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On the logarithmic frequence of the values of additive functions

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Introduction

In the present paper we shall consider the distributions of a set of integer-valued strongly additive functions f_x with respect to the logarithmic frequency. We will investigate the weak convergence of distribution functions

$$\mu_x\big(f_x(m) < u\big) = \Big(\sum_{m\leqslant x} \frac{1}{m}\Big)^{-1} \sum_{\substack{m\leqslant x\\ f_x(m)<u}} \frac{1}{m}$$

to some distribution function $F(u)$. We shall consider only additive functions f_x for which $f_x(p) \in \{0,1\}$ over primes p.

Troughout the paper we will denote prime numbers by $p, p_1, p_2, \ldots$. The function $\epsilon(x)$ tends to zero, as x tends to infinity. The absolute constants we will denote by $c_1, c_2, \ldots$. We use B to denote a quantity wich is bounded by an absolute constant. The expression $a \ll b$ is equivalent to $|a| \leqslant cb$. If the bounding constant or the vaniching function depend on a parameter a, we will write c_a, B_a, $\ll_a$, $\epsilon_a(x)$. The superscript * over the sign of sum means that the summation is expanded over primes for wich $f_x(p) = 1$.

The aim of this work is to prove the following assertion.

Theorem. *Let f_x, $x \geqslant 2$, be a set of strongly additive functions and $f_x(p) \in \{0,1\}$ for each prime number p. The distribution functions $\mu_x\big(f_x(m) < u\big)$ converge weakly as $x \to \infty$ if and only if the limits*

$$\lim_{x\to\infty} \sum_{p_1\leqslant x}^{*} \sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}}^{*} \cdots \sum_{\substack{p_{l-1}\leqslant x\\ p_{l-1}\neq p_1,p_2,\ldots,p_{l-2}}}^{*} \sum_{\substack{p_l\leqslant x\\ p_l\neq p_1,p_2,\ldots,p_{l-1}\\ p_1p_2\cdots p_l\leqslant x}} \frac{1}{p_1p_2\cdots p_l}\Big(1-\frac{\ln p_1p_2\ldots p_l}{\ln x}\Big) = g_l$$

exist for each natural number l.

Moreover, in this case the limiting distribution has characteristic function

$$1+\sum_{l=1}^{\infty} \frac{g_l}{l!}(\mathrm{e}^{it}-1)^l.$$

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The proof of necessity

Let $F(u)$ be a limit distribution function for $\mu_x(f_x(m) < u)$. This function is the distribution function of some discrete random variable with jumps at non-negative integer numbers. Asume that $\varphi_k = F(k+0) - F(k)$. Then from the weak convergence we have

$$\lim_{x\to\infty} \mu_x(f_x(m) = k) = \varphi_k \tag{1}$$

for each non-negative integer k. It is clear that $\varphi_{k^*} > 0$ for some k^*.

The inequality

$$\sup_y \mu_x(y \leqslant h(m) < y+u) \ll u\Bigg(\sum_{\substack{p^k \leqslant x\\ |h(p^k)|\leqslant u}} \frac{h^2(p^k)}{p^k} + u^2 \sum_{\substack{p^k \leqslant x\\ |h(p^k)|>u}} \frac{1}{p^k}\Bigg)^{-1/2}$$

holds for each real-valued additive function $h(m)$ (possibly dependent on x) and for $u > 0$, $x \geqslant 2$. The proof of this inequality we can find in [1] or [2].

Using the last inequality for $h(m) = f_x(m)$, $x \geqslant 2$ and $u = 1/2$, we obtain

$$\sup_y \mu_x(f_x(m) = y) \ll \Big(\sum_{p\leqslant x}{}^{*} \frac{1}{p}\Big)^{-1/2}.$$

According to (1),

$$\sum_{p\leqslant x}{}^{*} \frac{1}{p} \ll (\mu_x(f_x(m) = k^*))^{-2} \ll \frac{1}{\varphi_{k^*}^2}$$

for x large enough ($x \geqslant c_{1k^*}$). If $2 \leqslant x \leqslant c_{1k^*}$ the last sum does not exceed c_{1k^*}. Hence

$$\sum_{p\leqslant x}{}^{*} \frac{1}{p} \ll_F 1 \tag{2}$$

for all $x \geqslant 2$. Let

$$\beta_x(l) = \frac{1}{\ln x} \sum_{m\leqslant x} \frac{1}{m} f_x(m)(f_x(m)-1)\dots(f_x(m)-l+1)$$

for natural l.

It is clear that

$$\begin{aligned}\beta_x(l) &= \frac{1}{\ln x} \sum_{m\leqslant x} \frac{1}{m} \sum_{p_1|m}{}^{*} \sum_{\substack{p_2|m\\ p_2\neq p_1}}{}^{*} \cdots \sum_{\substack{p_l|m\\ p_l\neq p_1,p_2,\dots,p_{l-1}}}{}^{*} \\ &= \frac{1}{\ln x} \sum_{p_1\leqslant x}{}^{*} \sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}}{}^{*} \cdots \sum_{\substack{p_l\leqslant x\\ p_l\neq p_1,p_2,\dots,p_{l-1}\\ p_1p_2\cdots p_l\leqslant x}}{}^{*} \frac{1}{p_1p_2\cdots p_l} \sum_{d\leqslant x/(p_1p_2\cdots p_l)} \frac{1}{d}.\end{aligned} \tag{3}$$

Applying the estimate (2) and the equality

$$\sum_{m\leqslant x}\frac{1}{m}=\ln x+c_2+\frac{B}{x}, \tag{4}$$

where c_2 is Euler's constant, we get

$$\beta_x(l)\leqslant\Big(\sum_{p\leqslant x}{}^{*}\frac{1}{p}\Big)^{l}\ll_{l,F}1. \tag{5}$$

Suppose that l is fixed natural number and $K\geqslant l+10$. Using (1) we have that

$$\begin{aligned}\beta_x(l)&=\sum_{k=1}^{K}k(k-1)\dots(k-l+1)\frac{1}{\ln x}\sum_{\substack{m\leqslant x\\ f_x(m)=k}}\frac{1}{m}\\&\quad+\frac{1}{\ln x}\sum_{\substack{m\leqslant x\\ f_x(m)>K}}\frac{1}{m}f_x(m)\big(f_x(m)-1\big)\dots\big(f_x(m)-l+1\big)\frac{f_x(m)-l}{f_x(m)-l}\\&=\sum_{k=1}^{K}k(k-1)\dots(k-l+1)\big(\varphi_k+\epsilon_k(x)\big)+\frac{B}{K-l}\beta_x(l+1).\end{aligned}$$

From (5) we obtain

$$\liminf_{x\to\infty}\beta_x(l)=\limsup_{x\to\infty}\beta_x(l)=\limsup_{K\to\infty}\sum_{k=l}^{K}k(k-1)\dots(k-l+1)\varphi_k. \tag{6}$$

According to the estimate (5) the sequence

$$g_{lK}=\sum_{k=l}^{K}k(k-1)\dots(k-l+1)\varphi_k$$

is increasing and bounded. Therefore the limit

$$g_l=\lim_{K\to\infty}g_{lK}=\sum_{k=l}^{\infty}k(k-1)\dots(k-l+1)\varphi_k$$

exists for fixed natural l.

Hence from (6) we have

$$\lim_{x\to\infty}\beta_x(l)=g_l.$$

On the other hand (3) and (4) show that

$$\beta_x(l) = \sum_{p_1\leqslant x}^{*} \sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}}^{*} \cdots \sum_{\substack{p_l\leqslant x\\ p_l\neq p_1,\dots p_2,\dots,p_{l-1}\\ p_1p_2\cdots p_l\leqslant x}}^{*} \frac{1}{p_1p_2\dots p_l}\left(1-\frac{\ln p_1p_2\dots p_l}{\ln x}\right)$$

$$+\frac{B}{\ln x}\left(\sum_{p\leqslant x}^{*}\frac{1}{p}\right)^{l}. \qquad (7)$$

The obtained equalities and the estimate (2) ensure the validity of the conditions of our theorem.

The proof of sufficiency

Suppose that the limits

$$g_l = \lim_{x\to\infty} \sum_{p_1\leqslant x}^{*} \sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}}^{*} \cdots \sum_{\substack{p_l\leqslant x\\ p_l\neq p_1,\dots p_2,\dots,p_{l-1}\\ p_1p_2\cdots p_l\leqslant x}}^{*} \frac{1}{p_1p_2\dots p_l}\left(1-\frac{\ln p_1p_2\dots p_l}{\ln x}\right) \qquad (8)$$

exist for each fixed natural l. If $l=1$ we have

$$\lim_{x\to\infty}\sum_{p\leqslant x}^{*}\frac{1}{p}\left(1-\frac{\ln p}{\ln x}\right) = g_1. \qquad (9)$$

Hence

$$\sum_{p\leqslant x}^{*}\frac{1}{p} = \sum_{p\leqslant\sqrt{x}}^{*}\frac{1}{p} + \sum_{\sqrt{x}\leqslant p\leqslant x}^{*}\frac{1}{p} \leqslant 2g_1+\ln 2+\epsilon(x).$$

Therefore by (7) we obtain

$$\lim_{x\to\infty}\beta_x(l) = g_l$$

for each fixed natural l.

Suppose

$$\hat{\beta}_x(l) = \left(\sum_{m\leqslant x}\frac{1}{m}\right)^{-1}\sum_{m\leqslant x}\frac{1}{m}f_x(m)(f_x(m)-1)\dots(f_x(m)-l+1).$$

According to (4)

$$\lim_{x\to\infty}\hat{\beta}_x(l) = g_l \qquad (10)$$

for each fixed natural l.

Let $\psi_x(t)$ be the characteristic function of $\mu_x(f_x(m) < u)$. It is clear that

$$\psi_x(t) = \Big(\sum_{m\leqslant x} \frac{1}{m}\Big)^{-1} \sum_{m\leqslant x} \frac{\mathrm{e}^{itf_x(m)}}{m}$$

for $x \geqslant 2$, $t \in \mathbb{R}$.

If r and n are natural numbers, then

$$\Big|\mathrm{e}^{itr} - 1 - \sum_{j=1}^{n-1} \binom{r}{j} (\mathrm{e}^{it} - 1)^j\Big| \leqslant \binom{r}{n} |\mathrm{e}^{it} - 1|^n.$$

Hence

$$\psi_x(t) = 1 + \sum_{l=1}^{L} \frac{(\mathrm{e}^{it} - 1)^l}{l!} \hat{\beta}_x(l) + \frac{B}{(L+1)!} |\mathrm{e}^{it} - 1|^{L+1} \hat{\beta}_x(L+1),$$

where $L \in \mathbb{N}$.

Applying (8) and (9) we obtain

$$g_l \leqslant \lim_{x\to\infty} \Big(\sum_{p\leqslant x}^{*} \frac{1}{p}\Big(1 - \frac{\ln p}{\ln x}\Big)\Big)^l = g_1^l$$

for each natural number l.

Therefore from (10) we have

$$\psi_x(t) = 1 + \sum_{l=1}^{L} \frac{(\mathrm{e}^{it} - 1)^l}{l!} g_l + \epsilon_L(x) + \frac{B(2g_1)^{L+1}}{(L+1)!}, \tag{11}$$

where $t \in \mathbb{R}$, $x \geqslant 2$, $L \in \mathbb{N}$.

Inequality $g_l \leqslant g_1^l$, $l \in \mathbb{N}$, shows that the series

$$1 + \sum_{l=1}^{\infty} \frac{(\mathrm{e}^{it} - 1)^l}{l!} g_l$$

converges uniformly to some continuous function $\psi(t)$. By (11) we obtain

$$\lim_{x\to\infty} \psi_x(t) = \psi(t)$$

for each real number t.

Since $\psi(t)$ is continuous function, it follows from the last equality that distribution functions $\mu_x(f_x(m) < u))$ converge weakly to some distribution function $F(u)$, which has characteristic function $\psi(t)$. This completes the proof.

References

[1] E. Manstavičius, Distribution of additive arithmetic function, In: *Probability Theory and Mathematical Statistics: Proc. of Fifth Vilnius Conf.*, eds. B. Grigelionis *et al.*, vol.II, VSP/Utrecht, TEV/Vilnius, 139–149 (1990).

[2] J. Šiaulys, The logarithmic frequence of distributions of additive functions, *Liet. Matem. Rink.* (to appear).

Apie adityviųjų funkcijų reikšmių logaritminį dažnį

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Darbe nagrinėjama, kada pasiskirstymo funkcijos

$$\Big(\sum_{m\leqslant x}\frac{1}{m}\Big)^{-1}\sum_{\substack{m\leqslant x\\ f_x(m<u}}\frac{1}{m}$$

silpnai konverguoja. Čia f_x, $x \geqslant 2$, yra stipriai adityvios funkcijos, kurioms $f_x(p) \in \{0,1\}$ visiems pirminiams skaičiams p.