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Discrete limit theorems for general Dirichlet polynomials

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Let

$$p_n(t) = \sum_{m=1}^{n} a_m e^{i\lambda_m t} \tag{1}$$

be a Dirichlet polynomial with complex-valued coefficients a_m and real exponents λ_m. Discrete limit theorems for Dirichlet polynomials were proved in [2], however the explicit form of limit measures in mentioned theorems was obtained only in the case of ordinary Dirichlet polynomials

$$\sum_{m=1}^{n} \frac{a_m}{m^{it}}.$$

The aim of this note is to find the explicit form of the limit measure in the case of general Dirichlet polynomial (1).

Let, for $N \in \mathbb{N}$,

$$\mu_N(\ldots) = \frac{1}{N+1} \# \{0 \leqslant m \leqslant N\colon \ldots\},$$

where in place of dots a condition satisfied by m is to be written. We suppose that the exponents λ_m are real algebraic numbers, linearly independent over the field of rational numbers. Moreover, let $h > 0$ be such that $\exp\left\{\frac{2\pi}{h}\right\}$ is a rational number. Denote by $\mathcal{B}(S)$ the class of Borel of the space S, and let $\mathbb{C}$, as usual, be the complex plane.

Denote by γ the unit circle on $\mathbb{C}$, and let

$$\Omega_n = \prod_{m=1}^{n} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m = 1, \ldots, n$. Define a function $u\colon \Omega_n \to \mathbb{C}$ by the formula

$$u(x_1, \ldots, x_n) = \sum_{m=1}^{n} a_m x_m, \quad (x_1, \ldots, x_n) \in \Omega_n,$$

and let m_{nH} stands for the Haar measure on $(\Omega_n, \mathcal{B}(\Omega_n))$.

Theorem 1. *The probability measure*

$$P_N(A) = \mu_N\left(p_n(mh) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the measure $m_{nH}u^{-1}$ *as* $N \to \infty$.

Now let $s = \sigma + it$ be a complex variable, and let G be a region on $\mathbb{C}$. Denote by $H(G)$ the space of analytic on G functions equipped with the topology of uniform convergence on compacta. Consider a Dirichlet polynomial

$$q_n(s) = \sum_{m=1}^{n} a_m e^{-\lambda_m s}.$$

Let a function v: $\Omega_n \to H(G)$ be given by the formula

$$v(x_1, \dots, x_n) = \sum_{m=1}^{n} a_m e^{-\lambda_m s} x_m^{-1}, \quad (x_1, \dots, x_n) \in \Omega_n.$$

Theorem 2. *The probability measure*

$$Q_N(A) = \mu_N\left(q_n(s + imh) \in A\right), \quad A \in \mathcal{B}(H(G)),$$

converges weakly to the measure $m_{nH}v^{-1}$ *as* $N \to \infty$.

The main ingredient of the proof of Theorems 1 and 2 is the following lemma.

Lemma 1. *The probability measure*

$$\mu_N\left(\left(e^{i\lambda_1 mh}, \dots, e^{i\lambda_n mh}\right) \in A\right), \quad A \in \mathcal{B}(\Omega_n),$$

converges weakly to the Haar measure m_{nH} *as* $N \to \infty$.

Proof. The Fourier transform $g_N(k_1, \dots, k_n)$ of the measure of the lemma is

$$g_N(k_1, \dots, k_n) = \frac{1}{N+1} \sum_{m=0}^{N} e^{imh \sum_{l=1}^{n} k_l \lambda_l}.$$

Since the exponents λ_m are linearly independent over the field of rational numbers, we have that

$$g_N(k_1, \dots, k_n) = \begin{cases} 1, & (k_1, \dots, k_n) = (0, \dots, 0), \\ \frac{1}{N+1} \frac{1 - \exp\left\{i(N+1)h \sum_{l=1}^{n} k_l \lambda_l\right\}}{1 - \exp\left\{ih \sum_{l=1}^{n} k_l \lambda_l\right\}}, & (k_1, \dots, k_n) \neq (0, \dots, 0). \end{cases} \tag{2}$$

Really, we have that, for $(k_1, \dots, k_n) \neq (0, \dots, 0)$,

$$\exp\left\{ih\sum_{l=1}^{n} k_l\lambda_l\right\} \neq 1. \tag{3}$$

If

$$\exp\left\{ih\sum_{l=1}^{n} k_l\lambda_l\right\} = 1,$$

then

$$h\sum_{l=1}^{n} k_l\lambda_l = 2\pi k, \quad k \in \mathbb{Z},$$

and

$$\sum_{l=1}^{n} k_l\lambda_l = \frac{2\pi k}{h}.$$

However, by the Hermite-Lindemann theorem

$$\exp\left\{\sum_{l=1}^{n} k_l\lambda_l\right\}$$

is a transcendental number, while by the choice of h we have that

$$\exp\left\{\frac{2\pi k}{h}\right\}$$

is a rational number. Therefore inequality (3) is valid. From (2) we find that

$$\lim_{N\to\infty} g_N(k_1, \dots, k_n) = \begin{cases} 1, & (k_1, \dots, k_n) = (0, \dots, 0), \\ 0, & (k_1, \dots, k_n) \neq (0, \dots, 0). \end{cases}$$

This shows that the measure of the lemma converges weakly to the Haar measure m_{nH} as $N \to \infty$.

Note that the measure of Lemma 1 converges weakly to some limit measure without any restriction on the exponents λ_m. However, for applications we need the Haar measure.

Proof of Theorem 1. By the definition of the function $u(x_1, \dots, x_n)$ we have

$$p_n(mh) = u\left(e^{i\lambda_1 mh}, \dots, e^{i\lambda_n mh}\right).$$

The function u is continuous. Therefore, in view Theorem 5.1 from [1] and Lemma 1 we obtain that the measure of the theorem converges weakly to the Haar measure $m_{nH}u^{-1}$ as $N \to \infty$.

Proof of Theorem 2. We have that

$$q_n(s+imh) = v\left(e^{i\lambda_1 mh}, \dots, e^{i\lambda_n mh}\right),$$

and the function v is continuous. Therefore, the theorem follows in the same way as Theorem 1.

For applications to general Dirichlet series the following two assertions are useful. Let $g(m)$, $|g(m)| = 1$, be an arbitrary arithmetic function, and

$$\begin{aligned} p_n(t,g) &= \sum_{m=1}^{n} a_m g(m) e^{i\lambda_m t}, \\ q_n(s,g) &= \sum_{m=1}^{n} a_m g(m) e^{-\lambda_m s}. \end{aligned}$$

Theorem 3. *The probability measures P_N and*

$$\widetilde{P}_N(A) = \mu_N\left(p_n(mh,g) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}),$$

both converge weakly to the same limit measure as $N \to \infty$.

Theorem 4. *The probability measures Q_N and*

$$\widetilde{Q}_N(A) = \mu_N\left(q_n(s+imh,g) \in A\right), \quad A \in \mathcal{B}(H(G)),$$

both converge weakly to the same limit measure as $N \to \infty$.

Proof of Theorem 3. Let $\theta_m = \arg g(m)$, $m = 1, \dots, n$. Define a function $u_1\colon \Omega_n \to \Omega_n$ by the formula

$$u_1(x_1, \dots, x_n) = \left(x_1 e^{i\theta_1}, \dots, x_n e^{i\theta_n}\right).$$

By Theorem 1 the probability measures P_N and $\widetilde{P}_N$ converges weakly to the measures $m_{nH}u^{-1}$ and $m_{mH}\tilde{u}^{-1}$, respectively, where

$$\tilde{u}(x_1, \dots, x_n) = \sum_{m=1}^{n} a_m g(m) x_m, \quad (x_1, \dots, x_n) \in \Omega_n.$$

Hence we find

$$\tilde{u}(x_1, \dots, x_n) = \sum_{m=1}^{n} a_m\left(x_m e^{i\theta_m}\right) = u\left(u_1(x_1, \dots, x_n)\right).$$

Therefore

$$m_{nH}\tilde{u}^{-1} = m_{nH}\left(u(u_1)\right)^{-1} = \left(m_{nH}u_1^{-1}\right)u^{-1} = m_{nH}u^{-1},$$

since the Haar measure is invariant with respect to the translation by points in Ω_n. The theorem is proved.

Proof of Theorem 4. Define a function $v_1\colon \Omega_n \to \Omega_n$ by the formula

$$v_1(x_1,\ldots,x_n) = \left(x_1 e^{-i\theta_1},\ldots,x_n e^{-i\theta_n}\right).$$

By Theorem 2 the probability measures Q_N and $\widetilde{Q}_N$ converges weakly to the measures $m_{nH}v^{-1}$ and $m_{mH}\tilde{v}^{-1}$, respectively, where

$$\tilde{v}(x_1,\ldots,x_n) = \sum_{m=1}^{n} a_m g(m) e^{-\lambda_m s} x_m^{-1}, \quad (x_1,\ldots,x_n) \in \Omega_n.$$

Similarly as above we find

$$\tilde{v}(x_1,\ldots,x_n) = \sum_{m=1}^{n} a_m e^{-\lambda_m s}\left(x_m e^{i\theta_m}\right)^{-1} = v\left(v_1(x_1,\ldots,x_n)\right).$$

Hence

$$m_{nH}\tilde{v}^{-1} = m_{nH}v^{-1},$$

and the theorem is proved.

Note that in the last two theorems we applied essentially the properties of the Haar measure. If the limit measure in Lemma 1 is not the Haar measure, then Theorems 3 and 4 are not true.

References

[1] P. Billingsley, *Convergence of Probability Measures*, Wiley, New Yourk (1968).

[2] R. Kašinskaitė, Discrete limit theorems for trigonometric polynomials, *Proc. of XL Conf. of Lith. Math. Soc.*, 3, Vilnius (a spec. suplement of Liet. Matem. Rink.), Vilnius, 44–49 (1999).

Diskrečiosios ribinės teoremos bendriesiems Dirichlet polinomams

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Įrodytos diskrečios ribinės teoremos bendriesiems Dirichlet polinomams silpno matų konvergavimo prasme. Nurodytas išreikštinis ribinis matų pavidalas.