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Separable spatio-temporal covariance functions

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1. Introduction

Let $\{Z(s;t)\colon\ s \in D \in R^d;\ t \in [0,\infty]\}$ denote a spatio-temporal random process observed at N space-time coordinates $(s_1;t_1),\dots,(s_N;t_N)$. Optimal prediction (in space and time) of the unobserved parts of the process, based on the observations $Z \equiv (Z(s_1;t_1),\dots,(s_N;t_N))'$, is often the ultimate goal, but to achieve this goal, a model is needed for how various parts of the process co-vary in space and time.

In what follows, we assume that the spatio-temporal process $Z(s;t)$ satisfies the regularity condition, var $(Z(s;t)) < \infty$, for all $s \in D, t \geqslant 0$. Then we can define the mean function as

$$\mu(s;t) \equiv E(Z(s;t)),$$

and covariance function as

$$K(s,r;t,q) \equiv \text{cov}\,(Z(s;t), Z(r;q)), \quad s,r \in D, \quad t > 0, \quad q > 0.$$

Furthemore, the optimal (i.e., minimum MSPE (see, e.g., Cressie, 1993)) linear predictor of $Z(s_0;t_0)$ is

$$Z^*(s_0;t_0) = \mu(s_0;t_0) + C_0(s_0;t_0)'\Sigma^{-1}(Z-\mu), \tag{1}$$

where $\Sigma \equiv \text{cov}\,(Z)$, $C_0(s_0;t_0)' \equiv \text{cov}\,(Z(s_0;t_0))$, and $\mu \equiv E(Z)$, the MSPE is $C_0(s_0;t_0)'\Sigma^{-1}C_0(s_0;t_0)$. In the rest of this article, we assume that the covariance function is stationary in space and time, namely

$$K(s,r;t,q) = C(s-r;t-q), \tag{2}$$

for a certain function C. This assumption is often made so that the covariance function can be estimated from data. For any $(r_1;q_1),\dots,(r_m;q_m)$, any real $a_1,\dots,a_m$, and any positive integer m, C must satisfy

$$\sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j C(r_i - r_j; q_i - q_j) \geqslant 0. \tag{3}$$

To ensure positive definitness, one often specifies the covariance function C to belong to a parametric family whose members are known to be positive definite. That is, one assumes that

$$\operatorname{cov}\left(Z(s;t), Z\left(s+h_s; t+h_t\right)\right) = C^0\left(h_s; h_t|\theta\right), \tag{4}$$

where C^0 satisfies (3) for all $\theta \in \Theta \in R^p$.

Our goal in this article is to introduce new parametric families C^0 for (4) that will substantially increase the choices a modeler has for valid (i.e., positive-definite) spatio temporal stationary covariances. One commonly used class (see, e.g., Rodriguez–Iturbe and Mejia (1974)) consists of separable covariances.

Let C_s be a covariance function on R^m and C_t be a covariance function on T, then the product model for C^0 is

$$C_{st}(h_s; h_t) = C_s(h_s)C_t(h_t). \tag{5}$$

It can be re-written in terms of semivariograms as follows:

$$\gamma_{st}(h_s, h_t) = \left[C_t(0)\gamma_s(h_s) + C_s(0)\gamma_t(h_t) - \gamma_s(h_s)\gamma_t(h_t)\right]. \tag{6}$$

The "marginal" semivariograms are
1. $\gamma_{st}(h_s, 0) = C_t(0)\gamma_s(h_s)$
and
2. $\gamma_{st}(0, h_t) = C_s(0)\gamma_t(h_t)$.
$C_t(0)$ is the sill of $\gamma_t(h_t)$ and $C_s(0)$ is the sill of $\gamma_s(h_s)$.

EXAMPLE 1. $C_{st}(h_s; h_t) = \exp(-a||h_s|| - b|h_t|)$, where $\theta = (a, b)$.

EXAMPLE 2. Given the following variograms:

$$\gamma_s(h_s; b) = 1 - \frac{||h_s||^2}{1 + ||h_s||^2/b}, \quad b > 0,$$

$$\gamma_t(h_t; c, d) = 1 - \mathrm{e}^{\frac{-h_t}{c}} \cos\left(\frac{h_t}{d}\right), \quad c, d > 0,$$

that can be written

$$\gamma_{st}(h_s, h_t) = 1 - \frac{||h_s||^2}{1 + ||h_s||^2/b} \mathrm{e}^{\frac{-|h_t|}{c}} \cos\left(\frac{|h_t|}{d}\right), \quad \text{where } \theta = (b, c, d).$$

This function have a space-time Radial Basis function name.

2. Main Results

The kriging equations for space-time kriging are the same as for purely spatial problems, the difference is in the use of a space-time covariance in lieu of a purely spatial covariance. In the case of regression model of mean function

$$\mu(s;t) = E\left(Z(s;t)\right) = X_{s;t}^T \beta, \tag{7}$$

the optimal prediction is called universal kriging (see, e.g., Cressie (1993)).

Lemma. *Optimal linear prediction equation for product covariance function, defined in* (5), *is*

$$\widehat{Z}_{UK}(s_0, t_0) = x_{s_0,t_0}^T \widehat{\beta} + \delta^T \left[Z - X_{s,t} \widehat{\beta} \right], \tag{8}$$

where

$$\widehat{\beta} = \left(X_{s,t}^T \left(C_s^{-1} \otimes C_t^{-1} \right) X_{s,t} \right)^{-1} X_{s,t}^T \left(C_s^{-1} \otimes C_t^{-1} \right) Z, \tag{9}$$

$$\delta = \left(C_s^{-1} \otimes C_t^{-1} \right) \left(C_{s0} \otimes C_{t0} \right), \tag{10}$$

where $\otimes$ is the Kronecker product.

C_s and C_t is the separable covariance matrix for all observations. C_{s0} and C_{t0} is the covariance vectors of observations at the predicted point and observed points.

Proof. Expressions (9) and (10) were obtained by using (5) in universal kriging equation.

Then mean squared prediction error for the predictor, given in (8), is of the form

$$MSPE_{UK} = \delta(0) - 2b^T \left(C_{s0}(h_s) \otimes C_{t0}(h_t) \right) + b^T \left(C_s^{-1} \otimes C_t^{-1} \right) b, \tag{11}$$

where

$$\begin{aligned} b^T &= x_{s0,t0}^T \left(X_{s,t}^T (C_s^{-1} \otimes C_t^{-1}) X_{s,t} \right)^{-1} \\ &\quad + (C_s^{-1} \otimes C_t^{-1})(C_{s0} \otimes C_{t0}) \left(I - X_{s,t} \left((X_{s,t}^T (C_s^{-1} \otimes C_t^{-1}) X_{s,t}) \right)^{-1} \right) \\ &\quad \times X_{s,t}^T (C_s^{-1} \otimes C_t^{-1}), \end{aligned}$$

and $\delta(0) = C_s(0) C_t(0)$.

3. Example

In this section we apply the separable classes of spatio-temporal stationary covariance functions to the problem of prediction at the unobserved locations. The spatio-temporal data, used in this article, was collected in the Baltic sea, where the number of observations is taken in regularly time intervals (1994–1998), are considered 8 stations in the coastal zone. The observation we are working with is solinity in all the considered station.

The spatio-temporal separable covariance function we have after fitting product covariance model, defined in (5). The marginal covariance functions are

$$C^1(h_s) = 10.447 \exp\{-0.008||h_s||\}, \tag{12}$$

and

$$C^2(h_t) = 0.161 \exp\{-0.151|h_t|\}. \tag{13}$$

Then the general separable covariance model is

$$C^0(h_s, h_t|\theta) = 1.677 \exp\{-0.008||h_s|| - 0.151|h_t|\}. \tag{14}$$

Using prediction equation (8) and covariance model (5) we calculated prediction at an unobserved location. The ratio of MSPE (11) and predicted value of solinity is equal $0.0759/6.211 = 0.0122$. This result allows us to evalute procedure of covariance model fitting as sufficientity adequate to real data.

References

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Atskiriamos erdvės-laiko kovariacinės funkcijos

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Straipsnyje gautos analitinės išraiskos UK (universalaus krigingo) ir MSPE (vidutinės kvadratinės prognozės klaidos), kai erdvės-laiko kovariacinė funkcija yra atskiriama, naudojant sandaugos modelį. Paėmus realius duomenis, įvertinti erdvės ir laiko kovariacijų parametrai ir atlikta optimali prognozė laisvai pasirinktame taške.