Liet. matem. rink., T. 41, spec. nr., 106–112

Uniform distribution on the four-dimensional torus. II

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In the first part of the paper [7] we stated theorem on the uniform distribution on the four-dimensional torus and obtained some auxiliary results. We use the notation from [7].

Theorem. *Let the surface* $\Gamma = \{(x,y,z,w), z = \varphi_1(x,y), w = \varphi_2(x,y), (x,y) \in Q\}$ *have the non-zero curvatures* (1) *of components, and let the assumption* $K_1 \cdot K_2 \geqslant 0$ *hold. Moreover, let the characteristic polynomial of the matrix* V *be irreducible over the field of rational numbers with different real roots. Then for almost all points* $(x,y) \in Q$ *with respect to the Lebesgue measure the set of vectors*

$$\Big\{(x,y,\varphi_1(x,y),\varphi_2(x,y))V\Big\},\quad \Big\{(x,y,\varphi_1(x,y),\varphi_2(x,y))V^2\Big\},\dots$$

is uniformly distributed on the unit cube $[0,1]^4$ *of the space* $\mathbb{R}^4$.

3. Proof of the Theorem

In this proof we make use the ideas of D. Moskvin, V. Dubrovin and V. Leonov.

We suppose that the modulus of eigenvalues of matrix V are different, $|\theta_1| > |\theta_2| > |\theta_3| > |\theta_4|$, and $\vec{w}_i = (w_{i1}, w_{i2}, w_{i3}, w_{i4})$ is eigenvector corresponding to eigenvalue θ_i, $i = 1,2,3,4$. These vectors form the basis in $\mathbb{R}^4$, and therefore an arbitrary vector $\vec{x} = (x_1,x_2,x_3,x_4) \in \mathbb{R}^4$ multiplied by the m-th degree of matrix V may be expressed in the form

$$\vec{x}V^m = \sum_{k=1}^{4}(v_{i1}x_1 + v_{i2}x_2 + v_{i3}x_3 + v_{i4}x_4)\theta_i^m\vec{w}_i.$$

Here v_{ik} are real numbers defined by the matrix V and independent on m.

Let us introduce a linear form

$$L(\vec{x}) = v_{11}x_1 + v_{12}x_2 + v_{13}x_3 + v_{14}x_4. \tag{12}$$

It follows from Lemma 5 of [7] that an inner product

$$\vec{w}_1 \cdot \vec{m} = \sum_{j=1}^{4} w_{1k}m_k$$

is not zero for any integers m_k. Therefore for the function

$$f(x,y) \overset{def}{=} \frac{(x,y,\varphi_1(x,y),\varphi_2(x,y))V^m\vec{m}}{\theta_1^m(\vec{w}_1\cdot\vec{m})}$$

we have the following representation

$$f(x,y) = L(x,y,\varphi_1(x,y),\varphi_2(x,y)) + \left(\frac{\theta_2}{\theta_1}\right)^m \frac{f_1(x,y)}{\vec{w}_1\vec{m}}, \tag{13}$$

$f(x,y)$ being bounded on Q with its mixed derivatives of the third order.

We examine three cases:

1^0 $v_{13}\neq 0$ and $v_{14}=0$ or $v_{13}=0$ and $v_{14}\neq 0$;

2^0 $v_{13}=v_{14}=0$;

3^0 $v_{13}\neq 0$ and $v_{14}\neq 0$.

We begin with the case $v_{14}\neq 0$, $v_{13}=0$. We choose in (13) a vector $\vec{m}$ in such a way that

$$\left|\frac{\theta_2}{\theta_1}\right|\frac{1}{|\vec{w}_1\cdot\vec{m}|}\leqslant\frac{1}{\ln m},\quad m\geqslant 2. \tag{14}$$

This gives expression for the total curvature K_f of the surface $z=f(x,y)$:

$$K_f = \frac{(v_{13}\varphi''_{1x^2}+v_{14}\varphi''_{2x^2})(v_{13}\varphi''_{1y^2}+v_{14}\varphi''_{2y^2})-(v_{13}\varphi''_{1xy}+v_{14}\varphi''_{2xy})^2}{\left(1+(v_{11}+v_{13}\varphi'_{1x}+v_{14}\varphi'_{2x})^2+(v_{12}+v_{13}\varphi'_{1y}+v_{14}\varphi'_{2y})^2\right)^2}$$
$$+\mathrm{O}\left(\frac{1}{\ln m}\right)=\frac{(v_{14}^2(\varphi''_{2x^2}\varphi''_{2y^2}-\varphi_{2xy})}{(1+(v_{11}+v_{14}\varphi'_{2x})^2+(v_{12}+v_{14}\varphi'_{2y})^2}+\mathrm{O}\left(\frac{1}{\ln m}\right).$$

We note the boundedness of the derivatives φ'_{2x} and φ'_{2y} implies $|K_f|\geqslant c_1$ for sufficiently large m. Thus, we can apply the statement.

Theorem C. *Let $z=f(x,y)$ be a continuous function on Q with bounded partial derivatives of the third order. If the total curvature of the surface $z=f(x,y)$ is not zero, then the following estimate is true:*

$$I(n)=\iint\limits_Q \exp\left\{2\pi i n f(x,y)\right\}\,dxdy = c_2 n^{-1/3}.$$

For the proof see [6]. Taking $n=\theta_1^m(\vec{w}_1\cdot\vec{m})$ and having in mind the relation (14), we obtain

$$\iint\limits_Q \exp\{2\pi i\vec{\xi}V^m\cdot\vec{m}\}\,dxdy = c_3\left(\theta_1^m(\vec{w}_1\cdot\vec{m})\right)^{-1/3}, \tag{15}$$

where $\vec{\xi} = (x_1, y_1, \varphi_1(x,y), \varphi_2(x,y))$.

Applying Theorem 2 of W. Schmidt [9] we get that there exists a constant $c_4 > 0$ such that for any fixed vector $\vec{\varrho} = (\varrho_1, \varrho_2, \ldots, \varrho_s)$ and any non-zero integer vector $\vec{m}$ the estimate

$$|\vec{\varrho}\cdot\vec{m}| \geqslant \frac{c_4}{(\bar{m}_1\cdot\bar{m}_2\cdot\ldots\cdot\bar{m}_s)^{1+\varepsilon}}$$

is valid. Here $\varepsilon > 0$ depends only on c_4 and $\bar{m}_k$ is defined by the relation

$$\bar{m}_k = \begin{cases} 1 & \text{if} \quad m_k = 0, \\ |m_k| & \text{if} \quad m_k \neq 0. \end{cases}$$

This gives

$$|\vec{w}_1\cdot\vec{m}|^{-1} \leqslant \frac{1}{c_5}(\bar{m}_1\ldots\bar{m}_s)^{1+\varepsilon},$$

where $\varepsilon > 0$ is an arbitrary small number and $\varepsilon = \varepsilon(\varepsilon, \vec{w})$ is sufficiently small. Hence in view of (14)

$$\bar{m}_1\ldots\bar{m}_s \leqslant \left|\frac{c_6\theta_2^m}{\theta_1^m \ln m}\right|^{1/(1+\varepsilon)} \leqslant \left|\frac{\theta_1}{\theta_2}\right|^{m(1-\varepsilon_1)}, \tag{16}$$

and using (15), we get

$$\iint\limits_Q \exp\left\{2\pi i(\vec{m}\cdot\vec{\xi}V^m)\right\} \mathrm{d}x\mathrm{d}y = \frac{c_7(\bar{m}_1\ldots\bar{m}_s)^{(1+\varepsilon)/3}}{\theta_1^{m/3}}. \tag{17}$$

Let a function $g(\vec{x}) \in E_3^\alpha(c)$, $\alpha > 4/3$. Then we have representation

$$g(\vec{x}) = \int\limits_{\Omega_4}\!\!\ldots\!\!\int g(\vec{x})\,\mathrm{d}x_1\ldots\mathrm{d}x_4 + \sum_{\bar{m}_1\ldots\bar{m}_s=-\infty}^{\infty} c(\bar{m}_1\ldots\bar{m}_s)\exp\left\{2\pi i(\vec{m}\vec{x})\right\}.$$

Let μ_2 denote the Lebesgue measure in the plane. Hence we find

$$\left|\iint\limits_Q g(\vec{\xi}V^m)\,\mathrm{d}x\mathrm{d}y - \mu_2(Q)\int\limits_{\Omega_4}\!\!\ldots\!\!\int g(\vec{x})\,\mathrm{d}x_1\ldots\mathrm{d}x_4\right| \leqslant c_8 \sum_{\bar{m}_1\ldots\bar{m}_s>\Lambda} (\bar{m}_1\ldots\bar{m}_s)^{-\alpha}$$
$$+\, c_9 \sum_{\bar{m}_1\ldots\bar{m}_s\leqslant\Lambda(\bar{m}_1\bar{m}_s)^\alpha} \left|\int Q\exp 2\pi i(\vec{m}\cdot\vec{\xi}V^m)\,\mathrm{d}x\mathrm{d}y\right|, \tag{18}$$

where $\Lambda = |\theta_1/\theta_2|^{m(1-\varepsilon_1)}$. Now from Lemma 1 of [7] and (18) we deduce

$$\sum_{\bar{m}_1\ldots\bar{m}_s\leqslant\Lambda} (\bar{m}_1\ldots\bar{m}_s)^{-\alpha}\left|\iint\limits_Q \exp\left\{2\pi i(\vec{m}\cdot\vec{\xi}V^m)\right\}\mathrm{d}x\mathrm{d}y\right|$$

$$= c_{10}\theta^{-m/s} \sum_{\bar{m}_1\ldots\bar{m}_s \leqslant \Lambda} (\bar{m}_1 \ldots \bar{m}_s)^{-\alpha+(1+\varepsilon)/3} = c_{11}\theta_1^{-m/3}.$$

The later estimate together with (17) and (18) yields

$$\iint\limits_Q g(\vec{\xi}V^m)\, \mathrm{d}x\mathrm{d}y = \mu_2(Q) \int\limits_{\Omega_4}\!\!\ldots\!\!\int g(\vec{x})\, \mathrm{d}x_1 \ldots \mathrm{d}x_4 + c_{12}|\theta_1|^{-m/3} + c_{13}\left|\frac{\theta_2}{\theta_1}\right|^{m(1-\alpha)(1-\varepsilon_2)}, \tag{19}$$

where $\varepsilon_2 > 0$ is arbitrarily small. This relation proves the theorem in the first case.

Now we consider the case $v_{13} = v_{14} = 0$. In this case we have the expression

$$f(x,y) = v_{11}x + v_{12}y + \left(\frac{\theta_2}{\theta_1}\right)^m \frac{f_1(x,y)}{\vec{w}_1 \cdot \vec{m}},$$

and the total curvature of the surface tends to zero when $m \to \infty$, therefore we can not apply Theorem C. Therefore we will proceed in the following way. We denote $\omega = \vec{w}_1 \cdot \vec{m}$ and $a = v_{11}$, $s = v_{12}$. Suppose that $b \neq 0$, because the linear form $ax + by$ is not identically equal to zero. It is easy to check that

$$\iint\limits_Q \exp\left\{2\pi i(\vec{m} \cdot \vec{\xi}V^m)\right\} \mathrm{d}x\mathrm{d}y = \frac{c_{14}}{\vec{w}_1 \cdot \vec{m}}\left|\frac{\theta_1}{\theta_2}\right|^m. \tag{20}$$

This gives for $g \in E_3^\alpha(c)$, $\alpha > 2$,

$$\begin{aligned}\iint\limits_Q g(\vec{\xi}V^m)\, \mathrm{d}x\mathrm{d}y &= \mu_2(Q)\int\limits_{\Omega_4}\!\!\ldots\!\!\int g(\vec{x})\mathrm{d}x_1 \ldots \mathrm{d}x_4 \\ &\quad + c_{15}\left(\left|\frac{\theta_2}{\theta_1}\right|^m \sum_{\bar{m}_1\ldots\bar{m}_s} \frac{1}{|\vec{w}_1 \cdot \vec{m}|(\bar{m}_1 \ldots \bar{m}_s)^\alpha}\right) \\ &= \mu_2(Q) \int\limits_{\Omega_4}\!\!\ldots\!\!\int g(\vec{x})\mathrm{d}x_1 \ldots \mathrm{d}x_4 + c_{16}\left|\frac{\theta_2}{\theta_1}\right|^m,\end{aligned} \tag{21}$$

and if $g \in E_3^\alpha(c)$, $\alpha > 2$, then

$$\iint\limits_Q g(\vec{\xi}V^m)\, \mathrm{d}x\mathrm{d}y = \int\limits_{\Omega_4}\!\!\ldots\!\!\int g(\vec{x})\mathrm{d}x_1 \ldots \mathrm{d}x_4 \mu_2(Q) + c_{17}|\theta_1|^{-m/3} + c_{18}\left|\frac{\theta_2}{\theta_1}\right|^m. \tag{22}$$

Let $\vec{m}$ be a non-zero vector with integer components. It is sufficient to prove that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \exp\left\{2\pi i(\vec{m} \cdot \vec{\xi}V^k)\right\} = 0 \tag{23}$$

almost everywhere with respect to the measure μ_2, and the assertion of the theorem follows from the Weyl criterion.

Let us define two vector functions

$$C_n(\vec{x}) = \sum_{1\leqslant k\leqslant n} \cos 2\pi(\vec{m}\cdot\vec{x}V^k), \quad S_n(\vec{x}) = \sum_{1\leqslant k\leqslant n} \sin 2\pi(\vec{m}\cdot\vec{x}V^k).$$

It is well known that if the function $g(\vec{x})$, $\vec{x}\in\mathbb{R}^4$, is periodic with the period one with respect to every argument and has the derivatives of the fifth order bounded by the constant c, then $g\in E_3^{5/3}(c)$, see [4].

The derivative of the function $\vec{m}\cdot\vec{x}V^k$ and these of $\sin 2\pi(\vec{m}\cdot\vec{x}V^k)$ are bounded by θ_1^{5k}. Therefore the mixed derivatives of the fifth order do not exceed $c_{19}\theta_1^{5n}$, and for $|t|\leqslant\sqrt{n}$ the function

$$g(\vec{x}) = \exp\left\{itn^{-1/2}S_n(\vec{x})\right\}$$

belongs to the class $E_3^{5/3}(c)$, $c=\theta_1^{5n}$. Applying (22) to the latter function, we obtain, for $|t|\leqslant\sqrt{n}$,

$$\begin{aligned} f_n(t) &= \frac{1}{\mu_2(Q)}\iint\limits_Q \exp\left\{itn^{-1/2}S_n(\vec{\xi}V^m)\right\}\mathrm{d}x\mathrm{d}y \\ &= \int\limits_{\Omega_4}\!\!\dots\!\!\int \exp\left\{itn^{-1/2}S_n(\vec{x})\right\}\mathrm{d}x_1\dots\mathrm{d}x_4 + c_{20}\Bigg(\exp\left(\frac{m}{3}-5n\right)\ln|\theta_1|\Bigg) \\ &\quad + \exp\left\{-\left(m\ln\left|\frac{\theta_2}{\theta_1}\right| - 5n\right)\right\}, \end{aligned} \tag{24}$$

where $n=\gamma_0 m$ and $\gamma = 10^{-1}\min(1/3, \ln|\theta_1/\theta_2|) > 0$. We conclude that there exist constants $0<\gamma_0<1$ and c_0 such that, for $|t|\leqslant\sqrt{\gamma_0 m}$,

$$f_{\gamma_0 m}(t) = \int\limits_{\Omega_4}\!\!\dots\!\!\int \exp\left\{\frac{it}{\sqrt{\gamma_0 m}}S_{\gamma_0 m}(\vec{x})\right\}\mathrm{d}x_1\dots\mathrm{d}x_4 + c_{21}\exp\{-c_0 m\}. \tag{25}$$

For the unit matrix V_0 we obtain

$$\begin{aligned} &\int\limits_{\Omega_4}\!\!\dots\!\!\int \sin 2\pi(\vec{m}\cdot\vec{x})\sin 2\pi(\vec{m}\cdot\vec{x}V^m)\,\mathrm{d}x_1\dots\mathrm{d}x_4 \\ &= \frac{1}{2}\int\limits_{\Omega_4}\!\!\dots\!\!\int\Big(\cos 2\pi\big(\vec{m}\cdot\vec{x}(V_0 - V^m)\big) - \cos 2\pi\big(\vec{m}\cdot\vec{x}(V_0+V^m)\big)\Big)\,\mathrm{d}x_1\dots\mathrm{d}x_4 = 0, \end{aligned}$$

and there exist the limit

$$\lim_{n\to\infty}\int\limits_{\Omega_4}\!\!\dots\!\!\int\left(\frac{S_n(\vec{x})}{\sqrt{n}}\right)^2\mathrm{d}x_1\dots\mathrm{d}x_4 = \sigma^2 > 0. \tag{26}$$

We deduce from one central limit theorem of V. Leonov [5] that the integral in the right hand-side of (25) converges uniformly in finite intervals to the characteristic function $\exp\{-\sigma^2 t^2/2\}$ of the normal distribution as $m \to \infty$. Also, we deduce from (25) that

$$\lim_{m\to\infty} f_{\gamma_0 m}(t) = \exp\{-\sigma^2 t^2/2\}, \tag{27}$$

because $f_{\gamma_0 m}(t)$ is the characteristic function of the sum

$$(\gamma_0 m)^{-1/2} S_{\gamma_0 m}(\vec{\xi} V^m). \tag{28}$$

So we have

$$\lim_{m\to\infty} \iint\limits_Q \left(\frac{S_{\gamma_0 m}(\vec{\xi} V^m)}{\sqrt{\gamma_0 m}}\right)^{2k} \mathrm{d}x\,\mathrm{d}y = \frac{(2k)}{k!}\frac{\sigma^{2k}}{2^{2k}}\,\mu_2(Q),$$

that is even moments of the sum (28) tend to the moments of the normal distribution with the mean zero and the variance σ^2. From this we obtain that

$$\iint\limits_Q S_n^{2k}(\vec{\xi})\,\mathrm{d}x\mathrm{d}y = \mathrm{O}(n^k \ln^k n), \tag{29}$$

and

$$\iint\limits_Q \left(C_n(\vec{\xi})\right)^{2k} \mathrm{d}x\mathrm{d}y = \mathrm{O}(n^k \ln^k n), \tag{30}$$

as $n \to \infty$.

Finally we get for $n \to \infty$

$$\iint\limits_Q \left|\sum_{k=1}^{n} \exp\{2\pi i \vec{m}\cdot\vec{\xi} V^k\}\right|^{2k} \mathrm{d}x\mathrm{d}y$$
$$= \mathrm{O}\left(2^{2k} \iint\limits_Q C_n^{2k}(\vec{\xi})\,\mathrm{d}x\mathrm{d}y + 2^{2k} \iint\limits_Q S_n^{2k}(\vec{\xi})\,\mathrm{d}x\mathrm{d}y\right) = \mathrm{O}(n^k \ln^k n).$$

This estimate proves (23) and together the theorem in the second case.

The proof in the third case can be obtained by combining the proofs of the first and second cases. The case when the modulus of eigenvalues of matrix V coincide needs a slight modification of the proof.

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Tolygūs pasiskirstymai ant keturmačio toro. II

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Įrodoma teorema apie keturmačio toro dvimačių paviršių transformacijų tolygų pasiskirstymą. Įrodymas remiasi D. Moskvin, V. Dubrovin [2] ir V. Leonov [5] idėjomis.