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A joint limit theorems for trigonometric polynomials

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Let $s=\sigma+it$ is a complex variable. The Riemann zeta function $\zeta(s)$, for $\sigma>1$, is defined by

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s},$$

and by analytic continuation elsewhere.

Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S, and let, for $T>0$,

$$\nu_T^t(\ldots)=\frac{1}{T}meas\{t\in[0,T],\ldots\},$$

where $meas\{A\}$ stands for the Lebesgue measure of the set A, and in place of dots some condition satisfied by t is to be written.

Let N is a positive integer and

$$p_jN(t)=\sum_{m\leqslant N}a_{jm}\exp\{i\lambda_{jm}t\},\quad j=1,\ldots,n,$$

be arbitrary trigonometric polynomials and $\lambda_{jm}\in\mathbb{R}$, $a_{jm}\in\mathbb{C}$.

Let $\mathbb{C}$ is a complex plane and $\mathbb{C}^n=\underbrace{\mathbb{C}\times\mathbb{C}\times\ldots\mathbb{C}}_{n}$. On the probability space $(\mathbb{C},\mathcal{B}(\mathbb{C}))$ define the probability measure

$$P_{T,p_N}(A)=\nu_T^t\Big(\big(p_{1N}(tk_1),\ldots,p_{nN}(tk_n)\big)\in A\Big),\quad A\in\mathcal{B}(\mathbb{C}^n).$$

Here $k_1,\ldots,k_n$ are natural numbers.

Theorem 1. *On $(\mathbb{C}^n,\mathcal{B}(\mathbb{C}^n))$ here exists a probability measure P_{p_N} such that the measure P_{T,p_N} converges weakly to P_{p_N} as $T\to\infty$.*

Proof of Theorem 1. Instead of the mesure P_{T,p_N} we can consider the measure

$$Q_{T,p_N}(A)=\nu_T^t\Big(\big(\mathrm{Re}p_{1N}(tk_1),\mathrm{Im}p_{1N}(tk_1)\ldots,\mathrm{Re}p_{nN}(tk_n),\mathrm{Im}p_{nN}(tk_n)\big)\in A\Big),$$
$$A\in\mathcal{B}(\mathbb{R}^{2n}).$$

Let $J_k(x)$ is the Bessel functions, and

$$\varphi_{p_N}(\tau_1,\tau_1',\dots,\tau_n,\tau_n') = \sum\nolimits^* \prod_{j\leqslant n}\prod_{m\leqslant N} J_{k_{jm}}(|a_{jm}|\tau_j)J_{k'_{jm}}(|a_{jm}|\tau_j')$$
$$\times \exp\Big\{ i\sum_{j\leqslant n}\sum_{m\leqslant N}\Big(k_{jm}\Big(\varphi_{jm}+\frac{\pi}{2}\Big)+k'_{jm}\varphi_{jm}\Big)\Big\}. \quad (1)$$

Here $\varphi_{jm} = \arg a_{jm}$, $j = 1,\dots,n$, and the symbol $\sum^*$ means the summation which runs over all integers k_{jm} and k'_{jm}, $1 \leqslant j \leqslant n$, $1 \leqslant m \leqslant N$, satisfying the condition

$$\sum_{j\leqslant n}\sum_{m\leqslant N}(k_{jm}+k'_{jm})\lambda_{jm}k_j = 0.$$

The characteristic function of the measure Q_{T,p_N} is

$$\varphi_{T,p_N}(\tau_1,\tau_1',\dots,\tau_n,\tau_n') = \int_{\mathbb{R}^{2n}} \exp\{i(\tau_1x_1+\tau_1'x_1'+\dots+\tau_nx_n+\tau_n'x_n')\}\,\mathrm{d}Q_{T,p_N}$$
$$= \frac{1}{T}\int_0^T \exp\{i(\tau_1 \operatorname{Re}p_{1N}(tk_1)+\tau_1'\operatorname{Im}p_{1N}(tk_1)+\dots+\tau_n\operatorname{Re}p_{nN}(tk_n)+\tau_n'\operatorname{Im}p_{nN}(tk_n))\}\,\mathrm{d}t.$$

It is well known that

$$\operatorname{Re}p_{jN}(tk_j) = \sum_{m\leqslant N}|a_{jm}|\cos(\varphi_{jm}+tk_j\lambda_{jm}),$$
$$\operatorname{Im}p_{jN}(tk_j) = \sum_{m\leqslant N}|a_{jm}|\sin(\varphi_{jm}+tk_j\lambda_{jm}), \quad j=1,\dots,n.$$

Since [1]

$$\mathrm{e}^{ix\sin\theta} = \sum_{r=-\infty}^{\infty} J_r(x)\mathrm{e}^{ir\theta}$$

and

$$\mathrm{e}^{ix\cos\theta} = \sum_{r=-\infty}^{\infty} i^r J_r(x)\mathrm{e}^{ir\theta},$$

we have that

$$\mathrm{e}^{i\tau_j\operatorname{Re}p_{jN}(tk_j)} = \prod_{m\leqslant N}\sum_{k_{jm}=-\infty}^{\infty} J_{k_{jm}}(|a_{jm}|\tau_j)\mathrm{e}^{ik_{jm}(\lambda_{jm}tk_j+\varphi_{jm}+\frac{\pi}{2})} \quad (2)$$

and

$$e^{i\tau_j' \operatorname{Im} p_{jN}(tk_j)} = \prod_{m\leqslant N} \sum_{k'_{jm}=-\infty}^{\infty} J_{k'_{jm}}(|a_{jm}|\tau_j') e^{ik'_{jm}(\lambda_{jm}tk_j+\varphi_{jm})}. \tag{3}$$

Substituting (2) and (3) in (1) we obtain that

$$\begin{aligned} \varphi_{T,p_N}&(\tau_1,\tau_1',\dots,\tau_n,\tau_n') = \varphi_{p_N}(\tau_1,\tau_1',\dots,\tau_n,\tau_n') \\ &+ \sum\nolimits^{**} \prod_{j\leqslant n}\prod_{m\leqslant N} J_{k_{jm}}(|a_{jm}|\tau_j) J_{k'_{jm}}(|a_{jm}|\tau_j') \\ &\times \exp\left\{ i\left(\frac{\pi}{2}\sum_{j\leqslant n}\sum_{m\leqslant N} k_{jm} + \sum_{j\leqslant n}\sum_{m\leqslant N}(k_{jm}+k'_{jm})\varphi_{jm}\right)\right\} \\ &\times \frac{\exp\left\{ iT\sum\limits_{j\leqslant n}\sum\limits_{m\leqslant N}(k_{jm}+k'_{jm})\lambda_{jm}k_j)\right\} - 1}{iT\sum\limits_{j\leqslant n}\sum\limits_{m\leqslant N}(k_{jm}+k'_{jm})\lambda_{jm}k_j)}. \end{aligned}$$

Here the symbol $\sum^{**}$ means the summation which runs over all integers k_{jm} and k'_{jm}, $1 \leqslant j \leqslant n$, $1 \leqslant m \leqslant N$, satisfying the condition

$$\sum_{j\leqslant n}\sum_{m\leqslant N}(k_{jm}+k'_{jm})\lambda_{jm}k_j \neq 0.$$

Let $\varepsilon, c_1, c_1', \dots, c_n, c_n'$ be arbitrary positive numbers, and

$$A_0 = \left\{(\tau_1,\tau_1',\dots,\tau_n,\tau_n') \in \mathbb{R}^{2n} \colon |\tau_1| \leqslant c_1, |\tau_1'| \leqslant c_1', \dots, |\tau_n| \leqslant c_n, |\tau_n'| \leqslant c_n'\right\}.$$

Hence, using the properties of Bessel functions we find that for any $\varepsilon > 0$ and for any $(\tau_1,\tau_1',\dots,\tau_n,\tau_n') \in A_0$ the inequality

$$\left|\varphi_{T,p_N}(\tau_1,\tau_1',\dots,\tau_n,\tau_n') - \varphi_{p_N}(\tau_1,\tau_1',\dots,\tau_n,\tau_n')\right| < \varepsilon. \tag{4}$$

is satisfied. (4) formula shows, that φ_{T,p_N} converges weakly to φ_{p_N} as $T \to \infty$ and the convergence is uniform for $\tau_1,\tau_1',\dots,\tau_n,\tau_n'$ in every finite interval. This and the well known continuity theorem, see, for example [2], prove the theorem.

Let D be a region on $\mathbb{C}$ and $H(D)$ denote the space of analytic on D function equipped with the topology of uniform convergence on compacta and $H^n(D) = \underbrace{H(D) \times H(D) \times \ldots \times H(D)}_{n}$. Let

$$p_N = \sum_{m=1}^{N} \frac{a(m)}{m^s}.$$

Define a probability measure

$$P_{T,p_N}(A) = \nu_T^{\tau}\Big(\big(p_N(s+i\tau k_1),\dots,p_N(s+i\tau k_n)\big) \in A\Big), \quad A \in \mathcal{B}(H^n(D)).$$

Theorem 2. *There exists a probability measure P_{p_N} on $(H^n(D), \mathcal{B}(H^n(D))$ such that the measure P_{T,p_N} convereges weakly to P_{p_N} as $T \to \infty$.*

Proof of the Theorem 2. Let $p_1,\dots,p_r$ are the distinct primes which divide the product

$$\prod_{m=1,a(m)\neq 0}^{N} m,$$

and let

$$\Omega_r = \prod_{j=1}^{r} \gamma_{p_j}, \quad \gamma_{p_j} = \gamma = \{s \in \mathbb{C} : |s| = 1\}.$$

Let us define the function $h : \Omega_r \to H^n(D)$ by the formula

$$h(x_1,\dots,x_r) = \Bigg(\sum_{m=1}^{N} \frac{a(m)}{m^s}\Big(\prod_{p_j^{\alpha_j}\|m, j\leqslant r} x_j^{\alpha_j}\Big)^{-k_1},\dots,\sum_{m=1}^{N} \frac{a(m)}{m^s}\Big(\prod_{p_j^{\alpha_j}\|m, j\leqslant r} x_j^{\alpha_j}\Big)^{-k_n}\Bigg),$$

$$(x_1,\dots,x_r) \in \Omega_r.$$

The function h is continuous on Ω_r and

$$\big(p_N(s+ik_1\tau),\dots,p_N(s+ik_n\tau)\big) = h(p_1^{i\tau},\dots,p_r^{i\tau}). \tag{5}$$

Now we define the probability measure

$$Q_T(A) = \nu_T^{\tau}\big((p_1^{i\tau},\dots,p_r^{i\tau}) \in A\big).$$

on $(\Omega_r, \mathcal{B}(\Omega_r))$. The Fourier transform $g_T(l_1,\dots,l_r), l_j \in \mathbb{Z}, j = 1,\dots,r$ of Q_T is

$$g_T(l_1,\dots,l_r) = \int_{\Omega_r} x_1^{l_1},\dots,x_r^{l_r} \mathrm{d}Q_T = \frac{1}{T}\int_0^T \prod_{j=1}^{r} p_j^{il_j\tau} \mathrm{d}\tau$$

$$= \begin{cases} 1, & \text{if} \quad (l_1,\dots,l_r) = (0,\dots,0), \\ \dfrac{\exp\Big\{iT\sum\limits_{j=1}^{r} l_j \ln p_j\Big\} - 1}{iT\sum\limits_{j=1}^{r} l_j \ln p_j}, & \text{if} \quad (l_1,\dots,l_r) \neq (0,\dots,0). \end{cases}$$

Since the logarithms of prime numbers are lineary independent over the field of rational numbers, we find that

$$g_T(l_1, \dots l_r) = \begin{cases} 1, & \text{if} \quad (l_1, \dots, l_r) = (0, \dots, 0) \\ 0, & \text{if} \quad (l_1, \dots, l_r) \neq (0, \dots, 0). \end{cases}$$

as $T \to \infty$. Therefore by Theorem 1.3.19 from [1] the measure Q_T converges weakly to the Haar measure m_{rH} on $(\Omega_r, \mathcal{B}(\Omega_r))$ as $T \to \infty$. Taking into acount the continuity of the function h and the formula (5) and applying Theorem 1.1.16 from [1] we obtain that the probability measure P_{T,p_N} convereges weakly to the measure $m_{rH}h^{-1}$ as $T \to \infty$. The theorem is proved.

Theorems 1 and 2 will be applied to prove joint limit theorems for Dirichlet series.

References

[1] A. Laurinčikas, *Limit Theorems for the Riemann zeta-function*, Kluwer, Dodrecht (1996).
[2] P. Billingsley, *Convergence of Probability Measures*, John Wiley, New York (1996).

Daugiamatės ribinės teoremos trigonometriniams polinomams

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Straipsnyje įrodomos dvi daugiamatės ribinės teoremos trigonometriniams polinomams tikimybinių matų silpno konvergavimo prasme.