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An estimate for the Taylor coefficients

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We examine the Taylor coefficients of an analytic in $|z| < 1$ function $F(z)$ having a fairly particular form. Such functions appear frequently in analytic and probabilistic combinatorics as generating functions of the values of mappings defined on assemblies, multisets, selections or additive arithmetic semigroups (see [1–4], [6–8]).

Let

$$F(z) := \sum_{N\geqslant 0} m_N z^N = \sum_{k\geqslant 0} b_k z^k \exp\left\{\sum_{j\geqslant 1} \frac{a_j z^j}{j}\right\}$$
$$=: H(z)\exp\{U(z)\} =: H(z)G(z), \tag{1}$$

where $a_j, b_j \in \mathbb{C}$. Typically, the function $H(z)$ satisfies some smoothness conditions on the circumference $|z| = 1$ and is "better" than $U(z)$.

In the case $H(z) \equiv 1$, formal expansion of the series leads to the formula

$$m_N = m_N(a_1, \dots, a_N) = \sum_{L(\bar{k})=n} \prod_{j=1}^{n} \left(\frac{a_j}{j}\right)^{k_j} \frac{1}{k_j!}, \tag{2}$$

where the summation is extended over vectors $\bar{k} = (k_1, \dots, k_n)$ with nonnegative integer coordinates and satisfying the relation $L(\bar{k}) := 1k_1 + \dots Nk_N = N$. Given some initial information on a_j, it is rather difficult to use (2) to derive asymptotical properties of m_N as $N \to \infty$.

In [8], we have obtained a few estimates of m_N in terms of the Taylor coefficients $\tilde{m}_N$ of the function

$$D(z) := \sum_{n\geqslant 0} \tilde{m}_n z^n = \exp\left\{\sum_{j\leqslant N} \frac{d_j z^j}{j}\right\} =: \exp\{V(z)\},$$

provided that $|a_j| \leqslant d_j \leqslant d < \infty$ for each $1 \leqslant j \leqslant N$. This individual bound and a rather strong requirement $\sup_{|z|\leqslant 1} |H'(z)| \leqslant H < \infty$ were the main obstacles in some applications of the results. An instance of them is presented at the end of the paper. We now generalize Proposition 2 of our paper [8].

Theorem. *Let $a, b, d,$ and B be positive constants such that $|a_j| \leqslant a$, $0 \leqslant d_j \leqslant d$,*

$$\sum_{j\leqslant N} |b_j| \leqslant B, \quad j|b_j| \leqslant b, \tag{3}$$

and

$$\sum_{j\leqslant\delta N}\frac{|a_j|}{j}\leqslant\sum_{j\leqslant\delta N}\frac{d_j}{j}+C,\qquad\sum_{\delta N<j\leqslant N}\frac{d_j}{j}\geqslant c\log\frac{1}{\delta}\tag{4}$$

with some $c>0$, $C\geqslant 0$, *and arbitrary* $\delta\in(0,1]$. *Then there exists a positive constant* $c_1=c_1(c,d)$ *such that*

$$m_N\ll\exp\left\{\sum_{j\leqslant N}\frac{d_j-1}{j}-c_1\min_{|t|\leqslant\pi}\sum_{j\leqslant N}\frac{d_j-\Re(a_j e^{-itj})}{j}\right\}.$$

The constant in $\ll$, *the analog of the symbol* $O(\cdot)$, *depends on* a, b, d, B, *and* C *only.*

REMARK. The appearance of the first sum under the exponent is natural. To verify this, take $a_j\equiv d_j$ and assume the condition $0<d_0\leqslant d_j\leqslant d$. By Lemma 1 in [8] we have

$$m_N\asymp\exp\left\{\sum_{j\leqslant N}\frac{d_j-1}{j}\right\}$$

where the constants in $\asymp$ depend on d_0 and d only.

Actually, in the estimate of m_N one can take

$$c_1=\min\left\{\left(\sqrt{1+c}-1\right)^2,\ \left(\sqrt{1+c}-1\right)/2d\right\}\leqslant\min\{c^2/4,c/2d\}.$$

Note also that $c\leqslant d$.

Difficulties arising in the case of functions with unbounded coefficients a_j have been discussed in author's paper [6]. This article and remark [7] contain a few asymptotic formulas for m_N obtained under more restrictive conditions than those used in Theorem above.

Proof of Theorem. Without loss of generality, we may assume that $a_j=0$ and $b_j=0$ for $j>N$, nevertheless, even after this change, we leave the same notation of $U(z)$ and $H(z)$. Let $0<\alpha,\delta<1$ be arbitrary fixed numbers, $K=\delta N\geqslant 1$,

$$G_1(z):=\exp\left\{\alpha\sum_{j\leqslant K}\frac{a_j}{j}z^j\right\},\quad G_2(z):=\exp\left\{-\alpha\sum_{K<j\leqslant N}\frac{a_j}{j}z^j\right\},$$

and $G_3(z):=G^{\alpha}(z)-G_1(z)$.

By Cauchy's formula

$$m_N=\frac{1}{2\pi iN}\int\limits_{|z|=1}\frac{F'(z)}{z^N}\,dz$$

$$= \frac{1}{2\pi i N} \int\limits_{|z|=1} G^{1-\alpha}(z)\big(H(z)U'(z) + H'(z)\big)G_1(z)\,\frac{\mathrm{d}z}{z^N}$$

$$+ \frac{1}{2\pi i N} \int\limits_{|z|=1} G^{1-\alpha}(z)\big(H(z)U'(z) + H'(z)\big)G_3(z)\,\frac{\mathrm{d}z}{z^N} =: J_1 + J_2. \quad (5)$$

We have

$$|J_2| \leqslant \frac{B}{2\pi N} \max_{|z|=1} \big|G^{1-\alpha}(z)\big| \int\limits_{|z|=1} |U'(z)||G_3(z)|\,|\mathrm{d}z|$$

$$+ \frac{1}{2\pi N} \max_{|z|=1} |G^{1-\alpha}(z)| \int\limits_{|z|=1} |H'(z)||G_3(z)|\,|\mathrm{d}z| =: J_{21} + J_{22}. \quad (6)$$

Further,

$$J_{21} \leqslant \frac{BD^{1-\alpha}(1)}{2\pi N} \exp\Big\{(1-\alpha)\min_{|t|\leqslant\pi}\big(\Re U(\mathrm{e}^{it}) - V(1)\big)\Big\}$$

$$\times \bigg(\int\limits_{|z|=1} |U'(z)|^2|\mathrm{d}z|\bigg)^{1/2} \bigg(\int\limits_{|z|=1} |G_3(z)|^2\,|\mathrm{d}z|\bigg)^{1/2}. \quad (7)$$

Since $|a_j| \leqslant a$, by virtue of Parseval's equality, the first integral on the right-hand side does not exceed $2\pi a^2 N$. For the second integral, we apply (2) to get

$$G_3(z) = \sum_{n>K} \bigg(\sum_{\substack{1k_1+\ldots+nk_n=n \\ \exists j>K \text{ with } k_j\geqslant 1}} \prod_{j=1}^{n} \Big(\frac{\alpha a_j}{j}\Big)^{k_j} \frac{1}{k_j!}\bigg) z^n.$$

The sum in the braces does not exceed q_n defined via

$$\sum_{n\geqslant 0} q_n z^n = \exp\Big\{\alpha \sum_{j\leqslant N} \frac{|a_j|}{j} z^j\Big\} =: Q(z).$$

Hence by (4)

$$\int\limits_{|z|=1} |G_3(z)|^2\,|\mathrm{d}z| \leqslant \frac{2\pi}{K^2} \sum_{n\geqslant 1} q_n^2 n^2 = \frac{1}{K^2} \int\limits_{|z|=1} \big|Q'(z)\big|^2\,|\mathrm{d}z|$$

$$\leqslant \frac{2\pi \mathrm{e}^{2C} D^{2\alpha}(1)}{K^2} \sum_{j\leqslant N} |a_j|^2 \leqslant \frac{2\pi a^2 \mathrm{e}^{2C} D^{2\alpha}(1)}{\delta^2 N}.$$

Inserting these estimates of integrals into (7) we have

$$J_{21} \leqslant \frac{a^2 B \mathrm{e}^C D(1)}{\delta N} \exp\Big\{(1-\alpha)\min_{|t|\leqslant\pi}\big(\Re U(\mathrm{e}^{it}) - V(1)\big)\Big\}. \quad (8)$$

Similarly,

$$J_{22} \leqslant \frac{abe^C D(1)}{\delta N} \exp\left\{(1-\alpha) \min_{|t|\leqslant\pi} \left(\Re U(e^{it}) - V(1)\right)\right\}. \tag{9}$$

The estimates (8) and (9) imply the satisfactory bound for J_2 in (6).

Investigating J_1, we use the convolution arguments. Observe that

$$J_1 = \frac{1}{N} \sum_{\substack{n,s,k\geqslant 0\\ n+s+k\leqslant N-1}} g_n \tilde{g}_s b_k a_{N-s-n-k} + \frac{1}{N} \sum_{\substack{n,s\geqslant 0\\ n+s\leqslant N-1}} g_n \tilde{g}_s (N-s-n) b_{N-n-s}.$$

Here g_s and $\tilde{g}_s$ are the s-th Taylor coefficients of $G^{1-\alpha}(z)$ and $G_1(z)$ respectively. Thus by (4),

$$\sum_{s\leqslant N} |g_s| \leqslant \exp\left\{(1-\alpha)\sum_{j\leqslant N} \frac{|a_j|}{j}\right\} \leqslant \exp\left\{(1-\alpha)(C+V(1))\right\},$$

and

$$\sum_{s\leqslant N} |\tilde{g}_s| \leqslant \exp\left\{C\alpha + \alpha \sum_{j\leqslant K} \frac{d_j}{j}\right\}.$$

Exploiting the conditions of Theorem, we now obtain

$$\begin{aligned} |J_1| &\leqslant \frac{a}{N} \sum_{n\leqslant N} |g_n| \sum_{s\leqslant N} |\tilde{g}_s| \sum_{k\leqslant N} |b_k| + \frac{b}{N} \sum_{n\leqslant N} |g_n| \sum_{s\leqslant N} |\tilde{g}_s| \\ &\leqslant \frac{(aB+b)e^C D(1)}{N} \exp\left\{-\alpha \sum_{K<j\leqslant N} \frac{d_j}{j}\right\} \leqslant C_2 \frac{D(1)\delta^{c\alpha}}{N}, \end{aligned} \tag{10}$$

where $C_2 = (aB+b)e^C$.

Set $C_3 = \max\{C_2, (a^2B+ab)e^C\}$ and $E = \exp\left\{\min_{|t|\leqslant\pi}(\Re U(e^{it}) - V(1))\right\}$. It follows from (5), (6), (7), (9), and (10) that

$$m_N \leqslant \frac{C_3 D(1)}{N}\left(\frac{E^{1-\alpha}}{\delta} + \delta^{c\alpha}\right),$$

provided that $\delta N \geqslant 1$. The choice

$$\delta = \max\left\{\min\left\{1, E^{(1-\alpha)/(1+c\alpha)}\right\}, \frac{1}{N}\right\}$$

gives the estimate

$$m_N \leqslant \frac{2C_3 D(1)}{N}\left(\exp\left\{\frac{c\alpha(1-\alpha)}{1+c\alpha}\log E\right\} + \frac{1}{N^{c\alpha}}\right). \tag{11}$$

The desire to have the first factor under the exponent as large as possible leads to the choice $\alpha = (\sqrt{1+c}-1)/c$. Further, the conditions of Theorem yield $e^{-2d-C}N^{-2d} \leqslant E \leqslant e^{C}$. This enables us to get rid of the second term in (11). In this way we obtain

$$m_N \leqslant \frac{C_4 D(1)}{N}\Big(\exp\Big\{\min\{(\sqrt{1+c}-1)^2, (\sqrt{1+c}-1)/2d\}\log E\Big\}\Big),$$

where $C_4 > 0$ depends on C_3, C, and d.

Theorem is proved.

An application. Let $\mathcal{F}_N$ be the set of all mappings τ of an N set into itself, $k_j = k_j(\tau)$ be the number of the components in the functional digraph of τ, $1 \leqslant j \leqslant N$, and $f : \mathcal{F}_N \to \mathbb{C}$ be a completely multiplicative function, maybe, depending on N or other parameters. It has the following expression

$$f(\tau) = \prod_{j=1}^{N} f_j^{k_j(\tau)}, \quad 0^0 := 1,$$

where $f_j \in \mathbb{C}$. Assume that $|f(\tau)| \leqslant 1$. Then (see [1] or [8])

$$1 + \sum_{N=1}^{\infty} \frac{e^{-N} z^N}{N!} \sum_{\tau \in \mathcal{F}_N} f(\tau) = \exp\Big\{\sum_{j=1}^{\infty} \frac{\lambda_j f_j}{j} z^j\Big\},$$

where

$$\lambda_j = e^{-j}\sum_{s=0}^{j-1} \frac{j^s}{s!} = \frac{1}{2} + \frac{8\theta}{\sqrt{j}}, \quad |\theta| \leqslant 1.$$

The last estimate has been proved in [5]. Applying Theorem with $a_j = \lambda_j f_j$ and $d_j = 1/2$ for each $1 \leqslant j \leqslant N$ and Stirling's formula we obtain

$$N^{-N}\Big|\sum_{\tau \in \mathcal{F}_N} f(\tau)\Big| \leqslant C_5 \exp\Big\{-c_2 \min_{|t|\leqslant\pi} \sum_{j\leqslant N} \frac{1-\Re(f_j e^{itj})}{j}\Big\},$$

with absolute positive constants C_5 and c_2.

In its turn, this inequality could be used to estimate concentration of values of an additive function defined on $\mathcal{F}_N$.

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Tayloro koeficientų įvertis

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Darbe gautas analizinės vienetiniame skritulyje funkcijos, dažnai sutinkamos kombinatorikoje, Tayloro koeficientų įvertis. Šie koeficientai išreiškia atvaizdžių, apibrėžtų ansambliuose, kartotinėse svorinėse aibėse, adityviuose aritmetiniuose pusgrupiuose, reikšmių vidurkines savybes.