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Large deviations for counting processes

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Let, on a stochastic basis $(\Omega, \mathcal{F}, F, P)$, there be given a counting process $N(t) = \sum_{n\geqslant 1} \mathbf{1}(T_n \leqslant t)$, $t \in \mathbb{R}_+ = [0,\infty)$ where $\{T_n, n \geqslant 1\}$ is a nondecreasing sequence of positive F – stopping times with continuous distribution. Let $N(\infty) = \infty$ a.s.

Introduce the following assumption:

A. There exists a differentiable function $\psi_\beta(\lambda)$, $\lambda \in \mathbb{R}$ such that

$$\lim_{n\to\infty} n^{-1} \log \mathrm{E}e^{\lambda S_n(\beta)} = \psi_\beta(\lambda) < \infty \tag{1}$$

for some $\beta > 0$, where $S_n(\beta) = n^{1-\beta} T_n$.

We will use common notation from the theory of large deviations (see, e.g., [3], [4].)

Lemma 1 ([3], [1]). *Let condition A be satisfied. Then the family of measures* $P_n = \mathcal{L}(n^{-\beta}T_n)$, $n \in \mathbb{N}$ *satisfy the large deviation principle* (LDP) *with rate function* $I_\beta(x) = \sup_\lambda(\lambda x - \psi_\beta(\lambda))$, $x \geqslant 0$, *and:*

1) $n^{-\beta}T_n \to a = \psi_\beta{}'(0)$ *as* $n \to \infty$ $P-a.s.$,

2) *for all* $\gamma \in [0, a)$

$$\lim_{n\to\infty} n^{-1} \log P(n^{-\beta}T_n < \gamma) = -I_\beta(\gamma) \tag{2}$$

and for all $\gamma \in (a, \infty)$

$$\lim_{n\to\infty} n^{-1} \log P(n^{-\beta}T_n > \gamma) = -I_\beta(\gamma). \tag{3}$$

Lemma 2. *Under condition A*

$$t^{-\frac{1}{\beta}} N(t) \to a^{-\frac{1}{\beta}} \quad as \quad t \to \infty \quad P-a.s. \tag{4}$$

Proof. For every counting process $N(t)$ we have

$$T_{N(t)-1} \leqslant t < T_{N(t)+1}. \tag{5}$$

Remark that $N(t) \to \infty$ as $t \to \infty$ P – a.s. Dividing (5) by $N(t)^\beta$ from Lemma 1 we get

$$\lim_{t\to\infty} \frac{t}{N(t)^\beta} = a \quad P- \text{ a.s.}$$

Therefore

$$t^{-\frac{1}{\beta}}N(t) \to a^{-\frac{1}{\beta}} \quad \text{as} \quad t \to \infty \quad P-\text{a.s.}$$

The lemma is proved.

Theorem. *Under condition A a family of measures* $P_{t^{\frac{1}{\beta}}} = \mathcal{L}(t^{-\frac{1}{\beta}}N(t))$, $t \in \mathbb{R}_+$ *satisfy the large deviation principle with rate function* $I_{N,\beta}(x) = xI_\beta(x^{-\beta})$, $x \in \mathbb{R}_+$.

Proof. We have

$$P_{t^{\frac{1}{\beta}}}(B) = P(t^{-\frac{1}{\beta}}N(t) \in B), \quad B \in \mathcal{B}(\mathbb{R}_+).$$

Remark that

$$\{N(t) = n\} = \{T_n \leqslant t\} \setminus \{T_{n+1} \leqslant t\}.$$

Therefore, for arbitrary $0 < c < d$ we have

$$\begin{aligned} P_{t^{\frac{1}{\beta}}}([c,d]) &= P\left(N(t) \in \left[ct^{\frac{1}{\beta}}, dt^{\frac{1}{\beta}}\right]\right) = \sum_{n\in[n_1,n_2]} P(N(t) = n) \\ &= \sum_{n\in[n_1,n_2]} (F_n(t) - F_{n+1}(t)) = F_{[n_1]}(t) - F_{[n_2]+1}(t), \end{aligned} \tag{6}$$

and analogously

$$\begin{aligned} P_{t^{\frac{1}{\beta}}}((c,d)) &= P\left(N(t) \in \left(ct^{\frac{1}{\beta}}, dt^{\frac{1}{\beta}}\right)\right) = \sum_{n\in(n_1,n_2)} P(N(t) = n) \\ &= F_{[n_1]+1}(t) - F_{[n_2-0]+1}(t), \end{aligned} \tag{7}$$

where $n_1 = ct^{\frac{1}{\beta}}$, $n_2 = dt^{\frac{1}{\beta}}$, $[\ ]$ denotes the integer part of a number, and $F_n(t) = P(T_n \leqslant t)$.

Taking $n = t^{\frac{1}{\beta}}x(1 + o(1))$ as $t \to \infty$ in 2) Lemma 1, we get for all $x^{-\beta} \in [0, a)$

$$\lim_{t\to\infty} t^{-\frac{1}{\beta}} \log P(T_{t^{\frac{1}{\beta}}x} < t) = -xI_\beta(x^{-\beta}) \tag{8}$$

and for all $x^{-\beta} \in (a, \infty)$

$$\lim_{t\to\infty} t^{-\frac{1}{\beta}} \log P(T_{t^{\frac{1}{\beta}}x} > t) = -xI_\beta(x^{-\beta}). \tag{9}$$

Using (8) and (9) for (6) and (7) we get

$$\limsup_{t\to\infty} t^{-\frac{1}{\beta}} \log P_{t^{\frac{1}{\beta}}}([c,d]) \leqslant -\inf_{x\in[c,d]} I_{N,\beta}(x), \tag{10}$$

$$\liminf_{t\to\infty} t^{-\frac{1}{\beta}} \log P_{t^{\frac{1}{\beta}}}((c,d)) \geqslant -\inf_{x\in(c,d)} I_{N,\beta}(x). \tag{11}$$

Similarly, one can verify that

$$\limsup_{t\to\infty} t^{-\frac{1}{\beta}} \log P_{t^{\frac{1}{\beta}}}(F) \leqslant -\inf_{x\in F} I_{N,\beta}(x), \tag{12}$$

$$\liminf_{t\to\infty} t^{-\frac{1}{\beta}} \log P_{t^{\frac{1}{\beta}}}(G) \geqslant -\inf_{x\in G} I_{N,\beta}(x). \tag{13}$$

for all closed sets F and open sets G from $\mathbb{R}_+$. The theorem is proved.

COROLLARY (cf. [5]). Let $\{X_n,\, n \geqslant 1\}$ is a sequence of independent and identically distributed random variables such that $\psi(\lambda) = \mathrm{E}e^{\lambda X_1} < \infty$ for $\lambda > 0$ and $X_i \geqslant 0$ for all i. Then for the extended renewal process $N_\alpha(t) = \sum_{n\geqslant 1} \mathbf{1}(T_n(\alpha) \leqslant t)$ with $T_n(\alpha) = n^{-\alpha} S_n$, $S_n = \sum_{i=1}^{n} X_i$, $\alpha \in [0,1)$ is true the result of Theorem with $\beta = 1 - \alpha$.

REMARK 1. When $\beta = 1$ in Theorem we get result of Glynn and Whitt (see Theorem 1 [2]).

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Skaičiuojančių procesų Didieji nuokrypiai

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Šiame darbe nagrinėjami bendrieji skaičiuojantys procesai. Nustatytos bendros sąlygos, suformuluotos momentams T_n, esant kurioms galioja Didžiųjų nuokrypių principas šiems procesams. Taip pat gautas Didžiuosius nuokrypius atitinkančios greičio funkcijos pavidalas.