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Structural rules and cut admissibility in a sequent calculus of temporal logic with predicates = and >

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1. Introduction

Various mathematical theories have their own axioms. Addition of non-logical axioms to a sequent calculus has usually the consequence that cut is no more admissible in in this way obtained calculus. This is undesirable as the structural derivation analysis as well as derivation search become more complex.

We take a temporal logic sequent calculus as a basis and add axioms and rules for the predicates = and >. The reason of choosing > rather than $\leqslant$ is that, in fact, we wanted to get a calculus for a theory with $\leqslant$. Note that the axioms and rule for > are antecedent orientated, and this is needed for cut elimination. (We make use of the works [6] and [3] here.) $a \leqslant b$ is the same as $\neg(a > b)$, and we are enabled to work with $\leqslant$, taking $\neg(a > b)$ for $a \leqslant b$. If we had introduced $\leqslant$, we would not be able to prove, e. g., $\rightarrow ((a \leqslant b) \vee (b \leqslant a))$ as the axioms and rules are antecedent orientated. So, in our case, introducing of > in the calculus is intended for $\leqslant$. It is interesting, however, that the calculus is suitable for the > theory as well since all axioms of this theory are derivable in it.

The works [6] and [3] deal with general principles of introducing non-logical axioms in propositional and predicate, classical and intuitionistic logic sequent calculi, not loosing admissibility of structural rules and cut. Structural rule and cut elimination from sequent calculi of theories with equality is investigated in [4], [7], [9], [10] and [11]. Specialization of proof search in sequent calculi of theories with equality can be found in [5], [8] and [10]. Eventually, semantical admissibility of cut in a first order infinitary linear temporal logic with equality is proved in [12].

The present paper is organized as follows. After introduction, we present a sequent calculus of the temporal logic with time gaps (see [2]) with the predicates = and >. Next we prove admissibility of structural rules and cut in it. We end by giving some concluding remarks.

2. Calculus $LB^{\geq}_{=}$

The sequent calculus $LB^{\geq}_{=}$ is obtained from a variant of Gentzen's sequent calculus LK (without structural rules) by 1) adding some rules for temporal operators which are taken from [2] and slightly changed by us, and 2) adding some axioms and rules for the predicates 'equality' and 'more than'.

Axioms.

1. Logical axioms: $\Gamma, E \rightarrow E, \Delta$; $\Gamma, \mathcal{F} \rightarrow \Delta$.

2. Axioms for predicate 'more than': $t > t, \Gamma \rightarrow \Delta$; $t > t_1, t_1 > t, \Gamma \rightarrow \Delta$.

3. Axiom for predicate 'equality': $\Gamma \rightarrow \Delta, \alpha = \alpha$.

Here: E denotes an atomic formula; Γ, Δ denote finite, possibly empty, multisets of formulae; $\mathcal{F}$ denotes 'false'; t and t_1 denote arbitrary terms; α denotes a free variable.

Rules.

Logical rules:

$$\frac{\Gamma, A \rightarrow B, \Delta}{\Gamma \rightarrow A \supset B, \Delta} (\rightarrow\supset), \qquad \frac{\Gamma \rightarrow A, \Delta; B, \Gamma \rightarrow \Delta}{A \supset B, \Gamma \rightarrow \Delta} (\supset\rightarrow),$$

$$\frac{\Gamma \rightarrow A, \Delta; \Gamma \rightarrow B, \Delta}{\Gamma \rightarrow A \wedge B, \Delta} (\rightarrow \wedge), \qquad \frac{A, B, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta} (\wedge \rightarrow),$$

$$\frac{\Gamma \rightarrow A, B, \Delta}{\Gamma \rightarrow A \vee B, \Delta} (\rightarrow \vee), \qquad \frac{A, \Gamma \rightarrow \Delta; B, \Gamma \rightarrow \Delta}{A \vee B, \Gamma \rightarrow \Delta} (\vee \rightarrow),$$

$$\frac{\Gamma \rightarrow A(b), \Delta}{\Gamma \rightarrow \forall x A(x), \Delta} (\rightarrow \forall), \qquad \frac{A(t), \forall x A(x), \Gamma \rightarrow \Delta}{\forall x A(x), \Gamma \rightarrow \Delta} (\forall \rightarrow),$$

$$\frac{\Gamma \rightarrow A(t), \exists x A(x), \Delta}{\Gamma \rightarrow \exists x A(x), \Delta} (\rightarrow \exists), \qquad \frac{A(b), \Gamma \rightarrow \Delta}{\exists x A(x), \Gamma \rightarrow \Delta} (\exists \rightarrow).$$

Here: A, B denote arbitrary formulae; x denotes a bound variable; t denotes a term; in the $(\forall \rightarrow)$, $(\rightarrow \exists)$ rules, $A(x)$ is obtained from $A(t)$ by substituting x for at least one occurrence of t in $A(t)$; b denotes a free variable which does not occur in conclusions of the rules $(\rightarrow \forall)$, $(\exists \rightarrow)$, and $A(x)$ in these rules is obtained by substituting x for every occurrence of b in $A(b)$. We assume that different letters are used to denote bound and free variables and that terms do not contain bound variables.

Note that we do not introduce negation. Instead, we introduce $\mathcal{F}$ for 'false', and $\neg A$ can be expressed by $A \supset \mathcal{F}$.

Temporal rules:

$$\frac{\Gamma \rightarrow \Delta}{\Pi, \circ\Gamma \rightarrow \circ\Delta, \Lambda} (\circ), \qquad \frac{\Box\Gamma \rightarrow A}{\Pi, \Box\Gamma \rightarrow \Box A, \Lambda} (\Box),$$

$$\frac{A, \circ\Box A, \Gamma \rightarrow \Delta}{\Box A, \Gamma \rightarrow \Delta} (\Box \rightarrow), \qquad \frac{\Gamma \rightarrow A, \Delta; \Gamma \rightarrow \circ\Box A, \Delta}{\Gamma \rightarrow \Box A, \Delta} (\rightarrow \Box).$$

Here: if $\Gamma = A_1, \ldots, A_n$, then $\sigma\Gamma = \sigma A_1, \ldots, \sigma A_n$, where $\sigma \in \{\Box, \circ\}$; Π, Λ denote finite, possibly empty, multisets of formulae. The rule $(\rightarrow \Box)$ corresponds to the weak induction axiom: $(A \wedge \circ\Box A) \supset \Box A$.

Rule for 'more than':

$$\frac{t > t_1, t > t', \Gamma \rightarrow \Delta; t > t_1, t' > t_1, \Gamma \rightarrow \Delta}{t > t_1, \Gamma \rightarrow \Delta} (>).$$

Here t and t_1 are arbitrary terms, and t' occurs in the conclusion.

Rules for 'equality':

$$\frac{\alpha = \beta, [\Gamma \to \Delta]^{\alpha}_{\beta}}{\alpha = \beta, \Gamma \to \Delta} (=)_1, \qquad \frac{\alpha = \beta, [\Gamma \to \Delta]^{\beta}_{\alpha}}{\alpha = \beta, \Gamma \to \Delta} (=)_2.$$

Here: α and β are free variables; in the $(=)_1$ rule, $[\Gamma \to \Delta]^{\alpha}_{\beta}$ is obtained from $\Gamma \to \Delta$ by substituting α for one occurrence of β in $\Gamma \to \Delta$ (it is assumed that β occurs in $\Gamma \to \Delta$), similarly in the $(=)_2$ rule. We introduce the restriction, that no function symbols occur in equality.

Rule for the axiom $(\neg(a > b) \wedge \neg(b > a)) \supset a = b$:

$$\frac{a > b, \Gamma \to \Delta; b > a, \Gamma \to \Delta; a = b, \Gamma \to \Delta}{\Gamma \to \Delta} (*)$$

Here a and b occur in the conclusion.

3. Structural rule and cut admissibility in $LB^{\geq}_{=}$

Theorem 3.1. *The rule of weakening is admissible in* $LB^{\geq}_{=}$.

Proof. The theorem can easily be proved by induction on derivation height.

Let Σ be a sequent, multiset or a formula. $(\Sigma)^{\beta}_{\alpha}$ is obtain from Σ by substituting β for every occurrence of α in Σ.

Lemma 3.2. *Let $\Sigma(a)$ be a sequent with one or more occurrences of a free variable a in it. If $LB^{\geq}_{=} \vdash^{V} \Sigma(a)$, then there exists a derivation V^* such that $LB^{\geq}_{=} \vdash^{V^*} \Sigma(b)$ and $h(V^*) \leqslant h(V)$, here b is a free variable, and $\Sigma(b)$ is obtained from $\Sigma(a)$ by substituting b for every occurrence of a in $\Sigma(a)$.*

Proof. The lemma is proved by induction on derivation height. The base case is obvious. The inductive case:

$$\frac{a = c, [\Gamma \to \Delta]^{a}_{c}}{a = c, \Gamma \to \Delta} (=)_1.$$

Applying the inductive hypothesis to the premise, we get $b = c, ([\Gamma \to \Delta]^{a}_{c})^{b}_{a}$. It remains to apply $(=)_1$ to this sequent so that to get the required one.

As the proof is easy, we do not consider the other cases. See also [1].

Lemma 3.3. *If $LBJ^{\geq}_{=} \vdash^{V} \alpha = \alpha, \Gamma \to \Delta$, then there exists a derivation V^* such that $LBJ^{=} \vdash^{V^*} \Gamma \to \Delta$ and $h(V^*) \leqslant h(V)$.*

Proof. The lemma can easily be proved by induction on derivation height.

Theorem 3.4. *The structural rule of contraction is admissible in* $LB^{\geq}_{=}$.

Proof. See [1] and for more information on the proof. In the present work, we consider only a case which is not considered in the referred work. We denote a formula C with a particular occurrence of a free variable a in it by $C[a]$.

$$\frac{a = b, C[a], C[b], \Gamma \rightarrow \Delta}{a = b, C[b], C[b], \Gamma \rightarrow \Delta}\ (=)_1.$$

Applying Lemma 3.2 to the premise, we get $b = b, (C[b], C[b], \Gamma \rightarrow \Delta)^b_a$. (Note that we do not loose the possibility to apply the inductive hypothesis.) Applying the inductive hypothesis and Lemma 3.3 to this sequent, we get $(C[b], \Gamma \rightarrow \Delta)^b_a$. Applying Theorem 3.1 to this sequent, we get $a = b, (C[b], \Gamma \rightarrow \Delta)^b_a$. Now it suffices to apply $n \geqslant 0$ times the rule $(=)_2$ to this sequent in order to get the required one.

Theorem 3.5. *Cut is admissible in $LB^{\geq}_{=}$.*

Proof. See [1] and [6] for more information on the proof. In the present work, we consider only a case which is not considered in the referred works.

$$\frac{\dfrac{a = b, \Gamma \rightarrow b = c}{a = b, \Gamma \rightarrow a = c}\ (=)_2; \quad \dfrac{a = c, \Pi \rightarrow P(a)}{a = c, \Pi \rightarrow P(c)}\ (=)_1}{a = b, \Gamma, \Pi \rightarrow P(c)}\ (cut).$$

From $a = b, \Gamma \rightarrow b = c$, using Lemma 3.2, we get $a = a, (\Gamma)^a_b \rightarrow a = c$; using Lemma 3.3, we get $S : (\Gamma)^a_b \rightarrow a = c$; applying the inductive hypothesis to S and $a = c, \Pi \rightarrow P(c)$, we get $(\Gamma)^a_b, \Pi \rightarrow P(c)$; applying weakening to this sequent, we get $a = b, (\Gamma)^a_b, \Pi \rightarrow P(c)$; applying $(=)_1$ to this sequent $n \geqslant 0$ times we get the required one.

4. Concluding remarks

We introduced the restriction that only free variables can occur in equality. Elimination of contraction and cut become more complex if function symbols are present in equality. Contraction would not be admissible in a calculus, let us denote it by $LB_{=>}$, obtained from $LB^{\geq}_{=}$ by allowing function symbols to occur in equality:

$$\frac{g(b) = f(g(b)), \circ P(g(b)), \circ P(g(b)) \rightarrow \circ(\exists x P(f(g(x))) \wedge \exists x P(g(x)))}{g(b) = f(g(b)), \circ P(g(b)), \rightarrow \circ(\exists x P(f(g(x))) \wedge \exists x P(g(x)))}\ (Contr).$$

Note that the premise is derivable in $LB_{=>}$, but the conclusion is not. Cut is not admissible in $LB_{=>}$, either:

$$\frac{a = f(b), \circ\forall x C(f(x)) \rightarrow \circ C(a); \quad \circ C(a), a = g(b) \rightarrow \circ\exists x C(g(x))}{a = g(b), a = f(b), \circ\forall x C(f(x)) \rightarrow \circ\exists x C(g(x))}\ (cut).$$

Note that the both premises are derivable in $LB_{=>}$, but the conclusion is not.

Let $LB^{=>}$ be the calculus obtained from $LB_{=>}$ by introducing the following change: in the $(=)_1$ rule, $[\Gamma \rightarrow \Delta]^{\alpha}_{\beta}$ is obtained from $\Gamma \rightarrow \Delta$ by substituting α for *every* occurrence of β in $\Gamma \rightarrow \Delta$, similarly in the $(=)_2$ rule. Though contraction is admissible in $LB^{=>}$, cut is still not. Impossibility to eliminate cut from $LB_{=>}$ and $LB^{=>}$ is caused by the rules $(\circ)$ and $(\Box)$, application of which affects all formulas of the premises of these rules. This makes it that not every derivation can be transformed into a regular one (in a regular derivation, above any application of an equality rule, there are no applications of the rule $(\forall \rightarrow)$ or $(\rightarrow \exists)$, see also [4]). The possibility to have a regular derivation of every derivable sequent is, however, crucial for cut elimination from sequent calculi with equality (when function symbols are allowed to occur in it).

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Struktūrinių taisyklių bei pjūvio leistinumas laiko logikos su predikatais = ir > sekvenciniame skaičiavime

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Darbe pateikiamas laiko logikos su laiko tarpsniais bei predikatais „lygu" ir „daugiau nei" sekvencinis skaičiavimas ir nagrinėjama struktūrinių bei pjūvio taisyklių leistinumo šiame skaičiavime problema. Nurodomos priežastys, kodėl pjūvio taisyklė ne visada yra leistina tokio tipo skaičiavimuose.