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On parameter estimation for stochastic logistic growth laws through the maximum likelihood procedure

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Abstract. A three alternative stochastic logistic growth models (exponential, Verhulst, Gompertz) are used for modelling of growth processes. The aim of this paper is to develop stochastic growth curves for all three stochastic growth models. Estimates of parameters of the stochastic growth models are performed by the maximum likelihood procedure with the local linearization method. The Milshtein discrete time approximation for the solutions of stochastic differential equations is applied.

Keywords: stochastic differential equation, approximation, local linearization, simulation, stand.

1. Introduction

In this paper we present growth models whose incorporate stochastic behaviour of instantaneous relative growth rate. Our stochastic models are based on the premise that the growth process is managed by a scalar standard Brownian motion (white noise).

For stand growth modelling here we employ three concepts (see, [2], [7]):

i) one-parameter exponential model

$$\frac{1}{N} \cdot \frac{\mathrm{d}N}{\mathrm{d}t} = \alpha, \tag{1}$$

ii) two-parameter Verhulst model

$$\frac{1}{N} \cdot \frac{\mathrm{d}N}{\mathrm{d}t} = \alpha \cdot \left(1 - \frac{N}{\beta}\right), \tag{2}$$

iii) two-parameter Gompertz model

$$\frac{1}{N} \cdot \frac{\mathrm{d}N}{\mathrm{d}t} = \alpha \cdot \ln \frac{\beta}{N}. \tag{3}$$

These three well-known approaches serve as the basic dominant for growth modelling. The growth models by means of ordinary stochastic differential equations were introduced in forestry by Garcia [1], Rupšys [4]. It is really admissible that the deterministic logistic functions (1)–(3) have the shape of the stochastic form as (see, [4]):

$$\mathrm{d}(N(t)) = aN(t)\,\mathrm{d}t + \mu N(t)\,\mathrm{d}W(t), \quad N(t_0) = N_0,\ t_0 \leqslant t \leqslant T, \tag{4}$$

$$\mathrm{d}(N(t)) = \alpha N(t)\left(1 - \frac{N(t)}{\beta}\right)\mathrm{d}t + \mu N(t)\,\mathrm{d}W(t), \quad N(t_0) = N_0,\ t_0 \leqslant t \leqslant T, \tag{5}$$

$$\mathrm{d}N(t) = \alpha \ln \frac{\beta}{N(t)} N(t)\,\mathrm{d}t + \mu N(t)\,\mathrm{d}W(t), \quad N(t_0) = N_0,\ t_0 \leqslant t \leqslant T, \tag{6}$$

where $W(t)$ is a scalar standard Brownian motion (white noise).

2. Material and methods

Now consider the estimates of the parametersα, β, μ for all stochastic models (4)–(6). For estimating of the parametersα, β, μ of the stochastic models (4)–(6) we will apply a maximum likelihood procedure with the local linearization developed by Shoi and Ozaki [6], Shoi [5]. The basic idea of the method is that an original stochastic differential equation is first converted into a stochastic differential equation with a constant diffusion term, and then a drift term of the derived stochastic differential equation is locally approximated by a linear function of the time. This linearization enables us to derive the maximum likelihood function.

Our objective is to find a transformation

$$y(t) = \phi(N(t)), \tag{7}$$

that the diffusion term $\mu N(t)$ in the models (4)–(6) converts to μ. To achieve this, we have to choose the transformation as

$$\phi(N(t)) = \ln N(t). \tag{8}$$

Using stochastic integrals and Ito's formula (see, Oksendal [3]) for the transformation $\phi(N)$ we obtain the maximum likelihood function of the observed data$\{N_i, i = 1, 2, ..., n\}$. This function takes the form

$$L(\alpha, \mu) = -\frac{1}{2} \sum_{i=1}^{n} \left[\frac{(\ln N(t_i) - E_{i-1})^2}{V_{i-1}} + \ln(2\pi V_{i-1}) \right], \tag{9}$$

where $N(t_i) = N_i$, $\Delta t_{i-1} = t_i - t_{i-1}$, for the exponential stochastic growth model

$$E_{i-1} = \ln N(t_{i-1}) + \left(\alpha - \frac{1}{2}\mu^2\right)\Delta t_{i-1},$$

for the Verhulst stochastic growth model

$$\begin{cases} E_{i-1} = \ln N(t_{i-1}) + \left(\frac{\beta(\frac{\mu^2}{2\alpha} - 1)}{N(t_{i-1})} + 1\right)\left(e^{-\frac{\alpha}{\beta}N(t_{i-1})\Delta t_{i-1}} - 1\right) \\ \qquad - \frac{\beta\mu^2}{2\alpha N(t_{i-1})}\left(e^{-\frac{\alpha}{\beta}N(t_{i-1})\Delta t_{i-1}} - 1 + \frac{\alpha}{\beta}N(t_{i-1})\Delta t_{i-1}\right), \\ V_{i-1} = -\frac{\beta\mu^2(e^{-\frac{2\alpha}{\beta}N(t_{i-1})\Delta t_{i-1}} - 1)}{2\alpha N(t_{i-1})}, \end{cases}$$

for the Gompertz stochastic growth model

$$\begin{cases} E_{i-1} = \ln N(t_{i-1}) + \left(\ln N(t_{i-1}) - \ln\beta + \frac{1}{2\alpha}\mu^2\right)\left(e^{-\alpha\Delta t_{i-1}} - 1\right), \\ V_{i-1} = \frac{\mu^2}{2\alpha}\left(1 - e^{-2\alpha\Delta t_{i-1}}\right). \end{cases}$$

We have derived the exact maximum likelihood function (9) for the exponential, Verhulst, Gompertz models. The maximum likelihood estimates of the parameters α, β, μ for these models can be obtained by maximizing the function (9). In general closed-form solutions cannot be found. To illustrate the closed approximations numerically, we use the Monte Carlo simulation method and the observed data presented in Table 1. The results of estimates are summarized in Table 2.

The discrete time approximations by the Milstein method of the solutions of the stochastic models (4)–(6) in the weak sense using 200 Brownian paths and five discrete time approximations in the strong sense are plotted jointly with the observed data in Fig. 1.

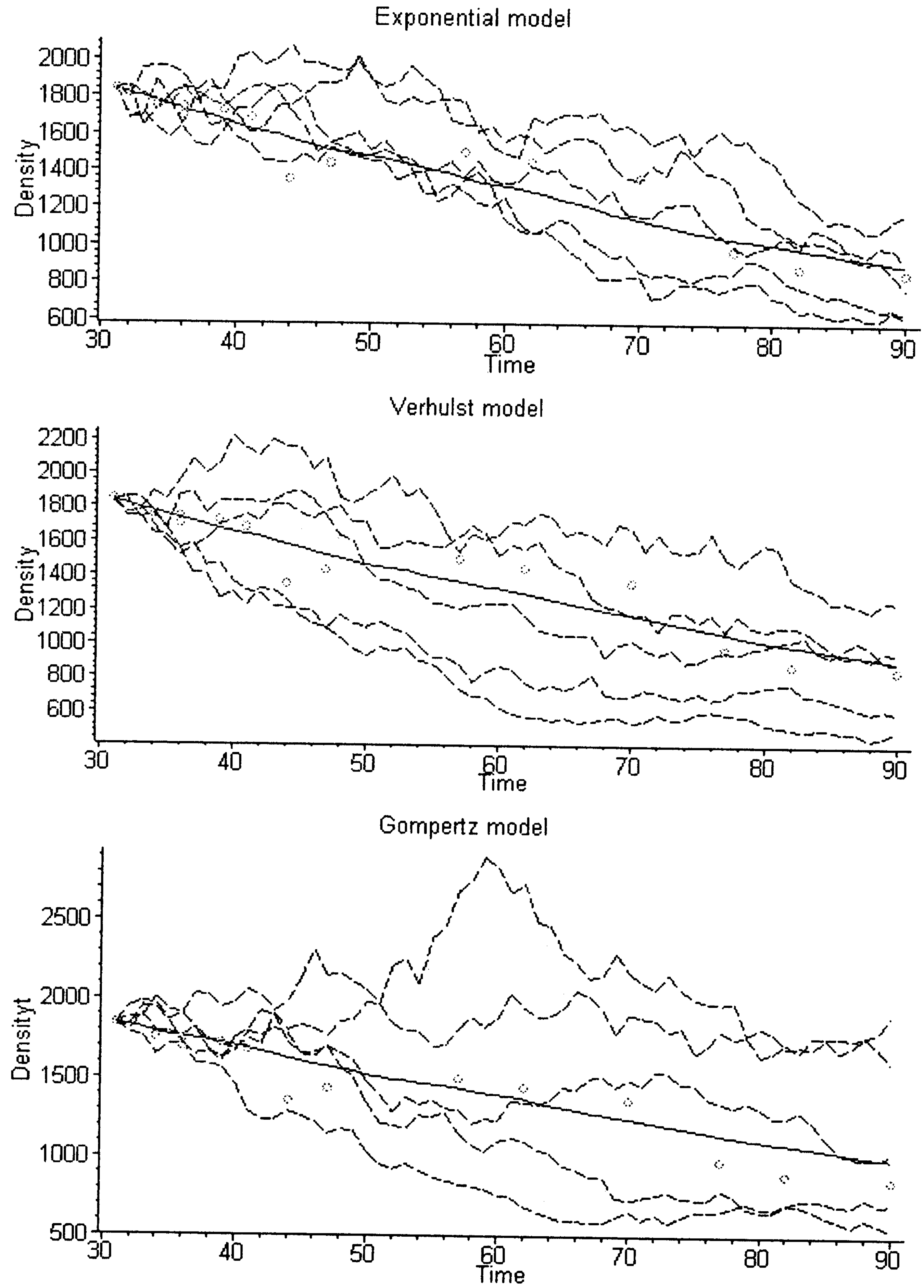

Fig. 1. Discrete time approximations of stochastic growth curves in the strong and weak senses (solid – stochastic growth curve in the weak sense, dash – stochastic growth curve in the strong sense).

Table 1. The observed data

Age	34	36	39	41	47	31	36	44	77	82	90	57	62	70
Stand's density	1756	1740	1724	1680	1436	1842	1696	1354	972	880	850	1496	1444	1362

Table 2. Maximum likelihood estimates of the parameters α, β, μ

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\mu}$
Exponential	−0.011440		0.080816
Verhulst	0.001215	124.8933	0.048819
Gompertz	0.010893	435.8625	0.048467

3. Conclussions

The method of maximum likelihood performs poorly results if there exists a few data points that are not explained by the model or the number of observations is large. Our interest to get the parameters that best fit the data could be implemented using the L^1 distance between the observed probabilities and the model.

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REZIUMĖ

P. Rupšys. Stochastinių logistinių augimo modelių parametrų įvertinimas maksimalaus tikėtinumo metodu

Darbe nagrinėjami trys stochastiniai logistiniai augimo modeliai: eksponentinis, Verhulsto, Gompertco. Pagrindinis tikslas yra rasti minėtų stochastinių augimo modelių parametrų įvertinimus. Parametrų įvertinimams gauti taikomas maksimalaus tikėtinumo metodas. Stochastinių logistinių augimo kreivių imitavimui naudojamos Milšteino skaitmeninės aproksimacijos.