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On weak solutions of Stratonovich integral equation driven by a continuous p-semimartingales

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Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P}), \mathbb{F} = \{\mathcal{F}_t, t \geqslant 0\}$, be a stochastic basis satisfying the usual conditions.

Definition 1. (see [7]). *For $p \in [1,2)$, an $\mathbb{F} = \{\mathcal{F}_t, t \geqslant 0\}$ adapted cadlag stochastic process Z is called a p-semimartingale if there exist stochastic processes M and A such that*

$$Z - Z(0) = M + A \qquad \text{almost surely,}$$

where $M(0) = A(0) = 0$, M is an $\mathbb{F}$ local martingale and A is an $\mathbb{F}$-adapted process with locally bounded p-variation, i.e., for any fixed $T > 0$, the process $A = \{A_t, 0 \leqslant t \leqslant T\}$ has bounded p-variation.

Let Y, Z be two p-semimartingales with continuous trajectories and let $Z = M + A$, where its summands are continuous processes. Then the Stratonovich integral $(S)\int Y(s)\,dZ(s)$ is defined by the formula

$$(S)\int_0^t Y(s)\,dZ(s) := \int_0^t Y(s)\,dZ(s) + \frac{1}{2}\,[Y,Z](t), \qquad t \geqslant 0.$$

The first integral we understand as a sum of two integrals

$$(SI)\int_0^t f(Y_s)\,dM_s \qquad \text{and} \qquad (RS)\int_0^t f(Y_s)\,dA_s,$$

where the symbol SI denotes the usual stochastic integral and the symbol RS denotes the Riemann–Stieltjes integral.

Consider the equation

$$X_t = \xi + (S)\int_0^t f(X_s)\,dZ_s, \qquad t \geqslant 0,$$

or equivalent equation

$$X_t = \xi + \int_0^t f(X_s)\,dZ_s + \frac{1}{2}\int_0^t ff'(X_s)\,d[M]_s, \qquad t \geqslant 0. \tag{1}$$

For short, we shall write $ff'(X_s)$ instead of $f(X_s)f'(X_s)$.

The purpose of this paper is to find conditions when the weak solution of the equation (1) exists.

Definition 2. *We say that the equation* (1) *has a weak solution if there exists a stochastic basis* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, \widetilde{\mathbf{P}})$ *and an* $\widetilde{\mathbb{F}}$ *adapted processes* $\widetilde{X}$, $\tilde{\xi}$, $\widetilde{Z}$, $\widetilde{[M]}$ *such that* $\mathcal{L}(\tilde{\xi}, \widetilde{Z}, \widetilde{[M]}) = \mathcal{L}(\xi, Z, [M])$ *and* (1) *holds for* $\widetilde{X}$, $\tilde{\xi}$, $\widetilde{Z}$, $\widetilde{[M]}$ *in place of* X, ξ, Z, $[M]$.

For $0 < \alpha \leqslant 1$, $C^\alpha(\mathbb{R})$ is the space of bounded Hölder functions g with the norm

$$\|g\|_\alpha := |g|_\infty + |g|_\alpha = \sup_x |g(x)| + \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x-y|^\alpha} < \infty.$$

The main result of this paper is the following theorem.

Theorem 3. *Let* $f \in C^1(\mathbb{R})$ *and* $\mathbf{E}\sup_{t\leqslant T}|M_t| < \infty$ *for every* $T > 0$. *Then there exists a weak solution of equation* (1).

1. Basic notions and auxiliary results

All facts mentioned below on the p-variation are taken from [1]

The p-variation, $0 < p < \infty$, of a real-valued function f on $[a, b]$ is defined as

$$v_p(f; [a,b]) = \sup_{\varkappa} \sum_{k=1}^{n} \left|f(x_k) - f(x_{k-1})\right|^p,$$

where the supremum is taken over all subdivisions $\varkappa = \{x_i\colon\ i = 0, \ldots, n\}$ of $[a, b]$ such that $a = x_0 < x_1 < \ldots < x_n = b$. If $v_p(f; [a,b]) < \infty$, f is said to have bounded p-variation on $[a, b]$. Let

$$\mathcal{W}_p([a,b]) := \{f\colon\ [a,b] \to \mathbb{R}\colon\ v_p(f; [a,b]) < \infty\}.$$

Define $V_p(f) := V_p(f; [a,b]) = v_p^{1/p}(f)$, which is a seminorm on $\mathcal{W}_p([a,b])$ provided $p \geqslant 1$ and $V_p(f)$ is 0 if and only if f is a constant.

Let $f \in \mathcal{W}_q([a,b])$ and $h \in \mathcal{W}_p([a,b])$ with $1 \leqslant p < \infty$, $q > 0$, $1/p + 1/q > 1$. If f and h have no common discontinuities then the RS integral $\int_a^b f\,dh$ exists and the Love–Young inequality

$$\left| \int_a^b f\,dh - f(y)\big[h(b) - h(a)\big] \right| \leqslant C_{p,q} V_q(f;[a,b]) V_p(h;[a,b]), \tag{2}$$

holds for any $y \in [a,b]$, where $C_{p,q} = \zeta(p^{-1} + q^{-1})$, $\zeta(s) = \sum_{n \geqslant 1} n^{-s}$. If, moreover, the function h is continuous, i.e., $h \in C\mathcal{W}_p([a,b])$, then the indefinite integral $\int_a^y f\,dh$, $y \in [a,b]$, is a continuous function.

Let τ and σ be a stopping times such that $\sigma < \tau \leqslant T$. Define $v_p(Y;[\sigma,\tau]) := v_p(Y^\tau - Y^\sigma;[0,T])$, where $Y^\tau = \{Y_{t\wedge\tau},\ t \geqslant 0\}$.

Any local martingale is locally of bounded q-variation for each $q > 2$ (see [6] and [8]). Moreover, for $q > 2$ and $1 \leqslant r < \infty$ there is a finite constants $K_{q,r}$ such that for every r-integrable martingale $M = \{M(t), 0 \leqslant t \leqslant T\}$, $T > 0$,

$$\mathbf{E}\big\{V_q(M;[0,T])\big\}^r \leqslant K_{q,r} \mathbf{E}\Big\{\sup_{0 \leqslant t \leqslant T} |M(t)|\Big\}^r.$$

Moreover, if M is a continuous martingale then by the Burkholder–Davis–Gundy inequality we get

$$\mathbf{E}\big\{V_q(M;[0,T])\big\}^r \leqslant K_{q,r} \ell_r \mathbf{E}\langle M\rangle_T^{r/2},$$

where ℓ_r is the constant from the Burkholder–Davis–Gundy inequality.

2. Proofs

Let $\varkappa^i = \{t_k^i\colon k \geqslant 0\}$ be a sequence of partitions of $[0,\infty)$, i.e., $0 = t_0^i < t_1^i < t_2^i < \cdots$, $\lim_{k\to\infty} t_k^i = \infty$, such that for every $T > 0$ we have $\max_{k \leqslant r^i(T)} |t_{k+1}^i - t_k^i| \to 0$ as $i \to +\infty$, where $r^i(T) = \max\{k\colon t_k^i \leqslant T\}$. For every $x \in D(\mathbb{R})$ and $\varkappa^i$ the sequence $\{x^{\varkappa^i}\}$ denotes the following discretizations of x:

$$x_t^{\varkappa^i} = x(t_k^i) \qquad \text{for } t \in [t_k^i, t_{k+1}^i),\ k \in \mathbb{N} \cup \{0\},\ i \in \mathbb{N}.$$

Define the approximations

$$X_t^n = \xi + \int_0^t f(X_{s-}^n)\,dZ_s^{\varkappa^n} + \frac{1}{2}\int_0^t ff'(X_{s-}^n)\,d[Z^{\varkappa^n}]_s, \quad t \geqslant 0,\ n \in \mathbb{N},$$

and

$$\widehat{X}_t^n = \xi + \int_0^t f(\widehat{X}_{s-}^{n,\varkappa^n})dZ_s + \frac{1}{2}\int_0^t ff'(\widehat{X}_{s-}^{n,\varkappa^n})d\langle M\rangle_s, \quad t \geqslant 0,\ n \in \mathbb{N}.$$

Lemma 4. *Let $f \in C^1(\mathbb{R})$ and let $q > 2$ be such that $\frac{1}{q} + \frac{1}{p} > 1$, where $1 \leqslant p < 2$. Then the sequence $\{V_q(\widehat{X}^n; [0,T])\}$ is tight in $\mathbb{R}$ for every $T > 0$ and the sequence $\{\widehat{X}^n\}$ is C-tight.*

Proof. Denote

$$\gamma_N = \inf\{t > 0\colon\ \langle M\rangle_t > N, V_p(A; [0,t]) > N\},$$

and

$$\widehat{X}_t^{n,N} = \xi + \int_0^t f(\widehat{X}_{s-}^{n,N,\varkappa^n})dZ_s^N + \frac{1}{2}\int_0^t ff'(\widehat{X}_{s-}^{n,N,\varkappa^n})d\langle M\rangle_s^N, \quad t \geqslant 0,\ n \in \mathbb{N},$$

where $Y^{n,N}(t) = Y^n(t \wedge \gamma_N)$.

Similarly as in [4] Lemma 1 one can get

$$\begin{aligned}\mathbf{E}V_q(\widehat{X}^{n,N}; [0,T]) &\leqslant \frac{1}{1-\alpha}\Big\{K_{q,1}\ell_1|f|_\infty \mathbf{E}\sqrt{\langle M\rangle_T^N} + C_{p,q/\alpha}|f|_\infty \mathbf{E}V_p(A^N;[0,T]) \\ &\quad + |f|_\infty|f'|_\infty \mathbf{E}\langle M\rangle_T^N\Big\} + \mathbf{E}\big\{C_{p,q/\alpha}|f|_\alpha V_p(A^N;[0,T])\big\}^{1/(1-\alpha)} \\ &\leqslant \frac{1}{1-\alpha}\Big\{K_{q,1}\ell_1|f|_\infty\sqrt{N} + C_{p,q/\alpha}|f|_\infty N \\ &\quad + |f|_\infty|f'|_\infty N + \big\{C_{p,q/\alpha}|f|_\alpha N\big\}^{1/(1-\alpha)}.\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{P}\big(V_q(\widehat{X}^n; [0,T]) > K\big) &\leqslant \mathbf{P}(\gamma_N < T) + K^{-1}\mathbf{E}V_p(\widehat{X}^{n,N}; [0,T]) \\ &\leqslant \mathbf{P}(\langle M\rangle_T > N, V_p(A;[0,T]) > N) + K^{-1}\mathbf{E}V_q(\widehat{X}^{n,N}; [0,T]),\end{aligned}$$

and the sequence $V_q(\widehat{X}^n; [0,T])$ is tight in $\mathbb{R}$.

Now we prove the tightness of $\{\widehat{X}^n\}$. We use the well known Aldous criterion. Let τ^n, $n \geqslant 1$, be stoping times such that $\tau^n \leqslant T$. Then by inequality (2)

$$\begin{aligned}\sup_{t\leqslant\delta}|\widehat{X}_{\tau^n+t}^n - \widehat{X}_{\tau^n}^n| &\leqslant \sup_{t\leqslant\delta}\Big|\int_{\tau^n}^{\tau^n+t} f(\widehat{X}_{s-}^{n,\varkappa^n})dM_s\Big| \\ &\quad + C_{p,q}\{|f'|_\infty V_q(\widehat{X}^n; [0,T]) + |f|_\infty\}V_p(A; [\tau^n, \tau^n+\delta]) \\ &\quad + |f|_\infty|f'|_\infty(\langle M\rangle_{\tau^n+\delta} - \langle M\rangle_{\tau^n}).\end{aligned}$$

By the Lenglart–Rebolledo inequality for every ε, η we have

$$\mathbf{P}\left(\sup_{t\leqslant\delta}\left|\int_{\tau^n}^{\tau^n+t} f(\widehat{X}^{n,\varkappa^n}_{s-})dM_s\right| > \varepsilon\right)$$

$$\leqslant \varepsilon^{-2}\mathbf{E}\left\{\left(\int_{\tau^n}^{\tau^n+\delta} f^2(\widehat{X}^{n,\varkappa^n}_{s-})d\langle M\rangle_s\right)\wedge\eta\right\}+\mathbf{P}\left(\int_{\tau^n}^{\tau^n+\delta} f^2(\widehat{X}^{n,\varkappa^n}_{s-})d\langle M\rangle_s\geqslant\eta\right)$$

$$\leqslant \varepsilon^{-2}\mathbf{E}\left\{|f|^2_\infty\left(\langle M\rangle_{\tau^n+\delta}-\langle M\rangle_{\tau^n}\right)\wedge\eta\right\}+\mathbf{P}\left(|f|^2_\infty\left(\langle M\rangle_{\tau^n+\delta}-\langle M\rangle_{\tau^n}\right)\geqslant\eta\right).$$

Thus we get the tightness of $\{\widehat{X}^n\}$.

Lemma 5. *Let $f \in C^1(\mathbb{R})$ and let $q > 2$ be such that $\frac{1}{q}+\frac{1}{p} > 1$, where $1 \leqslant p < 2$. Then the sequence $\{V_q(X^n;[0,T])\}$ is tight in $\mathbb{R}$ for every $T > 0$ and the sequence $\{X^n\}$ is tight in $D([0,\infty))$.*

Proof. The proof of the tightness of the sequence $\{V_q(X^n;[0,T])\}$ is similar as in previous lemma.

Since $X^n(t^n_i) = \widehat{X}^n(t^n_i)$ for all $i \geqslant 0$ then for every $T > 0$ we get

$$\sup_{t\leqslant T}|X^n_t - \widehat{X}^n_t|$$

$$\leqslant |f|_\infty \sup_{t\leqslant T}|Z(t) - Z^{\varkappa^n}(t)|$$

$$+|f|_\infty|f'|_\infty \sum_{i=1}^{r^n(T)} |M(t^n_i) - M(t^n_{i-1})| \cdot |A(t^n_i) - A(t^n_{i-1})|$$

$$+|f|_\infty|f'|_\infty \sum_{i=1}^{r^n(T)} |A(t^n_i)-A(t^n_{i-1})|^2+|f|_\infty|f'|_\infty \sup_{t\leqslant T}|\langle M\rangle(t)-\langle M\rangle^{\varkappa^n}(t)|$$

$$+\sup_{t\leqslant T}\left|\sum_{i=1}^{r^n(t)} ff'(\widehat{X}^n(t^n_{i-1}))\Big[(M(t^n_i)-M(t^n_{i-1}))^2-(\langle M\rangle(t^n_i)-\langle M\rangle(t^n_{i-1}))\Big]\right|.$$

Therefore $\sup_{t\leqslant T}|X^n_t - \widehat{X}^n_t| \xrightarrow{\mathbf{P}} 0$, as $n \to \infty$. By Lemma ?? we have that the sequence $\{\widehat{X}^n\}$ is tight. Thus by Lemma 3.31 in Section 6 in [2] we obtain that the sequence $\{X^n\}$ is tight.

Proof of Theorem 1. Define $M^n = M^{\varkappa^n}$ and $A^n = A^{\varkappa^n}$. The process $(M^n, \mathbb{F}^n)$ is a martingale, where $\mathbb{F}^n = (\mathcal{F}_{\rho^n(t)})$, $\rho^n(t) = \max\{t^n_k\colon t^n_k \leqslant t\}$. The process A^n has locally bounded p-variation since $V_p(A^n;[0,T]) \leqslant V_p(A;[0,T])$. By this inequality it follows that the sequence $\{v_p(A^n;[0,T])\}$ is tight in $\mathbb{R}$. Note that $M^n \to M$ a.s. and $A^n \to A$ a.s. in $C([0,T])$. Moreover,

$$\sup_{t\leqslant T}|[M^n]_t - \langle M\rangle_t| \xrightarrow{\mathbf{P}} 0, \qquad n \to \infty,$$

where

$$[M^n]_t = \sum_{k=1}^{r^n(t)} \left(M(t_k^n) - M(t_{k-1}^n)\right)^2.$$

By Lemma 5, by Corollary 3.33 in Section 6 in [2], and facts obtained above it follows that the sequence $\{(X^n, M^n, A^n, [M^n], \xi)\}$ is C-tight. Thus from every subsequence $\{n'\} \subset \{n\}$ we can choose a further subsequence $\{n''\}$ such that

$$(X^{n''}, M^{n''}, A^{n''}, [M^{n''}], \xi) \xrightarrow{D} (\widetilde{X}, \widetilde{M}, \widetilde{A}, [\widetilde{M}], \tilde{\xi}),$$

as $n'' \to \infty$, where $(\widetilde{X}, \widetilde{M}, \widetilde{A}, [\widetilde{M}], \tilde{\xi})$ is defined on some probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}})$ and $\mathcal{L}(\tilde{\xi}, \widetilde{M}, [\widetilde{M}], \widetilde{A}) = \mathcal{L}(\xi, M, [M], A)$. Since

$$\sup_{t \leqslant T} \left| [Z^{\varkappa^{n''}}]_t - [M^{n''}]_t \right| \xrightarrow{\mathbf{P}} 0,$$

as $n'' \to \infty$, and functions f and ff' are continuous, then by the continuous mapping theorem

$$\begin{aligned}&(X^{n''}, f(X^{n''}), ff'(X^{n''}), M^{n''}, A^{n''}, [Z^{\varkappa^{n''}}], \xi)\\ &\quad \xrightarrow{D} (\widetilde{X}, f(\widetilde{X}), ff'(\widetilde{X}), \widetilde{M}, \widetilde{A}, [\widetilde{M}], \tilde{\xi}).\end{aligned}$$

By Lemma 5, we get the tightness of the sequence $\{v_q(f(X^n); [0,T])\}$, $q > 2$, $T > 0$, in $\mathbb{R}$. Note that

$$\sup_n \mathbf{E} \sup_{t \leqslant T} |\Delta M_t^n| \leqslant 2\mathbf{E} \sup_{t \leqslant T} |M(t)|.$$

Thus conditions of Lemma 3 in [5] are satisfied and

$$\begin{aligned}&\left(X^{n''}, \int_0^{\cdot} f(X_{s-}^{n''}) dM_s^{n''}, \int_0^{\cdot} f(X_{s-}^{n''}) dA_s^{n''}, \int_0^{\cdot} ff'(X_{s-}^{n''}) d[Z^{\varkappa^{n''}}]_s, \xi\right)\\ &\quad \xrightarrow{D} \left(\widetilde{X}, \int_0^{\cdot} f(\widetilde{X}_s)\, d\widetilde{M}_s, \int_0^{\cdot} f(\widetilde{X}_s)\, d\widetilde{A}_s, \int_0^{\cdot} ff'(\widetilde{X}_s)\, d[\widetilde{M}]_s, \tilde{\xi}\right).\end{aligned}$$

Therefore

$$\begin{aligned}&\sup_{t \leqslant T} \left| X_t^{n''} - \xi - \int_0^t f(X_{s-}^{n''}) dZ_s^{\varkappa^{n''}} - \frac{1}{2} \int_0^t ff'(X_{s-}^{n''}) d[Z^{\varkappa^{n''}}]_s \right|\\ &\quad \xrightarrow{D} \sup_{t \leqslant T} \left| \widetilde{X}_t - \tilde{\xi} - \int_0^t f(\widetilde{X}_s)\, d\widetilde{Z}_s - \frac{1}{2} \int_0^t ff'(\widetilde{X}_s)\, d[\widetilde{M}]_s \right|.\end{aligned}$$

As a consequence

$$\widetilde{X}_t = \tilde{\xi} + \int_0^t f(\widetilde{X}_s)d\widetilde{Z}_s + \frac{1}{2}\int_0^t ff'(\widetilde{X}_s)\,d[\widetilde{M}]_s, \qquad t \leqslant T.$$

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Apie silpnus Stratanovičiaus integralinės lygties sprendinius, valdomus tolydžiu p-semimartingalu

K. Kubilius

Nagrinėjamas silpno Stratanovičiaus integralinės lygties sprendinio, valdomo tolydaus p-semimartingalo, egzistavimas.