tau\,.$$

Here we used such usual notations:

$$L_{k1}(t) := \frac{A^{\;\chi_k(t)-1}}{\Gamma(\chi_k(t))} h_{k1}(t), \quad L_{k2}(\tau) := \frac{A_1^{\;\chi_k(\tau)}}{\Gamma(-\chi_k(\tau))} h_{k2}(\tau),$$

$$\mu_{k1}(u) := \chi_k(u) - \gamma_0 - itE_k\,,\; \mu_{k2}(u) := -\chi_k(u) - \gamma_0 - itE_k\,.$$

Now, the integrals J_{k1} and J_{k2} are calculated using ideas proposed in the papers [1], [2].

It is relatively simple to prove, that the equations $\eta_k(t) = \gamma_0$ and $\eta_k(\tau) = -\gamma_0$ have only finite number of solutions (denoted by t_0 and τ_0 respectively), belonging to the interval $(-\pi, \pi]$.

Therefore, we can traditionaly split the interval $(-\pi, \pi]$ into the union of subintervals around the each t_0 or τ_0 respectively (for simplicity, we say that $t_0, \tau_0 \neq \pi$.) After the substitutions $t \to t_0 + t$ and $\tau \to \tau_0 + \tau$, the path of integration for each of integrals $J_{k1}(t_0)$ and $J_{k2}(\tau_0)$ about the solutions t_0, τ_0, becomes some neighbourhood of the zero point, say $D_j(0)$, $j = 1,2$.

Hence, we have

$$J_{k1} := \sum_{t_0} J_{k1}(t_0), \qquad J_{k2} := \sum_{\tau_0} J_{k2}(\tau_0),$$

where

$$J_{k1}(t_0) = \frac{e^{-it_0 \ln|m|}}{4\pi \ n^{\lambda_0}} \int_{D_1(0)} L_{k1}(t+t_0)\exp\left\{\lambda^2 \mu_{k1}(t) - ity_k\lambda\right\}dt,$$

$$J_{k2}(\tau_0) = \frac{I(G)(-1)^n e^{-i\tau_0 \ln|m|}}{4\pi A n^{\lambda_0}} \int_{D_2(0)} L_{k2}(\tau+\tau_0)\exp\left\{\lambda^2 \mu_{k1}(\tau) - i\tau y_k\lambda\right\}d\tau,$$

because from definition of numbers $t_{0,}\tau_0$, we obtain that

$$\chi_k(t+t_0) = \chi_k(t) \text{ and } \chi_k(\tau+\tau_0) = -\chi_k(\tau),$$

which implies

$$\lambda^2 \mu_{k1}(t+t_0) - i(t+t_0)y_k\lambda = \lambda^2 \mu_{k1}(t) - ity_k\lambda - it_0 \ln|m|,$$

and

$$\lambda^2 \mu_{k2}(\tau+\tau_0) - i(\tau+\tau_0)y_k\lambda = \lambda^2 \mu_{k1}(\tau) - i\tau y_k\lambda - i\tau_0 \ln|m|.$$

If $\eta_1(t_0) \neq \gamma_0$, or $\eta_1(\tau_0) \neq -\gamma_0$, then from continuity of functions $\eta_k(u)$ we derive, that

$$\sup_{u \in D_k(0)} \eta_1(u) = -\gamma_2 < 0, \tag{5}$$

provided that in this case there exists v such that $\lambda_v > 0$ and

$$\cos(u \ln|v|)\operatorname{sgn} v < 1,$$

when $u = t_0$ or $u = \tau_0$.

Estimation (5) implies that

$$J_{11} = O\left(n^{-\gamma_2}\right) \text{and } J_{12} = O\left(n^{-\gamma_2}\right). \tag{6}$$

Further, using expansion

$$\exp\left\{\lambda^2 \mu_{k1}(t)\right\} = \exp\left\{\lambda^2\left(\gamma_k - \gamma_0 - \frac{t^2\sigma_k^2}{2} + O\left(|t|^3\right)\right)\right\},$$

from inequality $\gamma_1 < \gamma_0$ it follows, that estimations (6) hold too. So, further we let $\gamma_1 = \gamma_0$, which implies

$$\chi_1(u) = \chi_0(u),\ \ E_1 = E_0,\ \ y_1 = y_0,\ \ \sigma_1 = \sigma_0\ .$$

On the other hand, in virtue of the conditions $\sigma_0^2 > 0$, (2) and provided uniqueness of solutions t_0 , τ_0 in intervals $D_j(0)$, $j=1,2$, we deduce, that when $|u| \le \varepsilon$, then for sufficiently smallnumber $\varepsilon > 0$

$$\exp\left\{\mu_1(iu)\lambda^2\right\} = \exp\left\{\left(-\frac{u^2}{2}\sigma_0^2 + O\left(|u|^3\right)\right)\lambda^2\right\} = O\left(\exp\left\{-\gamma_3\lambda^2\right\}\right), \tag{7}$$

with some $\gamma_3 = \gamma_3(\varepsilon) > 0$.

Now, following paper [2] , we can represent integrals $J_{k1}(t_0)$, $J_{k2}(\tau_0)$ in the form

$$J_{k1}(t_0) = \frac{e^{-it_0 \ln|m|}}{4\pi(An)^{\lambda_0}} \int_{|t|\le\varepsilon} L_{k1}(t+t_0)\exp\left\{\mu_{k1}(it)\lambda^2 - ity_k\lambda\right\}dt \ + O\left(\lambda^{-2}n^{-\lambda_0}\right)$$

and

$$J_{k2}(\tau_0) = \frac{I(G)(-1)^n e^{-i\tau_0 \ln|m|}}{4\pi A A_1^{\gamma_0} n^{\lambda_0}} \int_{|\tau|\le\varepsilon} L_{k2}(\tau+\tau_0)\exp\left\{\mu_{k1}(i\tau)\lambda^2 - i\tau y_k\lambda\right\}d\tau \ + O\left(\lambda^{-2}n^{-\lambda_0}\right),$$

since

$$L_{k1}(t+t_0) = O(1),\ L_{k2}(\tau+\tau_0) = O(1),$$

when $t \in D_1(0)$ and $\tau \in D_2(0)$.

Now, using condition (2), in the neighbourhood $|u| < \varepsilon$, we obtain estimations

$$\chi_k(u) = \gamma_0 + iuE_0 - \frac{u^2}{2}\sigma_0^2 + O(|u|^3) = \gamma_0 + O(|u|),\ \ \Gamma^{-1}(\chi_k) = \Gamma^{-1}(\gamma_0) + O(|u|),$$
$$A^{\chi_k - 1} = A^{-\lambda_0} + O(|u|),\ \ A_1^{\chi_k} = A_1^{-\gamma_0} + O(|u|)\ . \tag{8}$$

Finally, calculating as in papers [1], [2], we can obtain the following estimations

$$h_{k1}(t+t_0) = h_{k1}(t_0) + O(|t|),\ \ h_{k2}(\tau+\tau_0) = h_{k2}(\tau_0) + O(|\tau|). \tag{9}$$

The proof of (9) is based upon the equality

$$h_{kj}(u) := \prod_p \psi_{kj,p}(u)$$

$$= \prod_{\|p\|\le M} \psi_{kj,p}(u) \prod_{\|p\|>M} \psi_{kj,p}(u_0)\exp\left\{\sum_{\|p\|>M} \log\left(1 + \left(\frac{\psi_{kj,p}(u)}{\psi_{kj,p}(u_0)} - 1\right)\right)\right\} \tag{10}$$

When M is sufficiently large fixed number, then from conditions of Theorem 1 we derive, that

$$|\psi_{kj,p}(u)| > 1/2 \text{ and } \psi_{kj,p}^{\pm 1}(u) = 1 + O\left(\frac{1}{\|p\|}\right)$$

which implies

$$\prod_{\|p\| \geq M} \psi_{kj,p}(u) = \prod_{\|p\| \geq M} \psi_{kj,p}(u_0) + O(|u|)$$

with $u = t_0$ or $u = \tau_0$ and $|u| \leq \varepsilon$. For finite number of factors in (10) analogical expansions holds and we obtain the desired estimations.

According to the estimates of the type (6), (7), (8) and (9) we obtain

$$J_{k1}(t_0) = \frac{e^{-it_0 \ln|m|}}{2(An)^{\lambda_0}\Gamma(\gamma_0)} h_{k1}(t_0)\frac{\varphi(y_0/\sigma_0)}{\lambda\sigma_0} + O\left(\lambda^{-2} n^{-\lambda_0}\right)$$

and

$$J_{k2}(\tau_0) = \frac{I(G)(-1)^n e^{-i\tau_0 \ln|m|}}{2AA_1^{\gamma_0} n^{\lambda_0}} h_{k2}(\tau_0)\frac{\varphi(y_0/\sigma_0)}{\lambda\sigma_0} + O\left(\lambda^{-2} n^{-\lambda_0}\right).$$

Summing up these relations over all t_0 and τ_0 respectively, and putting into (4), we end the proof of Theorem 1.

5. Proof of Theorem 2

Denote $\varepsilon(m) = \operatorname{sgn}^2 g(m)$. Using definition of the class M(G), we obtain

$$\sum_{\substack{p,\delta(p)=l \\ \varepsilon(p)=0}} 1 = \pi(l)(\lambda_0 + \rho_0(l)),$$

$$\sum_{\substack{p,\delta(p)=l \\ \varepsilon(p)=1}} 1 = \pi(l)(1 - \lambda_0 + \rho\ (l)),$$

where condition (1) implies, that $\rho(l) = O\left(l^{-\alpha}\right)$.

In virtue, that $\varepsilon(a) \in M(G)$, and using Theorem 1 from [3], we have

$$\frac{1}{Aq^n}\sum_{\delta(a)=n}\varepsilon(a) = \frac{(An)^{\chi-1}}{\Gamma(\chi)}\prod_{p\in P}\left(1-\frac{1}{\|p\|}\right)^{\chi}\sum_{j=0}^{\infty}\frac{\varepsilon(p^j)}{\|p\|^j}$$

$$+\, \mathrm{I}(G)\frac{(-1)^n A_1^{\chi} n^{-\chi-1}}{A\Gamma(-\chi)}\prod_{p\in P}\left(1-\frac{(-1)^{\delta(p)}}{\|p\|}\right)^{\chi}\sum_{j=0}^{\infty}\frac{(-1)^{j\delta(p)}\,\varepsilon(p^j)}{\|p\|^j} + \mathrm{O}(n^{-\alpha}\ln n).$$

Now, using definition of number A_1 and applying formula

$$\nu_n(0) = 1 - \frac{1}{Aq^n} \sum_{\delta(a)=n} \varepsilon(a),$$

we obtain the assertion of theorem 2, because in this case $\chi = 1 - \lambda_0$.

Remark. If $\sigma_0^2 = 0$, then from $\lambda_\nu > 0$ it foollows $\nu = 0; \pm 1$, which implies $\chi_k = \gamma_k = \lambda_1 + (-1)^k \lambda_{-1}$ and

$$\nu_n(m) = \sum_{k=0}^{1} \operatorname{sgn}^k m \ \Delta_k(g) + O\left(n^{-\alpha} \ln n\right),$$

where

$$\Delta_k(g) := \frac{(An)^{\gamma_k - 1}}{\Gamma(\gamma_k)} \delta_{k1}(g) + \frac{I(G)(-1)^n}{A\Gamma(-\gamma_k)n} A_1^{\gamma_k} n^{-\gamma_k} \delta_{k2}(g)$$

and

$$\delta_{kj}(g) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \operatorname{Re}\left(h_{kj}(t) \exp\{-it \ln |m|\}\right) dt, \ j = 1; 2.$$

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Lokalinės ribinės teoremos multiplikatyvioms funkcijoms pusgrupėse

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Straipsnyje tęsiami aritmetinių funkcijų apibrėžtų specialiame "aritmetiniame" pusgrupyje reikšmių pasiskirstymo tyrimai. Įrodyta lokalioji ribinė teorema multiplikatyviosioms funkcijoms, tenkinančioms lokalumo sąlygas pusgrupio pirminių elementų aibėje.