LMD mokslo darbai, III tomas, 62–67

On the mean square of the Lerch zeta-function

Antanas LAURINČIKAS (VU, ŠU)*
e-mail: antanas.laurincikas@maf.vu.lt

1. Introduction

Let $\lambda \in \mathbb{R}$ and $0 < \alpha \leqslant 1$. The Lerch zeta–function $L(\lambda, \alpha, s)$, $s = \sigma + it$, is defined by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}$$

for $\sigma > 1$ if $\lambda \in \mathbb{Z}$, and for $\sigma > 0$ if $\lambda \notin \mathbb{Z}$. For $\lambda \in \mathbb{Z}$ the Lerch zeta-function reduces to the Hurwitz zeta-function

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}, \quad \sigma > 1.$$

Moreover, we have that $L(k, 1, s) = \zeta(s)$, $k \in \mathbb{Z}$, where $\zeta(s)$ denotes the classical Riemann zeta-function, and

$$L\left(k, \frac{1}{2}, s\right) = \zeta(s)(2^s - 1).$$

The function $L(\lambda, \alpha, s)$ was introduced by M. Lerch in [5].

The Hurwitz zeta-function is analytically continuable over the whole complex plane $\mathbb{C}$ except for a simple pole, with residue 1 at the point $s = 1$. If $\lambda \notin \mathbb{Z}$ (in this case we can assume without loss of generality that $0 < \lambda < 1$), then the function $L(\lambda, \alpha, s)$ is analytically continuable to an entire function.

In this note we consider the mean square of the Lerch zeta-function, i.e.

$$I_{\lambda,\alpha}(\sigma, T_0, T) = \int_{T_0}^{T} \left|L(\lambda, \alpha, \sigma + it)\right|^2 \mathrm{d}t, \quad \sigma \geqslant 1/2,\ T \to \infty.$$

In [3] it was obtained that, for $\lambda \notin \mathbb{Z}$,

$$I_{\lambda,\alpha}\left(\frac{1}{2}, 0, T\right) \sim T \log T$$

and, for $\frac{1}{2} < \sigma < 1$,

$$I_{\lambda,\alpha}(\sigma, 0, T) \sim T\zeta(2\sigma, \alpha)$$

as $T \to \infty$. In this note we improve the latter results. Let B_η denote a number bounded by a constant depending on η.

Theorem 1. *Suppose that* $\frac{1}{2} < \sigma < 1$ *is fixed, and* λ *is an arbitrary real number. Then for* $T \to \infty$

$$I_{\lambda,\alpha}(\sigma, 0, T) = T\zeta(2\sigma, \alpha) + B_{\lambda,\alpha,\sigma}T^{2-2\sigma}.$$

Theorem 2. *Let* $T \to \infty$*. Then for an arbitrary real* λ

$$I_{\lambda,\alpha}\Big(\frac{1}{2}, 0, T\Big) = T\log T + B_{\lambda,\alpha}T.$$

Theorem 3. *Let* $T \to \infty$*. Then for an arbitrary real* λ

$$I_{\lambda,\alpha}(1, 1, T) = T\zeta(2\sigma, \alpha) + B_{\lambda,\alpha}\log T.$$

2. Auxiliary results

We begin with an approximation of the function $L(\lambda, \alpha, s)$ by a finite sum.

Lemma 1. *Suppose that* $0 < \lambda < 1$*, and let* $0 < \sigma_0 \leqslant \sigma$*,* $|t| \leqslant \pi\lambda x$*. Then*

$$L(\lambda, \alpha, s) = \sum_{0 \leqslant m \leqslant x} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^s} + B_{\sigma_0,\lambda}x^{-\sigma}.$$

Proof is given in [1], [4].

Lemma 2. *Let* $0 < \sigma_0 \leqslant \sigma$ *and* $2\pi \leqslant |t| \leqslant \pi x$*. Then*

$$\zeta(s, \alpha) = \sum_{0 \leqslant m \leqslant x} \frac{1}{(m+\alpha)^s} + \frac{x^{1-s}}{s-1} + B_{\sigma_0}x^{-\sigma}.$$

Proof can be found in [2].

We also need a version of the Montgomery–Vanghan theorem [6].

Lemma 3. *Let* $a_m \in \mathbb{C}$*. Then there exists an absolute constant* $c > 0$ *such that*

$$\Big|\sum_{\substack{m=1\\m\neq k}}^{n}\sum_{k=1}^{n} a_m\bar{a}_k\Big(\log\frac{m+\alpha}{k+\alpha}\Big)^{-1}\Big| \leqslant c\sum_{m=1}^{n} m|a_m|^2.$$

Proof is given in [7].

3. Proofs

Proof of Theorem 1. First we will consider the case $\lambda \notin \mathbb{Z}$. Suppose $T/2 \leqslant t \leqslant T$ and take $x = T\lambda^{-1}$ in Lemma 1. Then we obtain that

$$L(\lambda, \alpha, s) = \sum_{0 \leqslant m \leqslant T\lambda^{-1}} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s} + R(s),$$

where $R(s) = B_\lambda T^{-\sigma}$. Since $|z| = z\bar{z}$, hence we have

$$\int_{T/2}^{T} |L(\lambda, \alpha, \sigma + it)|^2 \,\mathrm{d}t = \int_{T/2}^{T} \Big| \sum_{0 \leqslant m \leqslant T\lambda^{-1}} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^{\sigma+it}} \Big|^2 \mathrm{d}t$$
$$+ 2\mathrm{Re} \int_{T/2}^{T} \sum_{0 \leqslant m \leqslant T\lambda^{-1}} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^{\sigma+it}} \overline{R(\sigma+it)} \,\mathrm{d}t + B_\lambda T^{1-2\sigma}. \tag{1}$$

By Lemma 3 the first term in (1) is

$$\frac{T}{2} \sum_{0 \leqslant m \leqslant T\lambda^{-1}} \frac{1}{(m+\alpha)^{2\sigma}}$$
$$+ B \Bigg| \sum_{\substack{0 \leqslant m \leqslant T\lambda^{-1} \\ 0 \leqslant k \leqslant T\lambda^{-1} \\ m \neq k}} \sum \frac{e^{2\pi i \lambda m - 2\pi i \lambda k} \left(\left(\frac{m+\alpha}{k+\alpha}\right)^{-iT} - \left(\frac{m+\alpha}{k+\alpha}\right)^{\frac{-iT}{2}} \right)}{(m+\alpha)^\sigma (k+\alpha)^\sigma \log \frac{m+\alpha}{k+\alpha}} \Bigg|$$
$$= \frac{T}{2} \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{2\sigma}} + B_{\lambda,\sigma} T^{2-2\sigma} + B_{\lambda,\sigma} \sum_{0 \leqslant m \leqslant T\lambda^{-1}} \frac{m}{(m+\alpha)^{2\sigma}}$$
$$= \frac{T}{2} \zeta(2\sigma, \alpha) + B_{\lambda,\alpha,\sigma} T^{2-2\sigma}. \tag{2}$$

It is easily seen that the second term in (1) is estimated as

$$B_{\lambda,\alpha,\sigma} T^{2-2\sigma}. \tag{3}$$

Hence and from (1), (2) we have

$$\int_{T/2}^{T} |L(\lambda, \alpha, \sigma + it)|^2 \,\mathrm{d}t = \frac{T}{2} \zeta(2\sigma, \alpha) + B_{\lambda,\alpha,\sigma} T^{2-2\sigma}.$$

Taking $T \cdot 2^{-j}$ instead of T in the later formula and summing over $j = 0, 1, 2, \ldots$, we obtain the theorem.

When $\lambda \in \mathbb{Z}$, the proof remains the same and it uses Lemma 2.

Proof of Theorem 2. We have

$$\sum_{0 \leqslant m \leqslant T\lambda^{-1}} \frac{1}{m+\alpha} = B_\alpha + \log T + B_\lambda,$$
$$\sum_{0 \leqslant m \leqslant T\lambda^{-1}} \frac{m}{m+\alpha} = B_{\lambda,\alpha} T.$$

Hence, using (1)–(3) with $\sigma = \frac{1}{2}$, we find that

$$\int_{T/2}^{T} \left| L(\lambda, \alpha, \frac{1}{2} + it) \right|^2 \mathrm{d}t = \frac{T}{2} \log T + B_{\lambda,\alpha} T.$$

Consequently, the theorem follows in the same way as Theorem 1.

Proof of Theorem 3. Let $\lambda \notin \mathbb{Z}$. Then by Lemma 1

$$L(\lambda, \alpha, 1 + it) = \sum_{0 \leqslant m \leqslant T\lambda^{-1}} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^{1+it}} + B_\lambda T^{-1}.$$

Hence

$$\int_1^T |L(\lambda, \alpha, 1 + it)|^2 \mathrm{d}t = \int_1^T \left| \sum_{0 \leqslant m \leqslant T\lambda^{-1}} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^{1+it}} \right|^2 \mathrm{d}t$$
$$+ B_\lambda T^{-1} \int_1^T \left| \sum_{0 \leqslant m \leqslant T\lambda^{-1}} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^{1+it}} \right| \mathrm{d}t + B_\lambda T^{-1}. \tag{4}$$

By Lemma 3 the first term in the last equality is

$$(T-1) \sum_{0 \leqslant m \leqslant T\lambda_{-1}} \frac{1}{(m+\alpha)^2} + B_\alpha \sum_{m \leqslant T\lambda^{-1}} \frac{1}{m}$$
$$= \zeta(2, \alpha) T + B_\lambda + B_{\lambda,\alpha} \log T = \zeta(2, \alpha) T + B_{\lambda,\alpha} \log T. \tag{5}$$

This and the Cauchy–Schwarz inequality yield the estimate B_λ for the second term of the right-hand side of (4). Therefore, the theorem is a consequence of (4) and (5).

The case $\lambda \in \mathbb{Z}$ is more complicated. In this case Lemma 2 gives

$$\zeta(1+it,\alpha) = \sum_{0\leqslant m\leqslant T} \frac{1}{(m+\alpha)^{1+it}} + \frac{T^{-it}}{it} + BT^{-1}.$$

Thus,

$$\begin{aligned}\int_1^T \left|\zeta(1+it,\alpha)\right|^2 \mathrm{d}t &= \int_1^T \Big| \sum_{0\leqslant m\leqslant T} \frac{1}{(m+\alpha)^{1+it}}\Big|^2 \mathrm{d}t \\ &- 2\mathrm{Re}\Big(\frac{1}{i}\int_1^T \sum_{0\leqslant m\leqslant T} \frac{1}{m+\alpha}\Big(\frac{T}{m+\alpha}\Big)^{it}\frac{\mathrm{d}t}{t} \\ &+ BT^{-1}\int_1^T \sum_{0\leqslant m\leqslant T}\Big(\frac{1}{m+\alpha}\Big)^{1+it}\mathrm{d}t + B. \end{aligned} \tag{6}$$

The first and the third integrals in (6) were evaluated above, and it remains to calculate the second integral. Suppose that $2T^{-1} \leqslant c \leqslant \frac{2}{3}$. Then, integrating by parts, we find

$$\begin{aligned} &\int_1^T \sum_{0\leqslant m\leqslant T} \frac{1}{m+\alpha}\Big(\frac{T}{m+\alpha}\Big)^{it}\frac{\mathrm{d}t}{t} \\ &= \sum_{0\leqslant m\leqslant T(1-c)} \frac{1}{m+\alpha}\left(\frac{\left(\frac{T}{m+\alpha}\right)^{it}}{it\log\frac{T}{m+\alpha}}\Bigg|_1^T + \int_1^T \frac{\left(\frac{T}{m+\alpha}\right)^{it}}{it^2\log\frac{T}{m+\alpha}}\,\mathrm{d}t\right) \\ &\quad + B\sum_{T(1-c)<m\leqslant T}\frac{1}{m+\alpha}\int_1^T\frac{\mathrm{d}t}{t} \\ &= B\sum_{0\leqslant m\leqslant T(1-c)}\frac{1}{(m+\alpha)\log\frac{T}{m+\alpha}} + B\log T\sum_{T(1-c)\leqslant m\leqslant T}\frac{1}{m} \\ &= B_\alpha + B\int_1^{T(1-c)}\frac{\mathrm{d}u}{(u+\alpha)\log\frac{T}{u+\alpha}} + Bc\log T + B \\ &= B\int_{(1-c)^{-1}}^T \frac{\mathrm{d}v}{v\log v} + Bc\log T + B_\alpha \\ &= B\log\log T - \log\big(-\log(1-c)\big) + Bc\log T + B\alpha \\ &= B\log\log T + B\log\frac{1}{c} + Bc\log T + B_\alpha. \end{aligned}$$

Now, taking $c = 1/\log T$, we obtain the estimate

$$B \log\log T + B_\alpha$$

for the second integral in the righ–hand side of (6), and the theorem is also proved in the case $\lambda \in \mathbb{Z}$.

References

[1] R. Garunkštis, A. Laurinčikas, On the Lerch zeta-function, *Lith. Math. J.*, **36**(4), 337–346 (1996).
[2] A.A. Karatsuba, S.M. Voronin, *The Riemann Zeta-Function*, Walter de Gruyter, Berlin (1992).
[3] D. Klusch, Asymptotic equalities for the Lipschitz–Lerch zeta-function, *Arch. Math.*, **49**, 38–43 (1987).
[4] A. Laurinčikas, A limit theorem for the Lerch zeta-function in the space of analytic functions, *Lith. Math. J.*, **37**(2), 146–155 (1997).
[5] M. Lerch, Note sur la fonction $K(w, x, s) = \sum_{h \geqslant 0} \exp\{2\pi n x\}(n + w)^{-s}$, *Acta Math.*, **11**, 19–24 (1887).
[6] H.L. Montgomery, R.Vaughan, Hilbert's inequality, *J. London Math. Soc.*, **8**(2), 73–82 (1974).
[7] K. Ramachandra, *On the Mean-Value and Omega-Theorems for the Riemann Zeta-Function*, Tata Inst. Fund. Research, Springer–Verlag, Berlin (1995).

Apie Lercho dzeta funkcijos kvadrato vidurkį

A. Laurinčikas

Straipsnyje gauti Lercho dzeta funkcijos kvadrato vidurkio liekamojo nario įverčiai kritinėje juostoje.