Liet. matem. rink., **44**, *spec. nr.*, 2004, 839–842

Nonuniform estimate of the discounted limit theorem

Leonas SAULIS, Dovilė DELTUVIENĖ (VGTU)
e-mail:lsaulis@fm.vtu.lt

1. Introduction and summary

Let $X_0, X_1, X_2, \dots$ be a sequence of independent random variables (r.v.) with the common distribution function $F(x)$. Let v be a discount factor $(0 < v < 1)$. Then we define

$$S_v = \sum_{k=0}^{\infty} v^k X_k, \tag{1.1}$$

which may be interpreted as the present value of the sum of certain periodic and identically distributed payments X_k. We assume that the first three moments of X_k are finite:

$$\begin{aligned} \mu &= \int_{-\infty}^{\infty} x \, \mathrm{d}F(x) < \infty, \quad \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \, \mathrm{d}F(x) < \infty, \\ \rho &= \int_{-\infty}^{\infty} |x-\mu|^3 \, \mathrm{d}F(x) < \infty. \end{aligned} \tag{1.2}$$

Then it is easy to see that the mean and variance of the r.v. S_v are

$$\mathbf{E}S_v = \mu(1-v)^{-1}, \text{ and } \mathbf{D}S_v = \sigma^2(1-v^2)^{-1}, \tag{1.3}$$

respectively. It has been shown that the normalized random variable

$$Z_v = \sigma^{-1}(1-v)^{\frac{1}{2}}\left(S_v - \mu(1-v)^{-1}\right) \tag{1.4}$$

with the mean $\mathbf{E}Z_v = 0$ and variance $\mathbf{D}Z_v = (1+v)^{-1}$ is asymptotically normal for $v \to 1$. We denote the distribution function of the r.v. Z_v as $F_v(x)$ and that of the normal distribution with zero mean and variance $(1+v)^{-1}$ by

$$N_v(x) = \left(\frac{1+v}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{x} \exp\left\{-\frac{1+v}{2} y^2\right\} \mathrm{d}y. \tag{1.5}$$

Hans U. Gerber in [2] has proved the discounted version of the Berry–Esseen theorem: if (1.2) holds, then for all x,

$$\left|F_v(x) - N_v(x)\right| \leqslant 5.4(\rho/\sigma^3)(1-v)^{\frac{1}{2}}. \tag{1.6}$$

We consider a nonuniform estimate for the difference $F_v(x) - N_v(x)$ employing the cumulant method when centered moments $\mathbf{E}(X_0 - \mu)^s$ of the r.v.X_0 satisfy the condition: there exist quantities $\gamma \geqslant 0$ and $K > 0$ such that

$$\left|\mathbf{E}(X_0 - \mu)^s\right| \leqslant (s!)^{1+\gamma} K^{s-2}\sigma^2, \quad s = 3, 4, \dots l. \tag{B^*}$$

To this end we will use S.A. Achmedov's lemma (see [1], p. 624 or [4], p. 194).

2. Nonuniform estimate

Denote

$$\widehat{\Delta}_v = \frac{\sigma}{2(\sigma \vee K)\sqrt{1-v^2}}, \tag{2.1}$$

where $a \vee b = \max\{a, b\}$.

THEOREM 1. *Let identically distributed r.v.X_k with $\mathbf{E}X_k = \mu$ and $\sigma^2 = \mathbf{E}(X_k - \mu)^2$, $k = 0, 1, 2, \dots$ satisfy condition (B^*). Then for the distribution function $F_v(x)$ of the r.v. Z_v defined by* (1.4) *the following inequality holds:*

$$\left|F_v(x) - N_v(x)\right| \leqslant \frac{c(k,\gamma)\left(\ln \widehat{\Delta}_v^{\frac{1}{1+2\gamma}}\right)^{\frac{k}{2}}}{\left(1 + |x\sqrt{1+v}|^k\right)\widehat{\Delta}_v^{\frac{1}{1+2\gamma}}} \tag{2.2}$$

for all x and $k \leqslant l$ and

$$c(k,\gamma) = \max\left\{252(6/\sqrt{2})^{1/(1+2\gamma)}c(k);\ 12\,2^{2\gamma}(\sqrt{2}/6)^{2\gamma/(1+2\gamma)}k!\right\}, \tag{2.3}$$

where

$$c(k) \leqslant 2^{\frac{k}{2}}\left(5 + 2^{\frac{k}{2}}\sqrt{e/\pi}\,\Gamma\left((s+1)/2\right)\right), \quad k \geqslant 1. \tag{2.4}$$

Proof. Denote the characteristic function (ch.f.) of the r.v. X_k, $k = 0, 1, 2, \dots$, by $f(t) = \mathbf{E}\exp\{itX_k\}$. Then, recalling that the r.v. S_v was defined by equality (1.1) and taking into consideration that the r.v. X_k, $k = 0, 1, 2, \dots$, are independent, we obtain the expression of the ch.f. $f_{Z_v}(t)$ of the r.v. Z_v,

$$f_{Z_v}(t) = \mathbf{E}\exp\{itZ_v\} = \exp\left\{-it\mu\sigma^{-1}(1-v)^{-\frac{1}{2}}\right\}\prod_{k=0}^{\infty} f\left(\sigma^{-1}(1-v)^{\frac{1}{2}}v^k t\right). \tag{2.5}$$

Consequently,

$$\ln f_{Z_v}(t) = -it\mu\sigma^{-1}(1-v)^{-\frac{1}{2}} + \sum_{k=0}^{\infty}\ln f\left(\sigma^{-1}(1-v)^{\frac{1}{2}}v^k t\right). \tag{2.6}$$

The s-th order cumulant of the r.v. X is defined by

$$\Gamma_s(X) := \frac{d^s}{dt^s}\ln f_X(t)\Big|_{t=0}, \quad s = 1, 2, \dots.$$

Next, employing (2.6), we obtain $\Gamma_1(Z_v) = \mathbf{E}Z_v = 0$,

$$\Gamma_s(Z_v) = \left(\frac{(1-v)^{1/2}}{\sigma}\right)^s \frac{1}{1-v^s}\Gamma_s(X_0), \quad s = 2, 3, \ldots. \tag{2.7}$$

In particular, $\Gamma_2(Z_v) = \mathbf{D}Z_v = (1-v)^{-1}$,

$$\Gamma_3(Z_v) = \frac{(1-v)^{1/2}\Gamma_3(X_0)}{(1+v+v^2)\sigma^3} = \frac{1-v)^{1/2}}{1+v+v^2}\frac{\mathbf{E}(X_0-\mu)^3}{\sigma^3}, \ldots.$$

Our next step is to obtain the majorating upper estimates for the s-th order cumulant $\Gamma_s(Z_v)$, $s = 3, 4, \ldots, l$ of the r.v. Z_v.

PROPOSITION. *If for the r.v. X_k, $k = 0, 1, 2, \ldots$ with $\mu = \mathbf{E}X_k$ and $\sigma^2 = \mathbf{E}(X_k - \mu)^2$ the condition (B^*) is fulfilled, then*

$$|\Gamma_s(Z_v)| \leqslant \frac{1}{1+v+v^2}\frac{(s!)^{1+\gamma}}{\Delta_v^{s-2}}, \quad s = 3, 4, \ldots, l, \tag{2.8}$$

where

$$\Delta_v = \frac{\sigma}{2(\sigma \vee K)\sqrt{1-v}}. \tag{2.9}$$

Note that $\Delta_v \to \infty$ if $v \to 1$.

Noting that $\Gamma_s(X_0 - \mathbf{E}X_0) = \Gamma_s(X_0)$, $s = 2, 3, \ldots$, and making use of Lemma 1.8 (see [4], p. 195) from condition (B^*) we obtain

$$|\Gamma_s(X_0)| \leqslant (s!)^{1+\gamma}(2(\sigma \vee K))^{s-2}\sigma^2, \quad s = 3, 4, \ldots l. \tag{2.10}$$

Now, applying expression (2.7) of the cumulants $\Gamma_s(Z_v)$, $s = 2, 3, \ldots$, we obtain

$$\begin{aligned}
|\Gamma_s(Z_v)| &\leqslant \frac{(s!)^{1+\gamma}(2(\sigma \vee K))^{s-2}\sigma^2}{1-v^s}\left(\frac{(1-v)^{1/2}}{\sigma}\right)^s \\
&\leqslant \frac{(s!)^{1+\gamma}(1-v)^{s/2}}{1-v^3}\left(\frac{2(\sigma \vee K)}{\sigma}\right)^{s-2} \\
&= \frac{1}{1+v+v^2}\frac{(s!)^{1+\gamma}}{\Delta_v^{s-2}}, \quad s = 3, 4, \ldots, l,
\end{aligned} \tag{2.11}$$

where Δ_v is defined by (2.9).

Next, in order to apply S.A. Achmedovs lemma [1] to the random variable Z_v defined by equality (1.4), let $\widehat{Z}_v := \sqrt{1+v}Z_v$. The mean of this r.v. $\mathbf{E}\widehat{Z}_v = 0$, variance $\mathbf{D}\widehat{Z}_v = 1$, and s-th order cumulant $\Gamma_s(\widehat{Z}_v) = (1+v)^{s/2}\Gamma_s(Z_v)$, $s = 3, 4, \ldots$. On the basis of estimate (2.8) we find

$$|\Gamma_s(\widehat{Z}_v)| \leqslant \frac{1+v}{1+v+v^2}\frac{(s!)^{1+\gamma}}{\widehat{\Delta}_v^{s-2}}, \quad s = 3, 4, \ldots, l, \tag{2.12}$$

where $\widehat{\Delta}_v$ is defined by (2.1). Applying the above lemma to the r. variable $\xi = \widehat{Z}_v$, we get now

$$\left|\mathbf{P}(\widehat{Z}_v < x) - \mathbf{P}(\eta < x)\right| \leqslant \frac{c(k,\gamma)\left(\ln \widehat{\Delta}_v^{\frac{1}{1+2\gamma}}\right)^{\frac{k}{2}}}{\left(1+|x|^k\right)\widehat{\Delta}_v^{\frac{1}{1+2\gamma}}}, \tag{2.13}$$

for all x and $k \leqslant l$, where the r.v. $\eta \sim N(0,1)$ and quantities $c(k,\gamma)$, $\widehat{\Delta}_v$ are defined by (2.3) and (2.1), respectively. It is from this inequality that we obtain the proposition of the theorem.

References

1. S. Achmedov, On nonuniform estimations in the central limit theorem for dependent random variables, *Lith. Math. J.*, **30**(4), 623–629 (1990).
2. H.U. Gerber, The discounted central theorem and its Berry–Esseen analogue, *The Annals of Mathematical Statistics*, **42**, 389–392 (1971).
3. R. Rudzkis, L. Saulis, V. Statulevičius, A general lemma of large deviations, *Lith. Math. J.*, **18**(2), 169–179 (1978).
4. L. Saulis, V. Statulevičius, Limit theorems on large deviations, in: Yu.V. Prokhorov, V. Statulevičius (Eds.), *Limit Theorems of Probability Theory*, Springer-Verlag, Berlin, Heidelberg, New York (2000), pp. 185–266.
5. L. Saulis, D. Deltuviene, The discounted limit theorems for large deviations, *Lith. Math. J.*, **43**, 703–708 (2003).

REZIUMĖ

L. Saulis, D. Deltuvienė. Netolygūs įverčiai diskontavimo ribinėse teoremose

Tegul $S_v = \sum_{k=0}^{\infty} v^k X_k$, kur $0 < v < 1$ ir X_k, $k = 0, 1, 2, \ldots$, – nepriklausomi vienodai pasiskirstę atsitiktiniai dydžiai su vidurkiu $\mathbf{E}X_0 = \mu < \infty$ ir dispersija $\sigma^2 = \mathbf{E}(X_0 - \mu)^2 < \infty$. Darbe nagrinėjama normuoto at. dydžio $Z_v = \sigma^{-1}(1-v)^{\frac{1}{2}}(S_v - \mu(1-v)^{-1})$ pasiskirstymo funkcijos $F_v(x)$ aproksimacija normaliuoju dėsniu. Gautas netolygus įvertis, kai at.d. X_k, $k = 0, 1, 2, \ldots$ momentai tenkina S.N. Bernšteino sąlygą: egzistuoja dydžiai $\gamma \geqslant 0$ ir $K > 0$ tokie, kad $|\mathbf{E}(X_0 - \mu)^s| \leqslant (s!)^{1+\gamma} K^{s-2}\sigma^2$, $s = 3, 4, \ldots, l$. Rezultatas gautas kumuliantų metodu.