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On approximation by the Poisson law

Vydas ČEKANAVIČIUS* (VU)
e-mail: vydas.cekanavicius@maf.vu.lt

Let $\xi_{n1}, \xi_{n2}, \ldots, \xi_{nk_n}$, $k_n \to \infty$ as $n \to \infty$ be a sequence of series of mutually independent in each series random variables having three finite moments; $S_n = \xi_{n1} + \ldots + \xi_{nk_n}$; $\lambda = \mathbf{E} S_n$; Γ_2 be a second factorial cumulant of S_n; $\bar{\alpha}_{lj} = \mathbf{E} |\, \xi_{nj}(\xi_{nj} - 1) \ldots (\xi_{nj} - l + 1) \,|$; $\lambda_j = \mathbf{E}\xi_{nj} > 0$;

$$\Pi(x;\lambda) = \sum_{l=0}^{[x]} \frac{\lambda^l}{l!} \mathrm{e}^{-\lambda}; \quad \Pi_1\left([x];\lambda\right) = \frac{\lambda^{[x]}}{[x]!}\mathrm{e}^{-\lambda} - \frac{\lambda^{[x-1]}}{[x-1]!}\mathrm{e}^{-\lambda}.$$

Absolute constants are denoted by C_i.

Theorem 1. *Assume that* $\lambda \geqslant 1$ *and there exists a constant* $\Delta = \Delta(n)$, *satisfying inequalities*

$$|\, \bar{\alpha}_{lj} \,| \leqslant \frac{\lambda_j l!}{\Delta^{l-1}}, \quad l = 2, 3; \; j = 1, \ldots, k_n; \; n = 1, 2, \ldots \tag{1}$$

and

$$10 \leqslant \Delta_n \leqslant 1 / \max_{1 \leqslant j \leqslant k_n} \lambda_j. \tag{2}$$

Then

$$P(S_n \leqslant x) = \Pi(x;\lambda) + \frac{1}{2}\Gamma_2 \Pi_1\left([x];\lambda\right) + R_3(x) + R, \tag{3}$$

where

$$\sup_x |\, R_3(x) \,| \leqslant \begin{cases} \dfrac{C_1}{\Delta^2}\left(1 + \dfrac{\ln\Delta}{\sqrt{\lambda}}\right), & \textit{for any } S_n, \\ \dfrac{C_2}{\Delta^2}, & \textit{if all } \xi_j \textit{ are integral non-negative;} \end{cases} \tag{4}$$

and

$$|\, R \,| \leqslant 2\sum_{j=1}^{k_n} \left| \widetilde{F}_j^{(n)}(x + \varepsilon_{\Delta^2}) - \widetilde{F}_j^{(n)}(x) \right|, \quad \varepsilon_{\Delta^2} = \frac{\ln\Delta^2}{\Delta}.$$

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Here $\widetilde{F}_j$ is the part of the distribution of ξ_{nj} after rejecting jumps at the points $0, 1, 2, \ldots$ *Moreover, if ξ_{nj} is integral and non-negative, then $R = 0$.*

Theorem 1 is an improvement of Thm 1 from [1], for the case $\Delta \geqslant 10$ and $\lambda \geqslant 1$. Indeed, in [1], for this case, it was proved that

$$\sup_x |R_3(x)| \leqslant \begin{cases} \dfrac{8}{\Delta^2}\left(2,28\ln\Delta + \dfrac{2}{\pi}\ln\lambda\right), & \text{for any } S_n, \\ \dfrac{8}{\Delta^2}\left(0,78 + \dfrac{1}{2}\ln\lambda\right), & \text{if all } \xi_j \text{ are integral non-negative.} \end{cases} \tag{5}$$

Estimate (4) means that the logarithm factors in (5) are superfluous. Note that the logarithm factors can essentially change the order of approximation. Indeed, let us consider the binomial distribution with parameter $n^{-1/3} \leqslant p \leqslant 1/2$ and use the Poisson approximation with the parameter $\lambda = np$. The accuracy in (5) is of the order $\mathrm{O}(p^2\ln(np))$. Meanwhile, the well-known fact is that the right order should be $\mathrm{O}(p^2)$. On the other hand, (4) gives the accuracy $\mathrm{O}(p^2(1 + (np)^{-1/2}\ln p) = \mathrm{O}(p^2)$.

The main goal of this note is a demonstration how Levy's concentration function can be combined with the formulae of inversion. Such combination may be especially fruitful if used with Thm 1 from [2]. Thm 1 in above is just an example of such application. The main result of this note is the following lemma.

Lemma 1. *Let $f_0(t) \geqslant 0$, $f(t)$, $g(t)$ be characteristic functions, $|f(t) - g(t)| \leqslant \varrho f_0(t)$. Let $0 < T < T_1$,*

$$I_T = \frac{1}{2\pi}\int_{-T}^{T} \frac{|f(t) - g(t)|}{|t|}\,\mathrm{d}t, \quad l_{T_1} = \frac{T_1}{2\pi}\int_{-\infty}^{\infty} \frac{|f(t) - g(t)|}{|t|\sqrt{T_1^2 + t^2}}\,\mathrm{d}t. \tag{6}$$

Then, for any $\tau > 0$,

$$l_{T_1} \leqslant I_T + \varrho\frac{13}{2\pi}Q(F_0, \tau)\left(\frac{T_1}{T\sqrt{T_1^2 + t^2}} \cdot \frac{1}{\tau} + 2 + \ln\frac{T_1}{T}\right). \tag{7}$$

Here $Q(F_0, \tau)$ is Levy's concentration function

$$Q(F_0, \tau) = \sup_x F_0\{[x, x + \tau]\},$$

and F_0 is the distribution having non-negative characteristic function $f_0(t)$.

Proof of Lemma 1. It is obvious that

$$l_{T_1} = \frac{T_1}{2\pi}\int_{-T}^{T} + \frac{T_1}{2\pi}\int_{|t|>T} = l_1 + l_2; \tag{8}$$

$$l_1 = \frac{1}{2\pi}\int_{-T}^{T} \frac{|f(t)-g(t)|}{|t|}\cdot\frac{T_1}{\sqrt{T_1^2+t^2}}\,\mathrm{d}t \leqslant I_T. \tag{9}$$

Set

$$h(t) = \begin{cases} \dfrac{T_1}{|t|\sqrt{T_1^2+t^2}}, & |t|\geqslant T,\\ 0, & |t|<T. \end{cases} \tag{10}$$

Then

$$\sup_t h(t) = \frac{T_1}{T\sqrt{T_1^2+T^2}};$$

$$h_1(t) = \sup_{|s|\geqslant|t|} h(s) = \begin{cases} \dfrac{T_1}{|t|\sqrt{T_1^2+t^2}}, & |t|\geqslant T,\\ \dfrac{T_1}{T\sqrt{T_1^2+T^2}}, & |t|<T. \end{cases} \tag{11}$$

We need the following inversion formula.

Lemma 2 [3]. *Let $f_0(t)\geqslant 0$ be a characteristic function. Then for any measurable bounded function $h(t)$ and any $\tau>0$, the inequality*

$$\left|\int_{-\infty}^{\infty} h(t)f_0(t)\mathrm{d}t\right| \leqslant 13Q(F_0,\tau)\left(\frac{\sup_t|h(t)|}{\tau}+\int_0^{\infty}\sup_{|s|\geqslant|t|}|h(s)|\mathrm{d}t\right) \tag{12}$$

holds.

Applying Lemma 2 we get

$$\int_0^1 g_1(t)\,\mathrm{d}t = \int_0^T g_1(t)\,\mathrm{d}t + \int_T^{T_1} g_1(t)\,\mathrm{d}t + \int_{T_1}^{\infty} g_1(t)\,\mathrm{d}t$$

$$\leqslant \frac{T_1}{\sqrt{T_1^2+T^2}} + \int_T^{T_1}\frac{1}{t}\,\mathrm{d}t + \int_{T_1}^{\infty}\frac{T_1}{t^2}\,\mathrm{d}t \leqslant 2+\ln\frac{T_1}{T}. \tag{13}$$

Estimate (7) follows from 98), (9), (11), (12) and (13).

Proof of Theorem 1. As proved in [1], taking $T_1=\Delta^2, T=2\Delta$ we get

$$\sup_x\left|P(S_n\leqslant x)-\Pi(x;\lambda)-\frac{1}{2}\Gamma_2\Pi_1([x];\lambda)\right|\leqslant 1.345\,(I_T+l_{T_1}+|R|+3/T_1). \tag{14}$$

Taking into account (7), we see that it sufficies to estimate I_T. Set

$$g(t) = \exp\left\{\lambda(e^{it} - 1)\right\}\left(1 + 0.5\Gamma_2(e^{it} - 1)^2\right).$$

Let $f_S(t)$ denote the characteristic function of S_n. Proceeding as in the proof of (27) from [1], but retaining multiplier $\exp -0.2\lambda \sin^2(t/2)$ we get

$$|\, f_S(t) - g(t)\,| \leqslant C_3 \frac{\sqrt{\lambda}}{\Delta^2} |\, \sin(t/2)\,| \exp\left\{-0.2\lambda \sin^2(t/2)\right\}. \tag{15}$$

Note that, for $|\,t\,| \leqslant \pi$, $\sin^2(t/2) \geqslant t^2/\pi^2$. Consequently,

$$\begin{aligned} I_T &\leqslant C_4 \frac{\sqrt{\lambda}}{\Delta^2} \int_{-T}^{T} \frac{|\sin(t/2)|}{|\,t\,|} \exp\left\{-0.2\lambda \sin^2(t/2)\right\} \, dt \leqslant \\ &\leqslant C_5 \frac{\sqrt{\lambda}}{\Delta^2} \left(\int_0^{\pi} \exp\left\{-0.2\lambda \sin^2(t/2)\right\} \, dt + \int_{\pi}^{T} \frac{1}{t} \exp\left\{-0.2\lambda \sin^2(t/2)\right\} \lambda^{-1/2} \, dt \right) \\ &= I_1 + I_2. \end{aligned} \tag{16}$$

Obviously,

$$I_1 \leqslant C_6 \frac{\sqrt{\lambda}}{\Delta^2} \int_0^{\infty} \exp\left\{-0.2t^2/\pi^2\right\} \, dt \leqslant C_7 \frac{1}{\Delta^2}. \tag{17}$$

Applying Lemma 2 with

$$h(t) = \begin{cases} \dfrac{1}{t}, & \pi \leqslant t \leqslant T, \\ 0, & \text{elswhere.} \end{cases} \tag{18}$$

we get

$$I_2 \leqslant C_8 \frac{1}{\Delta^2} \left(1 + \ln \frac{T}{\sqrt{\lambda}}\right). \tag{19}$$

Estimates (14), (16)–(18) evidently complete the proof in the general case. For the integral non-negative ξ_j one should use (15) and Tsaregradskii's inequality.

References

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Apie aproksimavimą Puasono dėsniu

V. Čekanavičius

Pademonstruota, kaip Levy koncentracijos funkcija gali pagerinti liekamojo nario įverčius. Aproksimuojant Puasono dėsniu liekamajame naryje pašalinamas logaritminis daugiklis.