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# On stochastic differential equations driven by skew stable processes

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## Introduction

We consider the existence of weak solutions to the one-dimensional stochastic differential equation

$$\mathrm{d}X_t = \sigma_1(X_{s-})\mathrm{d}Z_s^{(1)} + \sigma_2(X_{s-})\mathrm{d}Z_s^{(2)}, \quad X_0 = x, \tag{1}$$

where $\sigma_1$ and $\sigma_2$ are bounded measurable nonnegative functions on $\mathbb{R} = (-\infty, \infty)$ and $Z_t^{(1)}, Z_t^{(2)}$ are independent skew stable process of order $\alpha \in (1,2)$ with the characteristic functions

$$\mathbf{E}\exp\left\{i\xi Z_t^{(j)}\right\} = \exp\left\{-ct|\xi|^\alpha\Big(1 + (-1)^j i \operatorname{sgn}\xi \operatorname{tg}\frac{\alpha\pi}{2}\Big)\right\}, \quad j = 1, 2, \tag{2}$$

respectively, $c > 0,\ t \geqslant 0,\ \xi \in \mathbb{R}$.

Stochastic differential equations with measurable coefficients are important in the stochastic optimal control theory, since, in many cases, their solutions are optimal processes. The study of such equations was originated by Krylov [3], in the case of driving Brownian motion, and developed by many authors (see, e.g., [5] and references therein).

The paper consists of two sections. In Section 1, an $L_2$-estimate for stochastic integrals is derived and, using this estimate, in Section 2, the existence of a weak solution to equation (1) is proved.

## 1. $L_2$-estimate of stochastic integrals

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ be a complete probability space with a right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)$, and let $(Z_t^{(1)}, \mathcal{F}_t)$ and $(Z_t^{(2)}, \mathcal{F}_t)$ be skew stable processes of order $\alpha \in (1,2)$ with the characteristic functions given by (2).

Consider the following optimal control problem. Fix $\alpha \in (1,2)$ and introduce a class of control strategies $\mathfrak{B}$ as the class of all predictable processes $1 \geqslant r_t \geqslant 0$ such that $|r_t - \frac{1}{2}| \leqslant \delta$, where

$$\delta = \frac{1}{2} \quad \text{if } \alpha \in \Big(\frac{3}{2}, 2\Big), \qquad 0 < \delta < 1/\Big(2\Big|\operatorname{tg}\frac{\alpha\pi}{2}\Big|\Big) \quad \text{if } \alpha \in (1, 3/2]. \tag{3}$$

The controlled process $X_t^{(r)}$ and the value function $v$ are defined by

$$X_t^{(r)} = \int_0^t r_s^{1/\alpha} \mathrm{d}Z_s^{(1)} + \int_0^t (1-r_s)^{1/\alpha} \mathrm{d}Z_s^{(2)}, \quad v(x) = \sup_{r\in\mathfrak{B}} \mathbf{E} \int_0^\infty \mathrm{e}^{-t} f(x + X_t^{(r)}) \mathrm{d}t,$$

where $f$ is a nonnegative smooth function on $\mathbb{R}$ with compact support.

**Lemma 1.** (i) *the function $v$ is bounded and Lipschitz continuous,*

(ii) *there exists a constant $N$ such that the function $v(x) + Nx^2$ is convex,*

(iii) *(Bellman principle) for each $x \in \mathbb{R}$*

$$v(x) = \sup_{r\in\mathfrak{B}} \mathbf{E} \left\{ \int_0^{\tau_r} \mathrm{e}^{-t} f(x + X_t^{(r)}) \mathrm{d}t + \mathrm{e}^{-\tau_r} v(x + X_{\tau_r}^{(r)}) \right\}, \tag{4}$$

*where $\tau_r$, $r \in \mathfrak{B}$, are arbitrary $\mathbb{F}$-stopping times,*

(iv) *(Bellman equation) for almost every $x \in \mathbb{R}$*

$$\sup_{|r-\frac{1}{2}|\leqslant\delta} \{ rL_1 v(x) + (1-r)L_2 v(x) - v(x) + f(x) \} = 0,$$

*where*

$$L_1 u(x) = \int_0^\infty \nabla_y^2 u(x) \frac{\mathrm{d}y}{y^{1+\alpha}}, \quad L_2 u(x) = \int_{-\infty}^0 \nabla_y^2 u(x) \frac{\mathrm{d}y}{|y|^{1+\alpha}},$$

$$\nabla_y^2 u(x) = u(x+y) - u(x) - u'(x)y.$$

The first two assertions of the lemma follow immediately from the definition of $v$. The assertions (iii) and (iv) can be proved using standard arguments from the stochastic optimal control theory (see [2]).

Let $\xi$ be a smooth nonnegative function on $\mathbb{R}$ such that $\xi(x) = 0$ if $|x| \geqslant 1$ and $\int \xi(x)\mathrm{d}x = 1$. Let $\xi^\varepsilon(x) = \xi(x/\varepsilon)/\varepsilon$. We further use the following notation:

$$u^{(\varepsilon)}(x) = \int \xi^\varepsilon(x-y) u(y) \mathrm{d}y, \qquad \|u\|_2 = \left\{ \int u^2(x) \mathrm{d}x \right\}^{1/2}.$$

**Lemma 2.** *For each $x \in \mathbb{R}$*

$$v(x) \leqslant N_1 \|v\|_2 \leqslant N_2 \|f\|_2, \tag{5}$$

*where the constants $N_1$ and $N_2$ depend on $\delta$ and $\alpha$ only.*

*Proof.* According to (5),

$$\frac{1}{2}(L_1 v + L_2 v) - v \leqslant 0 \qquad \text{a.e. in } \mathbb{R}. \tag{6}$$

By Lemma 3 of [4], this inequality implies

$$v(x) \leqslant \frac{g(0)}{g(|x-y|)} v(y), \qquad g(x) := \frac{1}{\pi} \int_0^\infty \frac{\cos(\xi x)}{2+\xi^\alpha} \mathrm{d}\xi,$$

and the first inequality in (6) is proved.

Assertions (i) and (ii) of Lemma 1 imply that

$$\nabla_y^2 v(x) \geqslant -N,$$

for almost every $x \in \mathbb{R}$ and each $y \in \mathbb{R}$ and that there exists a constant $c > 0$ such that $L_1 v \geqslant -c$ and $L_2 v \geqslant -c$ a.e. in $\mathbb{R}$. These estimates, together with (7), yield

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}} \big(|L_1 v| + |L_2 v|\big) < \infty. \tag{7}$$

By (4),

$$v(x) = \sup_{r\in\mathfrak{B}} \mathbf{E} e^{-\tau_R^{x,r}} v\Big(x + X^{(r)}_{\tau_R^{x,r}}\Big) \leqslant \sup_{r\in\mathfrak{B}} \mathbf{E} e^{-\tau_R^{x,r}} \sup_{y\in\mathbb{R}} |v(y)|,$$

where $\tau_R^{x,r} = \inf\{t \geqslant 0\colon x + X_t^{(r)} \in (-R,R)\}$ and $R$ is such that $f(x) = 0$ if $x \notin (-R,R)$. It is easy to prove that $\sup_{r\in\mathfrak{B}} \mathbf{E} e^{-\tau_R^{x,r}}$ is an integrable function. Therefore, $v$ is integrable on $\mathbb{R}$.

As can be easily seen, (5) is equivalent to the equation

$$\frac{1}{2}(L_1 v + L_2 v) + \delta|L_1 v - L_2 v| - v + f = 0,$$

a.e. $\mathbb{R}$. Taking the convolution with $\xi^\varepsilon$, we have

$$\frac{1}{2}\big(L_1 v^{(\varepsilon)} + L_2 v^{(\varepsilon)}\big) + \delta\big|L_1 v^{(\varepsilon)} - L_2 v^{(\varepsilon)}\big| - v^{(\varepsilon)} + f_\varepsilon = 0, \tag{8}$$

where $f_\varepsilon = f^{(\varepsilon)} + \delta[|L_1 v - L_2 v|^{(\varepsilon)} - |L_1 v^{(\varepsilon)} - L_2 v^{(\varepsilon)}|]$.

Since $v$ is bounded and integrable, $L_1 v^{(\varepsilon)}$ and $L_2 v^{(\varepsilon)}$ are integrable and, by (8), bounded uniformly with respect to $\varepsilon$. According to (9), the function $|L_1 v - L_2 v|^{(\varepsilon)}$ is integrable and bounded. This, together with the Fubini–Tonelli theorem, implies that $|L_1 v - L_2 v|$ is integrable and bounded.

By (9),

$$\Big[v^{(\varepsilon)} - \frac{1}{2}\big(L_1 v^{(\varepsilon)} + L_2 v^{(\varepsilon)}\big)\Big]^2 = \big[f_\varepsilon + \delta\big|L_1 v^{(\varepsilon)} - L_2 v^{(\varepsilon)}\big|\big]^2.$$

Applying Young's inequality, we have that, for each $\mu > 0$,

$$v^{(\varepsilon)2} - v^{(\varepsilon)}\big(L_1 v^{(\varepsilon)} + L_2 v^{(\varepsilon)}\big) + \frac{1}{4}\big(L_1 v^{(\varepsilon)} + L_2 v^{(\varepsilon)}\big)^2$$

$$\leqslant \Big(1+\frac{1}{\mu}\Big) f_\varepsilon^2 + \delta^2(1+\mu)\big(L_1 v^{(\varepsilon)} - L_2 v^{(\varepsilon)}\big)^2.$$

Integrating both sides of this inequality and using Parseval's equality, we conclude that

$$\|v^{(\varepsilon)}\|_2^2 \leqslant \Big(1+\frac{1}{\mu}\Big)\|f_\varepsilon\|_2^2, \tag{9}$$

if $\delta^2(1+\mu)(2\operatorname{tg}\frac{\alpha\pi}{2})^2 \leqslant 1$. According to (3), (10) holds for sufficiently small $\mu > 0$. Passing to the limit in (10) as $\varepsilon \to 0$ and using the Lebesgue dominated convergence theorem, we get the second inequality in (6). The lemma is proved.

Let $\sigma_t = (\sigma_{1t}, \sigma_{2t})$ be a pair of predictable nonnegative bounded processes such that

$$|\sigma_{1t}^\alpha/(\sigma_{1t}^\alpha + \sigma_{2t}^\alpha) - 1/2| \leqslant \delta, \tag{10}$$

where $\delta$ is defined by (3). Denote

$$X_t^\sigma = \int_0^t \sigma_{1s}\mathrm{d}Z_s^{(1)} + \int_0^t \sigma_{2s}\mathrm{d}Z_s^{(2)}, \qquad \phi_t^\sigma = \sigma_{1t}^\alpha + \sigma_{2t}^\alpha,\ \varphi_t^\sigma = \int_0^t \phi_s^\sigma \mathrm{d}s.$$

**Theorem 1.** *Let assumption* (11) *be satisfied. Then, for each $x \in \mathbb{R}$ and every nonnegative Borel function $f$ on $\mathbb{R}$,*

$$\mathbf{E}\int_0^\infty \mathrm{e}^{-\varphi_t^\sigma}\phi_t^\sigma f(x + X_t^\sigma)\mathrm{d}t \leqslant N\|f\|_2,$$

*where the constant $N$ depends on $\delta$ and $\alpha$ only.*

*Proof.* By assertion (iv) of Lemma 2, for all nonnegative numbers $\sigma_1$ and $\sigma_2$ such that $|\sigma_1^\alpha/(\sigma_1^\alpha + \sigma_2^\alpha) - 1/2| \leqslant \delta$,

$$\sigma_1^\alpha L_1 v + \sigma_2^\alpha L_2 v - v(\sigma_1^\alpha + \sigma_2^\alpha) + f(\sigma_1^\alpha + \sigma_2^\alpha) \leqslant 0,$$

a.e. in $\mathbb{R}$. Taking the convolution with $\xi^\varepsilon$, we have

$$\sigma_1^\alpha L_1 v^{(\varepsilon)} + \sigma_2^\alpha L_2 v^{(\varepsilon)} - v^{(\varepsilon)}(\sigma_1^\alpha + \sigma_2^\alpha) + f^{(\varepsilon)}(\sigma_1^\alpha + \sigma_2^\alpha) \leqslant 0,$$

in $\mathbb{R}$. This inequality, together with the Ito formula, yields

$$\mathbf{E}\mathrm{e}^{-\varphi_t^\sigma} v^{(\varepsilon)}(x + X_t^\sigma) - v^{(\varepsilon)}(x) = \mathbf{E}\int_0^t \mathrm{e}^{-\varphi_t^\sigma}\Big(\sigma_{1t}^\alpha L_1 v^{(\varepsilon)} + \sigma_{2t}^\alpha L_2 v^{(\varepsilon)}$$

$$-\phi_t^\sigma v\Big)(x + X_t^\sigma)\mathrm{d}t \leqslant -\mathbf{E}\int_0^t \mathrm{e}^{-\varphi_t^\sigma}\phi_t^\sigma f^{(\varepsilon)}(x + X_s^\sigma)\mathrm{d}s.$$

Letting $\varepsilon \to 0$ and $t \to \infty$, we have

$$\mathbf{E} \int_0^\infty e^{-\varphi_t^\sigma} \phi_t^\sigma f(x + X_t^\sigma) \mathrm{d}t \leqslant v(x),$$

and, in the case of smooth functions $f$ with compact support, the assertion of the theorem follows from Lemma 2. This result can be extended to nonnegative Borel functions $f$ by standard arguments, e.g., using the results of Sect. 1.2 of [3]. The theorem is proved.

## 2. Existence of weak solutions

**Theorem 2.** *Let $\alpha \in (1,2)$, and let nonnegative measurable functions $\sigma_1$ and $\sigma_2$ on $\mathbb{R}$ be such that $\mu \leqslant \sigma_1 + \sigma_2 \leqslant 1/\mu$ and $|\sigma_1^\alpha/(\sigma_1^\alpha + \sigma_2^\alpha) - 1/2| \leqslant \delta$, where $\mu$ is a positive constant and the constant $\delta$ is defined by* (3).

*Then equation* (1) *has a weak solution.*

*Proof.* The proof is similar to that of Lemma 3.2 of [5].

Let $\varepsilon_n \to 0$ as $n \to \infty$ be a sequence of positive numbers. Since $\sigma_{in} := \sigma_i^{(\varepsilon_n)}$, $i = 1, 2$, are smooth functions for each $n$, on each stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ with skew stable processes $(Z^{(1)}, \mathbb{F})$ and $(Z^{(2)}, \mathbb{F})$ there exists a unique strong solution $X_t^n$ to the equation

$$\mathrm{d}X_t^n = \sigma_{1n}(X_{t-}^n)\mathrm{d}Z_t^{(1)} + \sigma_{2n}(X_{t-}^n)\mathrm{d}Z_t^{(2)}, \qquad X_0^n = x.$$

It is easy to verify that the sequence $(X_t^n, Z_t^{(1)}, Z_t^{(2)})$ is tight in the Skorokhod topology. By virtue of the Skorokhod representation theorem there exist a complete probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{F}}, \overline{\mathbf{P}})$, a subsequence (still denoted by $n$) of processes $(\overline{X}_t^n, \overline{Z}_t^{(1)n}, \overline{Z}_t^{(2)n})$, and a process $(\overline{X}_t, \overline{Z}_t^{(1)}, \overline{Z}_t^{(2)})$ such that:

(i) for each $n$, the laws of $(\overline{X}_t^n, \overline{Z}_t^{(1)n}, \overline{Z}_t^{(2)n})$ and $(X_t^n, Z_t^{(1)}, Z_t^{(2)})$ coincide,

(ii) $(\overline{X}_t^n, \overline{Z}_t^{(1)n}, \overline{Z}_t^{(2)n}) \to (\overline{X}_t, \overline{Z}_t^{(1)}, \overline{Z}_t^{(2)})$ in probability as $n \to \infty$.

It is easy to prove that $\overline{Z}_t^{(1)n}$ and $\overline{Z}_t^{(2)n}$ are independent skew stable processes with the characteristic functions (2) as well as $\overline{Z}_t^{(1)}$ and $\overline{Z}_t^{(2)}$. Moreover,

$$\mathrm{d}\overline{X}_t^n = \sigma_{1n}(\overline{X}_{t-}^n)\mathrm{d}\overline{Z}_t^{(1)n} + \sigma_{2n}(\overline{X}_{t-}^n)\mathrm{d}\overline{Z}_t^{(2)n}, \qquad \overline{X}_0^n = x.$$

Hence, it suffices to prove that for each $t \geqslant 0$

$$\int_0^t \sigma_{in}(\overline{X}_{s-}^n)\mathrm{d}\overline{Z}_s^{(i)n} \longrightarrow \int_0^t \sigma_i(\overline{X}_{s-})\mathrm{d}\overline{Z}_s^{(i)}, \qquad i = 1, 2,$$

in probability as $n \to \infty$.

Consider, for example, $i = 1$. Obviously,

$$\begin{aligned}
&\mathbf{P}\left\{\left|\int_0^t \sigma_{1n}(\overline{X}^n_{s-})\mathrm{d}\overline{Z}^{(1)n}_s - \int_0^t \sigma_1(\overline{X}_{s-})\mathrm{d}\overline{Z}^{(1)}_s\right| > 3\varepsilon\right\}\\
&\quad\leqslant \mathbf{P}\left\{\left|\int_0^t \sigma_{1k}(\overline{X}^n_{s-})\mathrm{d}\overline{Z}^{(1)n}_s - \int_0^t \sigma_{1k}(\overline{X}_{s-})\mathrm{d}\overline{Z}^{(1)}_s\right| > \varepsilon\right\}\\
&\qquad+\mathbf{P}\left\{\left|\int_0^t (\sigma_{1n} - \sigma_{1k})(\overline{X}^n_{s-})\mathrm{d}\overline{Z}^{(1)n}_s\right| > \varepsilon\right\}\\
&\qquad+\mathbf{P}\left\{\left|\int_0^t (\sigma_1 - \sigma_{1k})(\overline{X}_{s-})\mathrm{d}\overline{Z}^{(1)}_s\right| > \varepsilon\right\}. \qquad (22)
\end{aligned}$$

The first term on the right side tends to zero as $n \to \infty$, since $\sigma_{1k}$ is a smooth function.

According to Theorem 2.1 of [6] and Theorem 1,

$$\begin{aligned}
\mathbf{P}\left\{\left|\int_0^{t\wedge\tau^n_R} (\sigma_{1n} - \sigma_{1k})(\overline{X}^n_{s-})\mathrm{d}\overline{Z}^{(1)n}_s\right| > \varepsilon\right\} &\leqslant N\varepsilon^{-\alpha}\mathbf{E}\int_0^{t\wedge\tau^n_R} |\sigma_{1n} - \sigma_{1k}|^\alpha(\overline{X}^n_{s-})\mathrm{d}s\\
&\leqslant N\varepsilon^{-\alpha}\left\||\sigma_{1n} - \sigma_{1k}|^\alpha \mathrm{I}_{(-R,R)}\right\|_2
\end{aligned}$$

for each $R > 0$, where $\tau^n_R = \inf\{t \geqslant 0\colon\ X^n_t \notin (-R, R)\}$. Letting first $n \to \infty$, then $k \to \infty$ and $R \to \infty$, we conclude that the second term on the right-hand side of (12) tends to zero as $n \to \infty$ and $k \to \infty$. The same arguments show that the last term on the right-hand side of (12) tends to zero as $k \to \infty$, since the estimate of Theorem 1 can be extended for the process $\overline{X}_t$. The theorem is proved.

## References

[1] N.V. Krylov, On Ito's stochastic integral equations, *Theory Prob. Appl.*, **14**(2), 330–336 (1969).

[2] N.V. Krylov, H. Pragarauskas, On traditional derivation of the Bellman equation for generaal stochastic equations, *Lith. Math. J.*, **21**(2), 101–110 (1981).

[3] P.A. Meyer, *Probability and Potentials*, Blaisdell, Waltham (1966).

[4] H. Pragarauskas, On $L^p$-estimate of stochastic integrals, In: *Probability Theory and Mathematical Statistics*, B. Grigelionis *et al.* (Eds.), VSP, Utrecht/TEV, Vilnius, 579–588, (1999).

[5] H. Pragarauskas, P.A. Zanzotto, On one-dimensional stochastic differential equations driven by stable processes, *Lith. Math. J.*, **40**(3), 361–385 (2000).

[6] L. Rosinski, W.A. Woyczynski, On Ito stochastic integration with respect to $p$-stable motion, *Ann. Probab.*, **14**, 271–286 (1986).

## Stochastinės diferencialinės lygtys pagal asimetrinius stabiliuosius procesus

H. Pragarauskas

Įrodytas vienmačių stochastinių diferencialinių lygčių pagal asimetrinius stabiliuosius procesus su mačiais koeficientais silpnųjų sprendinių egzistavimas. Įrodymas remiasi stochastinių integralų $L_2$-įverčiu.