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The definition of one complex-valued random variable

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Let $F(z)$ be a holomorphic cusp form of weight κ for the full modular group $SL(2,\mathbb{Z})$. This means that $F(z)$ is holomorphic function in the upper half-plane $\Im z > 0$, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ satisfies the functional equation

$$F\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^{\kappa} F(z),$$

and $\lim\limits_{\Im z\to\infty} F(z) = 0$. Moreover, we assume that $F(z)$ has the following Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m)\mathrm{e}^{2\pi imz}, \quad c(1) = 1.$$

Let $s = \sigma + it$ be a complex variable. The function

$$\varphi(s; F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}$$

is called the zeta-function attached to the cusp form $F(z)$. In view of the multiplicativity of the Fourier coefficients $c(m)$, $\varphi(s; F)$ also has an Euler product expansion over primes

$$\varphi(s; F) = \prod_p \Big(1 - \frac{\alpha(p)}{p^s}\Big)^{-1} \Big(1 - \frac{\beta(p)}{p^s}\Big)^{-1},$$

where $c(p) = \alpha(p) + i\beta(p)$. By Deligne's estimate [1]

$$|\alpha(p)| \leqslant p^{\frac{\kappa-1}{2}}, \quad |\beta(p)| \leqslant p^{\frac{\kappa-1}{2}}$$

it follows that the Euler product and the Dirichlet series for $\varphi(s, F)$ both converge absolutely for $\sigma > \frac{\kappa+1}{2}$. Hence $\varphi(s, F)$ is a non-vanishing holomorphic function in

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the half-plane $\sigma > \frac{\kappa+1}{2}$. Moreover, $\varphi(s, F)$ is analytically continuable to an entire function.

For the investigation of value-distribution of the function $\varphi(s, F)$, as for other zeta-functions, probabilistic methods are applied. We recall limit theorems on the complex plane $\mathbb{C}$ for the function $\varphi(s, F)$. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S.

Let $\gamma = \{s \in \mathbb{C}\colon\ |s| = 1\}$ denote the unit circle on the complex plane, and let

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. With product topology and pointwise multiplication the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H exists, and we obtain a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(p)$ stand for the projection of $\omega \in \Omega$ to the coordinate space γ_p. For positive integer m we put

$$\omega(m) = \prod_{p^\alpha \| m} \omega^\alpha(p),$$

where $p^\alpha \| m$ means that $p^\alpha | m$ but $p^{\alpha+1} \nmid m$. For $\sigma > \frac{\kappa}{2}$, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define the complex-valued random variable $\varphi(s, \omega; F)$ by

$$\varphi(s, \omega; F) = \sum_{m=1}^{\infty} \frac{c(m)\omega(m)}{m^\sigma} = \prod_p \left(1 - \frac{\alpha(p)\omega(p)}{p^\sigma}\right)^{-1} \left(1 - \frac{\beta(p)\omega(p)}{p^\sigma}\right)^{-1}.$$

Denote by P_φ the distribution of $\varphi(\sigma, \omega; F)$, i.e.,

$$P_\varphi(A) = m_H\big(\omega \in \Omega\colon\ \varphi(\sigma, \omega; F) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}).$$

Let $meas\{A\}$ denote the Lebesque measure of a measurable set $A \in \mathbb{R}$. Then we have the following statement.

THEOREM 1. *For $\sigma > \frac{\kappa}{2}$, the probability measure*

$$\frac{1}{T} meas\big\{t \in [0; T]\colon\ \varphi(\sigma + it; F) \in A\big\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_φ as $T \to \infty$.

Proof. The theorem is a consequence of a limit theorem in the space of analytic functions for $\varphi(s; F)$ obtained in [2].

Now we will state a discrete limit theorem for $\varphi(s; F)$. Let $h > 0$ be a fixed number.

THEOREM 2. *Suppose that* $\exp\{\frac{2\pi k}{h}\}$ *is irrational for all integers $k \neq 0$. Then, for $\sigma > \kappa/2$, the probability measure*

$$\frac{1}{N+1} \#\big(0 \leqslant m \leqslant N\colon\ \varphi(\sigma + imh; F) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_φ *as* $T \to \infty$.

Proof. In [3] a discrete limit theorem on the complex plane for the Matsumoto zeta-function was proved. Since the function $\varphi(s; F)$ is a particular case of the Matsumoto zeta-function, hence the theorem follows.

Now suppose that there exist integer numbers $k \neq 0$ such that $\exp\{\frac{2\pi k}{h}\}$ is rational. Clearly, it suffices to consider only positive integers k. Denote by k_0 the smallest of such k. Then it is not difficult to see that each k with above property is a multiple of k_0. Really, we can write $k = ak_0 + b$ with $0 \leqslant b < k_0$. Then

$$\exp\left\{\frac{2\pi k}{h}\right\} = \exp\left\{\frac{2\pi a k_0}{h}\right\}\exp\left\{\frac{2\pi b}{h}\right\},$$

where $\exp\{\frac{2\pi k}{h}\}$ and $\exp\{\frac{2\pi a k_0}{h}\}$ are rational numbers. Thus $\exp\{\frac{2\pi b}{h}\}$ must be also rational, and by the definition of k_0 we obtain that $b = 0$.

Let $\exp\{\frac{2\pi k_0}{h}\} = \frac{m_0}{n_0}$ with $m_0, n_0 \in \mathbb{N}$, $(m_0, n_0) = 1$, and let $\Omega_h = \{\omega \in \Omega : \omega(m_0) = \omega(n_0)\}$. Then Ω_h is a closed subgroup of Ω, and therefore it is a compact topological Abelian group. Therefore, as in the case of Ω, on $(\Omega_h, \mathcal{B}(\Omega_h))$ the probability Haar measure m_{hH} exists, and this leads to a probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_{hH})$.

The aim of this note is to define a complex-valued random variable on the probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_{hH})$. Let, for $\sigma > \frac{\kappa}{2}$,

$$\varphi(\sigma, \omega; F) = \sum_{m=1}^{\infty} \frac{c(m)\omega_h(m)}{m^\sigma}, \quad \omega_h \in \Omega_h.$$

THEOREM 3. $\varphi(\sigma, \omega_h; F)$ *is a complex-valued random variable defined on the probability space* $(\Omega_h, \mathcal{B}(\Omega_h), m_{hH})$.

Proof. If suffices to prove that, for $\sigma > \frac{\kappa}{2}$, the series

$$\sum_{m=1}^{\infty} \frac{c(m)\omega_h(m)}{m^\sigma} \tag{1}$$

converges almost surely with respect to the measure m_{hH}.

Without loss of generality we may suppose that the prime numbers $p_1, p_2, \ldots, p_l$ occur in the factorization of the numbers m_0 and n_0, and denote by α_i the exponent of p_i in $\frac{m_0}{n_0}$, $i = 1, \ldots, l$. Then we have that

$$\omega^{\alpha_1}(p_1)\omega^{\alpha_2}(p_2)\ldots\omega^{\alpha_l}(p_l) = 1.$$

Hence, say $\omega(p_1)$, can be expressed by $\omega(p_2), \ldots, \omega(p_l)$. Denote this expression by $\widehat{\omega}(p_1)$, where some fixed value is taken, for example, with the smallest argument. Define a function $y: \Omega \to \Omega_h$ by the formula

$$g(\omega) = \omega_h,$$

where

$$\omega = \big(\omega(p_1), \omega(p_2), \dots\big),$$

and

$$\omega_h = \big(\widehat{\omega}(p_1), \omega(p_2), \dots\big).$$

It is not difficult to see that

$$\int_{\Omega_h} \omega_h(m)\overline{\omega_h(n)}\,\mathrm{d}m_{hH} = 1,$$

if $m = n$ or $m = km_0$, $n = kn_0$, $k \in \mathbb{Z}$. The function g is measurable, it is even continuous. By the formula of change of variable we find

$$\int_{\Omega_h} \omega_h(m)\overline{\omega_h(n)}\,\mathrm{d}m_{hH} = \frac{1}{|\alpha_1|}\int_{\Omega} g(\omega)(n)\overline{g(\omega)(n)}\,\mathrm{d}m_H.$$

Since $\{\omega(m)\}$ is a sequence of pairwise orthogonal random variable, hence we obtain that

$$\int_{\Omega_h} \omega_h(m)\overline{\omega_h(n)}\,\mathrm{d}m_{hH} = 0,$$

if $m \neq n$ and $m \neq km_0$, $n \neq kn_0$, $k \in \mathbb{Z}$. Therefore we have that $\{\omega_h(m)\}$ is a sequence of pairwise orthogonal random variables on the probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_{hH})$.

Let

$$\varphi_k(\omega_h) = \frac{c(k)\omega_h(k)}{k^\sigma}, \quad k = 1, 2 \dots .$$

Denote by $E\xi$ the expectation of the random variable ξ. Then we deduce from above that

$$E(\varphi_j, \overline{\varphi_l}) = \begin{cases} \frac{|c(l)|^2}{l^{2\sigma}}, & \text{if } j = l, \\ \frac{c(km_0)c(kn_0)}{(km_0)^\sigma (kn_0)^\sigma}, & \text{if } j = km_0,\ l = kn_0, \\ 0, & \text{othervise.} \end{cases}$$

Since by the Deligne estimate [1]

$$|c(k)| \leqslant m^{\frac{\kappa-1}{2}} d(m),$$

where $d(m)$ is the divisor function, and $d(m) \ll m^\varepsilon$, $\varepsilon > 0$, hence we obtain that

$$\sum_{k=1}^{\infty} E|\varphi_k|^2 \log^2 k < \infty$$

for $\sigma > \frac{\kappa}{2}$. This and the Rademacher theorem on series of orthogonal random variables give the almost sure convergence of the series (1). The theorem is proved.

Note that the random variable $\varphi(\sigma, \omega_h; F)$ can be written in the form

$$\varphi(\sigma, \omega_h; F) = \prod_p \left(1 - \frac{\alpha(p)\omega_h(p)}{p^\sigma}\right)^{-1} \left(1 - \frac{\beta(p)\omega_h(p)}{p^\sigma}\right)^{-1}.$$

Theorem 3 can be applied in the investigation of weak convergence of the probability measure

$$\frac{1}{N+1} \#\big(0 \leqslant m \leqslant N\colon \varphi(\sigma + imh, F) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

in the case if $\exp\{\frac{2\pi k}{h}\}$ is rational for some $k \neq 0$.

References

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REZIUMĖ

A. Laurinčikas. Vieno kompleksinio atsitiktinio dydžio apibrėžimas

Nagrinėjamos ribinės teoremos kompleksinėje plokštumoje parabolinių formų dzeta funkcijoms. Apibrėžtas atsitiktinis dydis, kuris atsiranda tiriant tokių teoremų diskretųjį atvejį.