Liet. matem. rink., T. 43, spec. nr., 84 - 88
2003 Matematikos ir informatikos institutas


# The local behaviour of some additive functions

Jonas ŠIAULYS, Gediminas STEPANAUSKAS
*e-mail:* jonas.siaulys@mif.vu.lt, gediminas.stepanauskas@maf.vu.lt

## Introduction

Let $f_x,\ x \geqslant 3$, be the set of strongly additive integer-valued functions, and $\nu_x(n \in A_x)$ be the frequency of natural numbers not exceeding $x$ and satisfying the condition $A_x$. In this paper we will consider the local limiting behaviour of

$$\nu_x(f_x(n) = k), \tag{1}$$

when $x$ tends to infinity. If

$$\lim_{x\to\infty} \sum_{\substack{p\leqslant x\\ f_x(p)\neq 0}} \frac{1}{p} = \infty, \tag{2}$$

then the quantity (1) is usually approximating by the frequency of standard normal distribution, because the condition (2) guarantee virtually the vanishing of the rest in such approximation (see [1], [2], [3]). We give the typical example.

**Theorem 1** ([3]). *Suppose $f$ is a strongly additive integer-valued function,*

$$A_x = \sum_{p\leqslant x} \frac{f(p)}{p}, \quad B_x = \Bigg(\sum_{p\leqslant x} \frac{f^2(p)}{p}\Bigg)^{1/2},$$
$$\delta_x = \frac{\max_{p\leqslant x} |f(p)|}{B_x}, \quad \max_{p\in\tilde{\mathbb{P}}} |f(p)| = M,$$

*where $\tilde{\mathbb{P}}$ is some subset of primes for which*

$$\sum_{p\notin\tilde{\mathbb{P}}} \frac{1}{p} \ll 1.$$

*Suppose further that*

$$g.c.d.\Bigg\{b : \sum_{f(p)=b} \frac{1}{p} = \infty\Bigg\} = 1.$$

*Then there exists the positive constant* $c_1$ *such that*

$$\sup_{k\in\mathbb{Z}}\left|B_x\nu_x(f(n)=k)-\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(k-A_x)^2}{2B_x^2}\right\}\right| \ll \delta_x+\sum_{\substack{q\leqslant M\\ q\ \text{prime}}}\exp\left\{-c_1\sum_{\substack{p\leqslant x\\ f(p)\not\equiv 0 \bmod q}}\frac{1}{p}\right\}.$$

If the condition (2) is satisfied, then the approximation of the quantity (1) by the frequency of the normal distribution has no sense, because the rest in such approximation does not tend to zero. In similar cases we have to look for another distributions for the main term of (1). It needs to take the distribution to which the distribution functions $F_x(u)=\nu_x(f_x(n)<u)$ converge weakly. The set of the limiting distributions $F_x(u)$ is rich enough even in the most simple cases. We will consider here the approximation of the quantity (1) by the Poisson distribution $\Pi_\lambda$ in the case when $f_x(p)\in\{0,1\}$ for all primes $p$. The following theorem on the weakly convergence is true.

**Theorem 2** ([4]). *Suppose* $f_x,\ x\geqslant 3$, *is a set of strongly additive functions,* $f_x(p)\in\{0,1\}$ *for all primes p. Then*

$$\nu_x(f_x(n)<u)\Rightarrow\Pi_\lambda(u)$$

*if and only if*

$$\lim_{x\to\infty}\max_{\substack{p\leqslant x\\ f_x(p)=1}}\frac{1}{p}=0,\quad \lim_{x\to\infty}\sum_{\substack{p\leqslant x\\ f_x(p)=1}}\frac{1}{p}=\lambda,\quad \lim_{x\to\infty}\frac{1}{\ln x}\sum_{\substack{p\leqslant x\\ f_x(p)=1}}\frac{\ln p}{p}=0. \tag{3}$$

So, if the conditions (3) are satisfied, the equality

$$\lim_{x\to\infty}\nu_x(f_x(n)=k)=\frac{\lambda^k}{k!}\,\mathrm{e}^{-\lambda}$$

holds for every fixed $k$. In the following theorem we will estimate the rate of convergence in the last equality.

**Theorem 3.** *Suppose* $f_x, x\geqslant 3$, *is a set of strongly additive functions,* $f_x(p)\in\{0,1\}$ *for all primes p. Then*

$$\rho_1(\nu_x,\Pi_\lambda) := \sup_{k\in\mathbb{N}_0}\left|\nu_x(f_x(n)=k)-\frac{\lambda^k}{k!}\,e^{-\lambda}\right| \ll \min\left\{1, e^{2(\lambda+\Delta_x)}\left(\mu_x+\frac{1}{\ln x}\right)+\theta_x\lambda+\Delta_x\right\}, \tag{4}$$

$$\rho_2(\nu_x,\Pi_\lambda) := \left(\sum_{k=0}^{\infty}\left|\nu_x(f_x(n)=k)-\frac{\lambda^k}{k!}\,e^{-\lambda}\right|^2\right)^{1/2}$$

$$\ll \min\left\{1, e^{2(\lambda+\Delta_x)}\left(\mu_x+\frac{1}{\ln x}\right)+\theta_x\lambda+\Delta_x\right\}, \tag{5}$$

*where*

$$\theta_x = \max_{\substack{p\leqslant x\\ f_x(p)=1}} \frac{1}{p}, \quad \mu_x = \frac{1}{\ln x}\sum_{\substack{p\leqslant x\\ f_x(p)=1}} \frac{\ln p}{p}, \quad \Delta_x = \left|\lambda - \sum_{\substack{p\leqslant x\\ f_x(p)=1}} \frac{1}{p}\right|.$$

It can be easily seen that if the conditions of Theorem 2 are fulfilled then the values of $\rho_1(\nu_x, \Pi_\lambda)$ and $\rho_2(\nu_x, \Pi_\lambda)$ vanish, as $x$ tends to infinity.

## The proof of Theorem 3

The proof is based on one lemma about the distance between the distribution of the set of the additive integer-valued functions and the Poisson distribution.

**Lemma** ([5]). *Suppose $f_x$ is the set of additive functions, $f_x(p) \in \mathbb{N}_0$ for all primes $p$, $a_x \in \mathbb{N}_0$, $x \geqslant 3$. Then for $i = 1, 2$*

$$\rho_i(\nu_x, \Pi_x) \ll \min\left\{1, \beta_x\left(\gamma_x + \frac{1}{\ln x}\right) + \left(E_x + a_x^2\right)\min\left\{1, \frac{1}{\lambda_x}\right\}\right\},$$

*where*

$$\lambda_x = \sum_{p\leqslant x} F_{p1}, \qquad E_x = \sum_{p\leqslant x}\left(F_{p1}^2 + F_{p2}\right),$$

$$\beta_x = \exp\left\{2\sum_{\substack{p\leqslant x\\ f_x(p)\neq a_x}} \frac{1}{p}\right\}, \qquad \gamma_x = \frac{1}{\ln x}\sum_{\substack{p\leqslant x\\ f_x(p)\neq a_x}} \frac{\ln p}{p},$$

$$F_{p1} = \sum_{r=1}^{\gamma_p} f_x(p^r)\pi(p^r), \qquad F_{p2} = \sum_{r=1}^{\gamma_p} f_x(p^r)(f_x(p^r)-1)\pi(p^r),$$

$$\pi(p^r) = \begin{cases} \frac{1}{p^r}\left(1-\frac{1}{p}\right) & \text{if } r = 0, 1, \ldots, \gamma_p - 1,\\ \frac{1}{p^r} & \text{if } r = \gamma_p, \end{cases} \qquad \gamma_p = \left[\frac{\ln x}{\ln p}\right].$$

Let $a_x = 0$. It is evident that

$$F_{p1} = \frac{f_x(p)}{p} \quad \text{for } p \leqslant x, \quad F_{p2} = 0,$$

$$\lambda_x = \sum_{\substack{p\leqslant x\\ f_x(p)=1}} \frac{1}{p}, \quad E_x = \sum_{\substack{p\leqslant x\\ f_x(p)=1}} \frac{1}{p^2} \leqslant \theta_x\lambda_x,$$

$$\beta_x = \exp\left\{2\sum_{\substack{p\leqslant x\\ f_x(p)=1}} \frac{1}{p}\right\} = \mathrm{e}^{2\lambda_x}, \quad \gamma_x = \frac{1}{\ln x}\sum_{\substack{p\leqslant x\\ f_x(p)=1}} \frac{\ln p}{p} = \mu_x.$$

Hence from lemma we get

$$\rho_1(\nu_x, \Pi_{\lambda_x}) \ll \min\left\{1, \mathrm{e}^{2(\lambda+\Delta_x)}\left(\mu_x + \frac{1}{\ln x}\right) + \theta_x\lambda + \theta_x\Delta_x\right\}.$$

Since $|\mathrm{e}^z - 1| \leqslant |z|$, if $z \in \mathbb{C}$, $\Re z \leqslant 0$, and

$$\frac{\lambda^k}{k!}\,\mathrm{e}^{-\lambda} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{itk}\mathrm{e}^{\lambda(\mathrm{e}^{it}-1)}\,\mathrm{d}t, \quad k = 0, 1, 2, \ldots,$$

in the case $\lambda \geqslant \lambda_x$ we have

$$\begin{aligned}\rho_1(\Pi_{\lambda_x}, \Pi_\lambda) &\ll \int_{-\pi}^{\pi} \left|\mathrm{e}^{\lambda_x(\mathrm{e}^{it}-1)} - \mathrm{e}^{\lambda(\mathrm{e}^{it}-1)}\right|\,\mathrm{d}t \\ &= \int_{-\pi}^{\pi} \left|\mathrm{e}^{(\lambda-\lambda_x)(\mathrm{e}^{it}-1)} - 1\right|\,\mathrm{d}t \leqslant \int_{-\pi}^{\pi} |\lambda - \lambda_x||\mathrm{e}^{it} - 1|\,\mathrm{d}t \ll \Delta_x.\end{aligned}$$

If $\lambda < \lambda_x$, similarly

$$\rho_1(\Pi_{\lambda_x}, \Pi_\lambda) \ll \int_{-\pi}^{\pi} \left|\mathrm{e}^{(\lambda-\lambda_x)(\mathrm{e}^{it}-1)} - 1\right|\,\mathrm{d}t \ll \Delta_x.$$

The relation (4) follows from the obtained estimates and the triangle inequality for the distance $\rho_1$.

It follows from lemma also that

$$\rho_2(\nu_x, \Pi_{\lambda_x}) \ll \min\left\{1, \mathrm{e}^{2(\lambda+\Delta_x)}\left(\mu_x + \frac{1}{\ln x}\right) + \theta_x\lambda + \theta_x\Delta_x\right\}.$$

From the inequality $|\mathrm{e}^z - 1| \leqslant |z|$, $z \in \mathbb{C}$, $\Re z \leqslant 0$, and the Parseval equality

$$\rho_2(\Pi_{\lambda_x}, \Pi_\lambda) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\mathrm{e}^{\lambda_x(\mathrm{e}^{it}-1)} - \mathrm{e}^{\lambda(\mathrm{e}^{it}-1)}\right|^2\,\mathrm{d}t\right)^{1/2}$$

we obtain that $\rho_2(\Pi_{\lambda_x}, \Pi_\lambda) \ll \Delta_x$. Using the triangle inequality again but for the distance $\rho_2$, we get the desired estimate (5). The proof is complete now.

## Examples

**I.** Let

$$f_x(p) = \begin{cases} 1 \text{ if } \ln x < p \leqslant \ln^2 x, \\ 0 \text{ otherwise.}\end{cases}$$

From Theorem 3 we have

$$\rho_i\left(\nu_x, \Pi_{\ln 2}\right) \ll \mathrm{e}^{-c_2\sqrt{\ln\ln x}}, \quad i = 1, 2,$$

where $c_2$ is a positive constant.

Hence

$$\#\left\{n \leqslant x : \text{the primes from } \left(\ln x, \ln^2 x\right] \text{ do not divide } n\right\} = x\left(\frac{1}{2} + \mathrm{O}\left(\mathrm{e}^{-c_2\sqrt{\ln\ln x}}\right)\right),$$

$$\#\left\{n \leqslant x : n \text{has exactly one prime divisor from } \left(\ln x, \ln^2 x\right]\right\} = x\left(\frac{\ln 2}{2} + \mathrm{O}\left(\mathrm{e}^{-c_2\sqrt{\ln\ln x}}\right)\right).$$

**II.** Let

$$f_x(p) = \begin{cases} 1 \text{ if } \ln\ln x < p \leqslant (\ln\ln x)^3, \\ 0 \text{ otherwise.} \end{cases}$$

In this case from Theorem 3 we obtain

$$\rho_i\left(\nu_x, \Pi_{\ln 3}\right) \ll \mathrm{e}^{-c_2\sqrt{\ln\ln\ln x}}, \quad i = 1, 2.$$

Therefore similarly

$$\#\left\{n \leqslant x\text{: the primes from } \left(\ln\ln x, (\ln\ln x)^3\right] \text{ do not divide } n\right\} = x\left(\frac{1}{3} + \mathrm{O}\left(\mathrm{e}^{-c_2\sqrt{\ln\ln\ln x}}\right)\right),$$

$$\#\left\{n \leqslant x\text{: } n \text{has exactly two prime divisors from } \left(\ln\ln x, (\ln\ln x)^3\right]\right\} = x\left(\frac{\ln^2 3}{6} + \mathrm{O}\left(\mathrm{e}^{-c_2\sqrt{\ln\ln\ln x}}\right)\right).$$

## References


[1] И. Кубилюс, Об асимптотических локальных законах распрелеления арифметических функций, *Liet. matem. rink.*, **11**(4), 809–816 (1971).

[2] G. Halász, On the distribution of additive and the mean values of multiplicative arithmetic functions, *Studia Scient. Math. Hung.*, **6**, 213–233 (1971).

[3] Г. Стяпанаускас, Локальная предельная теорема для одного класса арифметических функций, *Liet. matem. rink.*, **20**(2), 159–174 (1980).

[4] J. Šiaulys, The von Mises theorem in number theory, In: *New Trends in Probability and Statistics*, Vol. **2**, F. Schweiger and E. Manstavičius (Eds), Utrecht/VSP, Vilnius/TEV (1992), pp. 293–310.

[5] И. Шяулис, Целозначные аддитивные арифметические функции и распрелеление Пуассона, *Liet. matem. rink.*, **28**(2), 384–398 (1988).


## Kai kurių aritmetinių funkcijų lokalusis elgesys

J. Šiaulys, G. Stepanauskas

Darbe gauti lokaliosios ir $l_2$ metrikos įverčiai tarp adityviųjų funkcijų sekos skirstinio ir Puasono skirstinio su pastoviu parametru.