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Value concentration of additive functions on semigroups

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Let $\mathbb{G}$ be a commutative multiplicative semigroup with identity element 1, which contains a countable subset $\mathbb{P}$ such that every element $a \neq 1$ admits unique factorization into a finite product of powers of elements of $\mathbb{P}$. Suppose that the completely additive degree function $\delta\colon\ \mathbb{G} \to \mathbb{N} \cup \{0\}$ such that $\delta(p) \geqslant 1$ for each $p \in \mathbb{P}$ is defined. The main assumption on the semigroup $\mathbb{G}$ accepted in this paper is the following asymptotic formula.

Condition P. *For some* $\gamma > 2$,

$$\pi(j) := \bigl|\bigl\{p \in \mathbb{P}\colon\ \delta(p) = j\bigr\}\bigr| = \frac{q^j}{j} + \mathrm{O}\biggl(\frac{q^j}{j\log^{\gamma}(j+1)}\biggr), \quad j \geqslant 1.$$

The corollary of Theorem 3 in [5] shows that Condition P implies

$$|\mathbb{G}_n| := \bigl|\{a \in \mathbb{G}\colon\ \delta(a) = n\}\bigr| = Aq^n + \mathrm{O}\bigl(q^n \log^{2-\gamma} n\bigr).$$

The class of arithmetical semigroups satisfying Condition P contains the semigroup of monic polynomials over a finite field and many other examples listed in [2] as well.

Let ν_n be the uniform probability measure on the set $\mathbb{G}_n$. If $\alpha_p(a)$ denotes the multiplicity of a prime element $p \in \mathbb{P}$ in the canonical product representation of $a \in \mathbb{G}$, then $\nu_n(\alpha_p(a) = k) \sim p^{-k\delta(p)}$ for $k \in \mathbb{N}$ as $n \to \infty$. In other words, $\alpha_p(a)$ is asymptotically distributed as the geometric random variable ξ_p with $P(\xi_p \geqslant 1) = q^{-\delta(p)}$. Moreover, $\alpha_p(a)$, $p \in \mathbb{P}$, are dependent random variables (r. vs) with respect to ν_n. So, a function $h\colon\ \mathbb{G} \to \mathbb{R}$ (later called *additive*) having the expression

$$h(a) = \sum_{p \in \mathbb{P}} h_p(\alpha_p(a))$$

for some double sequence $\{h_p(k)\}$ with the property $h_p(0) \equiv 0$, where $p \in \mathbb{P}$ and $k \geqslant 0$, can be regarded as the sum of dependent r. vs. Nevertheless, dealing with its distribution, we can achieve results close to that known for sums independent r. vs. In this remark, we demonstrate such possibility by obtaining an analog of the Kolmogorov-Rogozin inequality for the Lévy concentration function

$$Q_n(l) = \sup_{x \in \mathbb{R}} \nu_n(x \leqslant h(a) < x + l), \quad l \geqslant 0.$$

Note that $\sum_{p\in\mathbb{P}} \alpha_p(a)\delta(p) = \delta(a) = n$ if $a \in \mathbb{G}_n$. The functions proportional to $\delta(a)$ can appear as components in an additive function under consideration. This phenomenon is taken into account in the formulation of our result. Recently the author [4], using the ideas of I.Z. Ruzsa [6], obtained a similar estimate for the concentration function of an additive functions defined on the symmetric group. We now exploit this experience.

Let $x \wedge y = \min(x, y)$. For an additive function $h(a)$ and $\lambda \in \mathbb{R}$, we set $h_p(1) = a(p)$,

$$D_n(l;\lambda) = \sum_{\delta(p)\leqslant n} \frac{l^2 \wedge (a(p) - \lambda\delta(p))^2}{q^{\delta(p)}}, \qquad D_n(l) = \min_{\lambda\in\mathbb{R}} D_n(l;\lambda).$$

Throughout the paper $c, c_1, \ldots, C, C_1, \ldots$ will denote positive constants depending on q and the constant in the remainder of Condition P. The main result of the paper is the following theorem.

Theorem. *We have*

$$Q_n(l) \leqslant Cl(D_n(l))^{-1/2}.$$

Of course, if $D_n(l) = \mathrm{o}(l^2)$ as $n \to \infty$, the trivial estimate $Q_n(l) \leqslant 1$ is better. Observe that the Kolmogorov–Rogozin theorem (see [1]) applied for the sum

$$S_n := \sum_{\delta(p)\leqslant n} h_p(\xi_p),$$

where ξ_p are the above mentioned independent geometrically distributed r. vs, yields the estimate

$$\sup_{x\in\mathbb{R}} P(x \leqslant S_n < x + l) \leqslant C_1 l(D_n(l;0))^{-1/2}.$$

Thus, with a successful choice of λ, our concentration estimate for $h(a) - \lambda\delta(a)$ is comparable with that for S_n.

Proof of Theorem is split into a few steps.

1. It suffices to deal with $Q_n(1)$ only. By Lemma 2.2.1 of [1], we have

$$Q_n(1) \leqslant \frac{C_2}{|\mathbb{G}_n|} \int_{-1}^{1} \Bigg| \sum_{a\in\mathbb{G}_n} \mathrm{e}^{2\pi i t h(a)} \Bigg| \,\mathrm{d}t.$$

2. Set $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. This group may be identified with the additive group of real numbers in the interval $[0, 1)$ with addition modulo one.

Lemma 1. *Set*

$$m(u,t) := \sum_{\delta(p)\leqslant n} \frac{1 - \cos 2\pi\big(a(p)t - u\delta(p)\big)}{q^{\delta(p)}}, \qquad t \in \mathbb{R}, \quad u \in \mathbb{T}.$$

Then

$$\left|\frac{1}{|\mathbb{G}_n|}\sum_{a\in\mathbb{G}_n}\mathrm{e}^{2\pi ith(a)}\right|\leqslant C_3\exp\big\{-c\min_{u\in\mathbb{T}}m(u,t)\big\}.$$

Proof. Let $f(a):=\mathrm{e}^{2\pi ith(a)}$, $f_p(k):=\mathrm{e}^{2\pi ith_p(k)}$, and $|z|\leqslant 1$. Then

$$\begin{aligned}\sum_{a\in\mathbb{G}}f(a)(q^{-1}z)^{\delta(a)}&=\sum_{n=0}^{\infty}\Big(q^{-n}\sum_{\delta(a)=n}f(a)\Big)z^n\\ &=\prod_{p\in\mathbb{P}}\Big(1+f_p(1)(q^{-1}z)^{\delta(p)}+f_p(2)(q^{-1}z)^{2\delta(p)}+\cdots\Big)\\ &:=H(z)\exp\Big\{\sum_{p\in\mathbb{P}}f_p(1)(q^{-1}z)^{\delta(p)}\Big\}\\ &=H(z)\exp\Big\{\sum_{j=1}^{\infty}\Big(jq^{-j}\sum_{\delta(p)=j}f_p(1)\Big)\frac{z^j}{j}\Big\}.\end{aligned}$$

An appropriate estimate of the n-th Taylor coefficient of such type series has been obtained in Theorem of author's remark [3]. Condition P above assures its applicability. We leave the details for the reader. Lemma 1 is proved.

So, we can conclude this step by the estimate

$$Q_n(1)\leqslant C_4\int_{-1}^{1}\exp\Big\{-c\min_{u\in\mathbb{T}}m(u,t)\Big\}\,\mathrm{d}t. \tag{1}$$

3. To evaluate the last integral, we apply the arguments originated in the papers [4] and [6]. Set

$$X_k=\{t\in[-1,1]\colon \min_{u\in\mathbb{T}}m(u,t)\leqslant k\},\quad k=1,2,\ldots.$$

These sets are nonempty measurable, symmetric with respect to the origin, and having the Lebesgue measure $\mu_k:=\mathrm{meas}(X_k)>0$. The main task is to obtain a satisfactory estimate of this measure.

Lemma 2. *If $X\subset[-1,1]$ is a set of positive Lebesgue measure, symmetric to the origin and containing it, then we have*

$$X^r:=\{x_1+\cdots+x_r\colon x_1,\ldots,x_r\in X\}\supset[-1,1]$$

provided that $r=[12/\mathrm{meas}(X)]$.

Proof see in [6].

Lemma 3. *If $r\geqslant 12/\mu_k$, then, for every $t\in[-1,1]$, there exist $u_1,\ldots,u_r\in\mathbb{T}$ such that*

$$m(u,t)\leqslant kr^2 \tag{2}$$

with $u = u_1 + \cdots + u_r \bmod 1$.

Proof. Apply Lemma 2 with $X = X_k$ and the inequality

$$1 - \cos(x_1 + \cdots + x_r) \leqslant r\big((1 - \cos x_1) + \cdots + (1 - \cos x_r)\big), \quad x_j \in \mathbb{R}. \qquad (3)$$

Lemma 3 is proved.

Thus, by (2), to get an upper estimate of μ_k, we have to concentrate on the the values of $u = u(t) \in \mathbb{T}$ for which $m(u,t)$ attains its minimum. The standard analysis, via a criteria for implicit functions shows that $u(t)$ is well defined continuous function in some nontrivial neighborhood of the point $t = 0$, $u(0) = 0$. Beyond it, if several values of $u(t)$ appear for a fixed t, we can choose the smallest of them and so obtain the function $u(t)$ defined on the whole interval [-1,1] and taking values in $\mathbb{T}$.

4. We now relate $u(t)$ with a homomorphism of the additive group $\mathbb{R}$ to $\mathbb{T}$. Observe that the group $\mathbb{T}$ is the complete metric space with respect to the metric defined via the distance to the nearest integer $\|x\| = \{x\} \wedge (1 - \{x\})$ which is not a norm. Nevertheless, the solution of the approximate Cauchy equation with respect to it has similar properties as in Banach spaces.

Lemma 4. *Let* $v\colon [-1,1] \to \mathbb{T}$ *be continuous at the point* $t = 0$ *function,* $v(0) = 0$. *Suppose that, for some* $0 < \eta < 1/18$, *we have* $\|v(t_1+t_2) - v(t_1) - v(t_2)\| \leqslant \eta$ *whenever* $t_1, t_2, t_1 + t_2 \in [-1,1]$. *Then* $\|v(t) - \lambda t\| \leqslant 3\eta$ *for some* $\lambda \in \mathbb{R}$ *and all* $t \in [-1,1]$.

Proof see [4].

Lemma 5. *Let* $m(u,t)$ *and* $u(t)$ *be the above defined functions and* r *be as in Lemma 3. Then, for some* $\lambda \in \mathbb{R}$, *we have* $m(\lambda t, t) \leqslant 20kr^2 + C_5$ *uniformly in* $t \in [-1,1]$.

Proof. For $\rho := \mathrm{e}^{-1/n}$, set

$$\Psi(y) := \sum_{j=1}^{\infty} \frac{1 - \cos 2\pi j y}{j} \rho^j = \frac{1}{2} \log\left(1 + \frac{4\rho}{(1-\rho)^2} \sin^2 \pi y\right)$$

and observe that $\Psi(\theta x) \leqslant \Psi(x) + C_6$ uniformly in $0 \leqslant \theta \leqslant 10$. Moreover (see [4] for details),

$$\left| \sum_{j=1}^{n} \frac{1 - \cos 2\pi j y}{j} - \Psi(y) \right| \leqslant 3.$$

Hence, via Condition P, we obtain $|m(y,0) - \Psi(y)| \leqslant C_7$ and

$$m(\theta y, 0) \leqslant m(y, 0) + C_8. \qquad (4)$$

We now return to inequality (2). By the definition of $u(t)$, we have

$$m(u(t), t) \leqslant kr^2 \qquad (5)$$

as well. Set

$$\alpha = \sup\{\|u(t_1 + t_2) - u(t_1) - u(t_2)\|\colon t_1, t_2, t_1 + t_2 \in [-1,1]\}.$$

If $\alpha = 0$, then by Lemma 4, $\|u(t) - \lambda t\| = 0$ and the task is done. If $\alpha > 0$, we chose $t_1, t_2, t_1 + t_2 \in [-1, 1]$ so that

$$\beta := \|u(t_1 + t_2) - u(t_1) - u(t_2)\| \geqslant \frac{9}{10}\alpha.$$

For arbitrary $t \in [-1, 1]$, by Lemma 4 with $\eta = \alpha$, we have $\beta_1 := \|u(t) - \lambda t\| \leqslant 9\alpha \leqslant 10\beta$. Since the first inequality is trivial for $\alpha \geqslant 1/18$, here applying Lemma 4, we have avoided the condition on α. Now, by (4), (3), and (5), we obtain

$$m(\beta, 0) \leqslant 3\Big(m(u(t_1+t_2), t_1+t_2)+m(u(t_1), t_1)+m(u(t_2), t_2)\Big) \leqslant 9kr^2. \qquad (6)$$

Further by (3), (4), and (6), we arrive at

$$m(\lambda t, t) \leqslant 2m(u(t), t) + 2m(\beta_1, 0) \leqslant 2kr^2 + 2m(\beta, 0) + 2C_8 \leqslant 20kr^2 + C_9.$$

Lemma 5 is proved.

5. This is the final step of the proof of Theorem. Integrating over $[0, 1]$ the function $m(\lambda t, t)$ and using the inequality obtained in Lemma 5 together with the estimate $1 - (\sin x)/x \geqslant c_1 \min\{1, x^2\}$, where $x \in \mathbb{R}$, we obtain $D_n(1, \lambda) \leqslant C_{10}kr^2$. Hence, $\mu_k \leqslant C_{11}\big(k/D_n(1)\big)^{1/2}$. This and (1) imply

$$Q_n(1) \leqslant C_4 \sum_{k \geqslant 1} \mathrm{e}^{-c(k-1)}\mu_k \leqslant C_{12}(D_n(1))^{-1/2}.$$

The theorem is proved.

References

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Pusgrupių adityviųjų funkcijų reikšmių koncentracija

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Nagrinėjama adityviųjų funkcijų, apibrėžtų aritmetiniuose pusgrupiuose, reikšmių koncentracija. Levy koncentracijos funkcijai įrodytas Kolmogorovo-Rogozino nelygybės analogas.