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The existence and uniqueness of the solution of the integral equation driven by fractional Brownian motion

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Introduction

In this note we consider a non-linear stochastic integral equation (SIE)

$$X_t = \xi + \int_0^t f(X_s)\,ds + \int_0^t g(X_s)\,dB_s^H, \qquad 0 \leqslant t \leqslant T, \tag{1}$$

where B^H is a fractional Brownian motion (fBm) with the Hurst index $1/2 < H < 1$. It is known that almost all sample paths of fBm B^H, $1/2 \leqslant H < 1$, have bounded p-variation for $p > 1/H$. Thus the integrals on the right side of (1) will exist pathwise as the Riemann-Stieltjes integrals.

A solution of the stochastic integral equation (1), on a given filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{F})$ and with respect to the fixed fBm $(B^H, \mathbf{F})$, $1/2 < H < 1$, and initial condition ξ, is an adapted to the filtration $\mathbf{F}$ continuous process $X = \{X_t\colon 0 \leqslant t \leqslant T\}$ such that $X_0 = \xi$ a.s., $\mathbf{P}(\int_0^t |f(X_s)|\,ds + |\int_0^t g(X_s)\,dB_s^H| < \infty) = 1$ for every $0 \leqslant t \leqslant T$, and its almost all sample paths satisfy (1).

For $0 < \alpha \leqslant 1$, $\mathcal{H}_{1+\alpha}$ will denote the set of all C^1-functions $g\colon \mathbf{R} \to \mathbf{R}$ such that

$$\sup_x |g'(x)| + \sup_{x \neq y} \frac{|g'(x) - g'(y)|}{|x-y|^\alpha} < \infty.$$

Denote by $\mathcal{W}_p([a,b])$, $1 < p < 2$, the class of all functions defined on $[a,b]$ with bounded p-variation. Let $CW_p([a,b])$ denote the subclass of $\mathcal{W}_p([a,b])$ of all continuous functions.

Theorem. *Let f be a Lipshitz function and $g \in \mathcal{H}_{1+\alpha}$, $0 < \alpha \leqslant 1$. For $1 \leqslant p < 1+\alpha$ there exists a unique solution of the equation* (1) *with almost all sample paths in the* $CW_p([0,T])$.

Existence and uniqueness of the solution

All facts mentioned bellow about the p-variation are taken from [1] and [6].

For $p \geqslant 1$, denote by $v_p(f;[a,b])$ the p-variation of the function f on $[a,b]$ and define $V_p(f;[a,b]) = v_p^{1/p}(f;[a,b])$, which is a seminorm on $\mathcal{W}_p([a,b])$. Let $V_{p,\infty}(f;[a,b]) = v_p(f;[a,b]) + \sup_{a\leqslant x\leqslant b}|f(x)|$. Then $V_{p,\infty}(f;[a,b])$ is a norm and $\mathcal{W}_p([a,b])$ equipped with this norm is a Banach space.

Note that

$$V_{p,\infty}(f;[a,b]) \leqslant V_{p,\infty}(f;[a,c]) + V_{p,\infty}(f;[c,b]), \tag{2}$$

where $a < c < b$. The rest inequality follows from the inequality

$$V_p(f;[a,b]) \leqslant V_p(f;[a,c]) + V_p(f;[c,b]). \tag{3}$$

Let $f \in \mathcal{W}_q([a,b])$ and $h \in \mathcal{W}_p([a,b])$ with $p > 0$, $q > 0$, $1/p + 1/q > 1$. If f and h have no common discontinuities then the Riemann – Stieljes integral $\int_a^b f\,dh$ exists and the Love–Young inequality

$$V_p\left(\int_a^{\cdot} f(x)\,dh(x)\right) \leqslant C_{p,q} V_{q,\infty}(f;[a,b]) V_p(h;[a,b]) \tag{4}$$

holds, where $C_{p,q} = \zeta(p^{-1}+q^{-1})$, $\zeta(s)$ denotes the Riemann zeta function, i.e., $\zeta(s) = \sum_{n\geqslant 1} n^{-s}$.

Let f be a function on $[0,T]$ and let $\lambda = \{\lambda_m\colon m \geqslant 1\}$ be a sequence of dyadic partitions $\lambda_m = \{i2^{-m}\colon i = 0,\dots,([T]+1)2^m\}$ of $[0,[T]+1)$. For $0 < p < \infty$ and $0 < t \leqslant T$, let

$$v_p(f;\lambda_m)(t) := \max\Big\{ \sum_{j=1}^{k} |f(s_j\wedge t) - f(s_{j-1}\wedge t)|^p\colon \\ \{0,([T]+1)2^m\} \subset \{s_j\colon j = 0,\dots,k\} \subset \lambda_m \Big\},$$

which is the p-variation over the finite set $\{i2^{-m}\wedge t\colon i = 0,\dots,([T]+1)2^m\}$.

Since λ is a sequence of nested partitions, the sequence $v_p(f;\lambda_m)(t)$, $m \geqslant 1$, is non-decreasing for each $0 < t \leqslant T$. For $0 \leqslant t \leqslant T$, let

$$v_p(f)(t) := \sup_{m\geqslant 1} v_p(f;\lambda_m)(t) = \lim_{m\to\infty} v_p(f;\lambda_m)(t).$$

For a stochastic process $Y = \{Y(t)\colon 0 \leqslant t \leqslant T\}$ and each $0 \leqslant t \leqslant T$, $v_p(Y)(t,\omega) := v_p(Y(\cdot,\omega))(t)$ is possibly unbounded but measurable function of $\omega \in \Omega$. Let Y be a cadlag process. If $v_p(Y)(T) < \infty$ almost surely, then $\{v_p(Y;[0,t])\colon 0 \leqslant t \leqslant T\}$ is a stochastic process indistinguishable from $v_p(Y)$. Moreover, $\{v_p(Y;[0,t])\colon 0 \leqslant t \leqslant T\}$ is a cadlag stochastic process.

The proof of the Theorem 1 is similar to the proof in the case when an integral equation is driven by a deterministic function of bounded p-variation (see [2]-[4]). Here we have to prove in addition that a solution is indeed adapted.

The existence of a solution is proved using the Picard iteration method, i.e. we consider the iteration

$$X_t^{n+1} = \xi + \int_0^t f(X_s^n)ds + \int_0^t g(X_s^n)dB_s^H, \qquad n \geqslant 0, \tag{5}$$

where $X^0 = \xi$. The integrals on the right side of (5) are clearly well defined for $n = 0$. Moreover, the processes X^1 is continuous, $\mathbf{F}$-adapted and $v_p(X^1)(T) < \infty$ a.s. By induction one can prove that for any $n \geqslant 1$ the process X^n has these properties.

First we will prove two lemmas. Define a sequence of stopping times

$$\tau_n = \inf\left\{t > \tau_{n-1}\colon V_p\big(B^H; [\tau_{n-1}, t]\big) > \frac{1}{4C_{p,p}} \min\left\{1, L^{-1}, \left(2|g'|_\infty\right)^{-1}\right\}\right\}$$
$$\wedge\left(\tau_{n-1} + \frac{1}{4C_{p,p}} \min\left\{1, L^{-1}, \left(2|g'|_\infty\right)^{-1}\right\}\right), \quad n \in \mathbf{N}, \qquad \tau_0 = 0,$$

where L is the Lipshitz constant of the function f in (5) and $|g'|_\infty = \sup_x |g'(x)|$.

Lemma 1. *For any $m, n \in \mathbf{N}$ the inequality*

$$V_p\big(X^{n+1}; [0, \tau_m]\big) \leqslant 2^{m+1} \max\left\{1, |\xi|\right\}\left[1 + |f(0)| + |g(0)|\right] \tag{6}$$

holds.

Proof. For any n and k, by the Love-Young inequality (4), we have

$$\begin{aligned}
&V_p\big(X^{n+1}; [\tau_{k-1}, \tau_k]\big)\\
&\leqslant V_1\bigg(\int_0^\cdot f(X^n)\,ds; [\tau_{k-1}\tau_k]\bigg) + V_p\bigg(\int_0^\cdot g(X^n)\,dB_s^H; [\tau_{k-1}\tau_k]\bigg)\\
&\leqslant \int_{\tau_{k-1}}^{\tau_k} |f(X_s^n)|\,ds + C_{p,p}V_{p,\infty}\big(g(X^n); [\tau_{k-1}, \tau_k]\big)V_p\big(B^H; [\tau_{k-1}, \tau_k]\big)\\
&\leqslant \Big[|f(0)| + L|X^n(\tau_{k-1})| + L\cdot V_p\big(X^n; [\tau_{k-1}, \tau_k]\big)\Big]\big(\tau_k - \tau_{k-1}\big)\\
&\quad + C_{p,p}\Big[2|g'|_\infty V_p\big(X^n; [\tau_{k-1}, \tau_k]\big) + |g(0)| + |g'|_\infty |X^n(\tau_{k-1})|\Big]\\
&\quad \times V_p\big(B^H; [\tau_{k-1}, \tau_k]\big).
\end{aligned} \tag{7}$$

To prove (6) we use the inequality (7) for induction on k. First we estimate the quantity $V_p(X^{n+1}; [0, \tau_1])$. Denote $R := 2\max\{1, |\xi|\}(1 + |f(0)| + |g(0)|)$. By the definition of the stopping time τ_1 it is obvious that for all $n \geqslant 0$

$$\begin{aligned}
V_p\big(X^{n+1}; [0, \tau_1]\big) &\leqslant |f(0)| + |g(0)| + |\xi| + \frac{1}{2} V_p\big(X^n; [0, \tau_1]\big)\\
&\leqslant \Big(1 + \frac{1}{2} + \cdots + \frac{1}{2^n}\Big)\big(|f(0)| + |g(0)| + |\xi|\big)\\
&\quad + \frac{1}{2^{n+1}} V_p\big(X^0; [0, \tau_1]\big)
\end{aligned}$$

$$\leqslant 2\big(|f(0)| + |g(0)| + |\xi|\big) \leqslant R.$$

Let

$$V_p\big(X^{n+1}; [\tau_{k-1}, \tau_k]\big) \leqslant 2^{k-1}R, \qquad n \geqslant 0.$$

Then by the inequality (7), we get

$$\begin{aligned}
&V_p\big(X^{n+1}; [\tau_k, \tau_{k+1}]\big) \\
&\quad \leqslant \Big[|f(0)| + C_{p,p}|g(0)| + \big(L + C_{p,p}|g'|_\infty\big)\Big(V_p\big(X^n; [0, \tau_k]\big) + |\xi|\Big) \\
&\qquad + \big(L + 2C_{p,p}|g'|_\infty\big) \cdot V_p\big(X^n; [\tau_k, \tau_{k+1}]\big)\Big] \\
&\qquad \times \max\Big\{\tau_{k+1} - \tau_k, V_p\big(B^H; [\tau_k, \tau_{k+1}]\big)\Big\} \\
&\quad \leqslant |f(0)| + |g(0)| + \frac{1}{2}\Big[\sum_{i=1}^{k} 2^{i-1}R + |\xi|\Big] + \frac{1}{2}V_p\big(X^n; [\tau_k, \tau_{k+1}]\big) \\
&\quad \leqslant 2\Big[|f(0)| + |g(0)| + \frac{2^k - 1}{2}R + \frac{1}{2}|\xi|\Big] \leqslant 2^k R.
\end{aligned} \tag{8}$$

Thus (3) and (8) imply the result. □

For fixed $m \geqslant 1$, define a sequence of stopping times $\sigma_m = \gamma_m \wedge \tau_m \wedge \theta_m$, where

$$\begin{aligned}
\theta_m &= m \cdot \mathbf{1}\{|\xi| \leqslant m\}, \\
\gamma_m &= \inf\Big\{t > \gamma_{m-1}\colon V_p\big(B^H; [\gamma_{m-1}, t]\big) > \frac{1}{8C_{p,p/\alpha}} \\
&\qquad \times \big[2|g'|_\infty + |g'|_\alpha 2^{m+1}\widehat{R}_m\big]^{-1}\Big\} \wedge \Big(\gamma_{m-1} + \frac{1}{8L}\Big), \\
\widehat{R}_m &= m \cdot \big[1 + |f(0)| + |g(0)|\big].
\end{aligned}$$

Lemma 2. *For fixed* m, *we have*

$$\begin{aligned}
&V_{p,\infty}\big(X^{n+1} - X^n; [0, \sigma_m]\big) \\
&\qquad \leqslant \frac{2}{2^n} 2^{m-1}\big[1 + n + \cdots + n^{m-1}\big]\big[|f(\xi)| \cdot \sigma_m + |g(\xi)| \cdot V_p\big(B^H; [0, \sigma_m]\big)\big].
\end{aligned}$$

Proof. Denote $Z^{n+1} = X^{n+1} - X^n$, $n \geqslant 0$. Note that for any $k, n \in \mathbf{N}$

$$\begin{aligned}
V_{p,\infty}\ &\big(Z^{n+1}; [\sigma_{k-1}, \sigma_k]\big) \\
&\leqslant 2V_p\big(Z^{n+1} - Z^{n+1}(\sigma_{k-1}); [\sigma_{k-1}, \sigma_k]\big) + \big|Z^{n+1}(\sigma_{k-1})\big|.
\end{aligned} \tag{9}$$

By the Love-Young inequality and Lemma 2 in [5], we have

$$
\begin{aligned}
&V_p\big(Z^{n+1}-Z^{n+1}(\sigma_{k-1});[\sigma_{k-1},\sigma_k]\big)\\
&\quad\leqslant \int_{\sigma_{k-1}}^{\sigma_k}\big|f(X_s^n)-f(X_s^{n-1})\big|\,ds\\
&\qquad+C_{p,p/\alpha}V_{p/\alpha,\infty}\big(g(X^n)-g(X^{n-1});[\sigma_{k-1},\sigma_k]\big)V_p\big(B^H;[\sigma_{k-1},\sigma_k]\big)\\
&\quad\leqslant L\sup_{\sigma_{k-1}\leqslant s\leqslant\sigma_k}|Z_s^n|\cdot(\sigma_k-\sigma_{k-1})\\
&\qquad+C_{p,p/\alpha}\big\{2|g'|_\infty+|g'|_\alpha\cdot V_p^\alpha\big(X^{n-1};[\sigma_{k-1},\sigma_k]\big)\big\}\\
&\qquad\times V_{p,\infty}\big(Z^n;[\sigma_{k-1},\sigma_k]\big)\cdot V_p\big(B^H;[\sigma_{k-1},\sigma_k]\big). \qquad (10)
\end{aligned}
$$

It is obvious that

$$
V_{p,\infty}\big(Z^1;[0,\sigma_1]\big)\leqslant 2V_p\big(Z^1;[0,\sigma_1]\big)\leqslant 2|f(\xi)|\cdot\sigma_1+2|g(\xi)|\cdot V_p\big(B^H;[0,\sigma_1]\big)
$$

and by (10), it follows that

$$
\begin{aligned}
V_{p,\infty}\big(Z^{n+1};[0,\sigma_1]\big)&\leqslant \frac{1}{2^n}V_{p,\infty}\big(Z^1;[0,\sigma_1]\big)\\
&\leqslant \frac{2}{2^n}\big[|f(\xi)|\cdot\sigma_1+|g(\xi)|\cdot V_p\big(B^H;[0,\sigma_1]\big)\big].
\end{aligned}
$$

Denote $A:=|f(\xi)|\cdot\sigma_k+|g(\xi)|\cdot V_p\big(B^H;[0,\sigma_k]\big)$. Similarly, we have

$$
\begin{aligned}
&V_{p,\infty}\big(Z^{n+1}-Z^{n+1}(\sigma_{m-1});[\sigma_{m-1},\sigma_m]\big)\\
&\quad\leqslant \frac{1}{2}V_{p,\infty}\big(Z^{n+1}-Z^{n+1}(\sigma_{m-1});[\sigma_{m-1},\sigma_m]\big)+\frac{1}{2}\big|Z^n(\sigma_{m-1})\big|\\
&\quad\leqslant \frac{1}{2^n}V_{p,\infty}\big(Z^1-Z^1(\sigma_{m-1});[\sigma_{m-1},\sigma_m]\big)+\sum_{i=1}^n\frac{1}{2^i}\big|Z^{n-i+1}(\sigma_{m-1})\big|\\
&\quad\leqslant \frac{2}{2^n}\Big[\big|f(\xi)\big|(\sigma_m-\sigma_{m-1})+\big|g(\xi)\big|V_p\big(B^H;[\sigma_{m-1},\sigma_m]\big)\Big]\\
&\qquad+\sum_{i=1}^n\frac{1}{2^i}\sum_{j=1}^{m-1}V_{p,\infty}\big(Z^{n-i+1}-Z^{n-i+1}(\sigma_{j-1});[\sigma_{j-1},\sigma_j]\big)\\
&\quad\leqslant \frac{2A}{2^n}+\sum_{i=1}^n\frac{1}{2^i}\bigg\{\frac{2A}{2^{n-i}}+\frac{2A}{2^{n-i}}[1+n-i]\\
&\qquad+\frac{2A}{2^{n-i}}\sum_{j=2}^{m-1}2^{j-2}\big[1+(n-i)+\cdots+(n-i)^{j-1}\big]\bigg\}\\
&\quad\leqslant \frac{2A}{2^n}\bigg\{1+n+n^2+\sum_{j=2}^{m-1}2^{j-2}\big[n+n^2+\cdots+n^j\big]\bigg\}\\
&\quad\leqslant \frac{2A}{2^n}\cdot 2^{m-2}\big[1+n+\cdots+n^{m-1}\big].
\end{aligned}
$$

Then by (9) and (2), we have

$$\begin{aligned}
V_{p,\infty}\big(Z^{n+1};[\sigma_{m-1},\sigma_m]\big) &\leqslant \sum_{j=1}^{m} V_{p,\infty}\big(Z^{n+1}-Z^{n+1}(\sigma_{j-1});[\sigma_{j-1},\sigma_j]\big)\\
&\leqslant \frac{2A}{2^n}+\frac{2A}{2^n}n+\sum_{j=3}^{m}\frac{2A}{2^n}2^{j-2}\big[1+n+\cdots+n^{j-1}\big]\\
&\leqslant \frac{2A}{2^n}2^{m-1}\big[1+n+\cdots+n^{m-1}\big].
\end{aligned}$$

The proof of the lemma is complete. □

Proof of Theorem. Existence of the solution. By Lemma 2, it follows that there exists a stochastic process Y with almost all trajectories in $CW_p([0,T])$ such that for any fixed k, $V_{p,\infty}(X^{n,\sigma_k}-Y^{\sigma_k};[0,T])\to 0$ as $n\to\infty$ since $V_{p,\infty}(X^n-Y;[0,\sigma_k])=V_{p,\infty}(X^{n,\sigma_k}-Y^{\sigma_k};[0,T])$, where $X_t^{n,\sigma_k}=X^n(t\wedge\sigma_k)$, $Y_t^{\sigma_k}=Y(t\wedge\sigma_k)$.

The process Y^{σ_k} is $\mathbf{F}^{\sigma_k}=\{\mathcal{F}(t\wedge\sigma_k),0\leqslant t\leqslant T\}$-adapted. We still have to show that

$$Y_t=\xi+\int_0^t f(Y_s)\,ds+\int_0^t g(Y_s)\,dB_s^H \qquad t\in[0,\sigma_k]. \tag{11}$$

By the definition of the stopping times σ_k and Lemma 2 in [5], we have

$$\begin{aligned}
&V_{p,\infty}\Big(Y-\xi-\int_0^{\cdot} f(Y_s)\,ds-\int_0^{\cdot} g(Y_s)\,dB_s^H;[0,\sigma_k]\Big)\\
&\quad\leqslant V_{p,\infty}(Y-X^n;[0,\sigma_k])+V_{p,\infty}\Big(\int_0^{\cdot}[f(Y_s)-f(X_s^{n-1})]ds;[0,\sigma_k]\Big)\\
&\qquad+V_{p,\infty}\Big(\int_0^{\cdot}[g(Y_s)-g(X_s^{n-1})]dB_s^H;[0,\sigma_k]\Big)\\
&\quad\leqslant V_{p,\infty}(Y-X^n;[0,\sigma_k])+2Lk\sup_{0\leqslant s\leqslant\sigma_k}|Y_s-X_s^{n-1}|\\
&\qquad+2kC_{p,p/\alpha}\big[|g'|_\infty+2^{k+1}|g'|_\alpha\widehat{R}_k\big]V_{p,\infty}(Y-X^{n-1};[0,\sigma_k]).
\end{aligned}$$

The equality (11) follows from the above inequality.

Uniqueness of the solution. Let X be another adapted, continuous solution of (11). By the Love-Young inequality and definition of the stopping times (σ_m), we have $V_{p,\infty}(X-Y;[0,\sigma_1])\leqslant\frac{1}{4}V_{p,\infty}(X-Y;[0,\sigma_1])$. Thus $X=Y$ on the interval $[0,\sigma_1]$. Similarly, we prove that $V_{p,\infty}(X-Y;[\sigma_{k-1},\sigma_k])=0$. Thus $X=Y$ on $[0,\sigma_m]$.

Since the sequence of stopping times (σ_m) goes to infinity as $m\to\infty$ then we have existence and uniqueness of the solution of equation (1) on the interval $[0,T]$.

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Integralinės lygties valdomos trupmeninio Brauno judesio sprendinio egzistavimas ir vienatis

K. Kubilius

Nagrinėkime trupmeninį Brauno judesį, kurio Hursto indeksas $1/2 < H < 1$. Rastos sąlygos, kada nagrinėjama lygtis turi vienintelį sprendinį ir sprendinio trajektorijos yra tolydžių, turinčių baigtinę p-variaciją, $p > 1/H$, funkcijų klasėje.