Liet. matem. rink., **44**, *spec. nr.*, 2004, 753–758

On the identification of Hammerstein systems

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1. Introduction

A special class of nonlinear systems applied in engineering is Hammerstein systems with hard input nonlinearities. They, usually, consist of a static input nonlinearity and a linear dynamic system that are coupled together. Ordinary examples of hard nonlinearities are the saturation, preload, relay, dead-zone, hysteresis-relay, and hysteresis nonlinearities [1]. However, the assumptions that the nonlinearity is invertible or linear in a small region around the origin are not satisfied for most hard nonlinearities, because they cannot be described by polynomials and are noninvertible in general. On the other hand, the Hammerstein systems are common in nonlinear control applications [3]. Frequently as an input nonlinearity the piecewise saturation-like nonlinearity is used here, too. Assuming the nonlinearity to be piecewise linear, one could let the nonlinear part of the Hammerstein system be represented by different regression functions with some parameters, that are unknown beforehand. In such a case, observations of the input of a Hammerstein system could be partitioned into distinct data sets according to different descriptions. The boundaries of sets of observations depend on the value of the unknown threshold a – observations are divided into regimes subject to whether the some observed threshold variable is smaller or larger than a [4, 5]. Therefore according to [6], the problem of the identification of unknown parameters of nonlinear and linear blocks of the Hammerstein systems could be solved, if a simple way of partitioning the available data sets were found in the case of unknown a. Afterwards, the estimates of parameters of regression functions could be calculated by processing particles of observations to be determined.

2. Statement of the problem

The Hammerstein system given in Fig. 1 consists of a static nonlinearity $f(\cdot, \eta)$ followed by a linear part $G(q, \Theta)$. The linear part of the Hammerstein system is dynamic, time invariant, causal, and stable. It can be represented by a time invariant dynamic system (LTI) with the transfer function $G(q, \Theta)$ as a rational invertible function of the form

$$G(q, \Theta) = \frac{b_0 + b_1 q^{-1} +, \ldots, + b_m q^{-m}}{1 + a_1 q^{-1} +, \ldots, + a_m q^{-m}} = \frac{B(q, \mathbf{b})}{1 + A(q, \mathbf{a})} \tag{1}$$

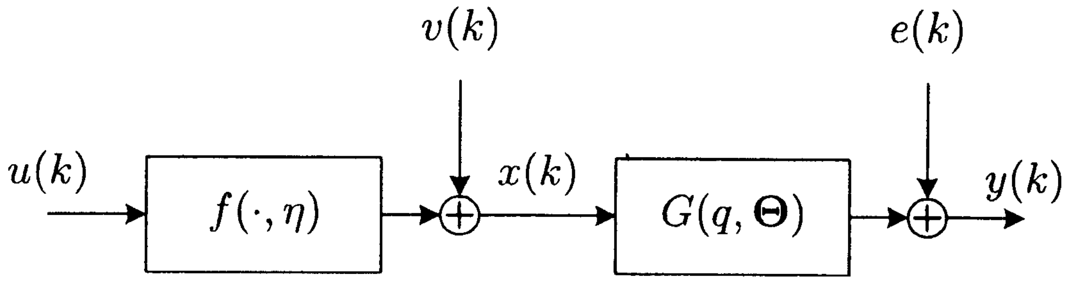

Fig. 1. The Hammerstein system with the process noise $v(k)$ and that of the measurement $e(k)$. The linear dynamic part $G(q,\Theta)$ of the Hammerstein system is parameterised by Θ, while the static nonlinear part $f(\cdot,\eta)$ – by η. Signals: $u(k)$ is input, $y(k)$ is output, $x(k)$ is an unmeasurable intermediate signal.

with a finite number of parameters

$$\Theta^T = (b_0, b_1, \dots, b_m, a_1, \dots, a_m),$$
$$\mathbf{b}^T = (b_0, b_1, \dots, b_m), \quad \mathbf{a}^T = (a_1, \dots, a_m), \tag{2}$$

that are determined from the set Ω of permissible parameter values Θ. Here q is a time-shift operator, the set Ω is restricted by conditions on the stability of the respective difference equation. The output signal

$$y(k) = \frac{B(q,\mathbf{b})}{1 + A(q,\mathbf{a})} x(k) + e(k), \tag{3}$$

is generated by the linear part of the Hammerstein system (1) as a response to the unknown intermediate signal

$$x(k) = f\big(u(k), \eta\big) + v(k) \tag{4}$$

with $u(k)$ as an input. Here the nonlinear part $f(\cdot,\eta)$ with the vector of parameters η is a saturation-like function of the form [7]

$$f(u(k),\eta) = \begin{cases} c_0 + c_1 u(k) & \text{if } u(k) \leqslant -a, \\ u(k) & \text{if } -a < u(k) \leqslant a, \\ d_0 + d_1 u(k) & \text{if } u(k) > a \end{cases} \tag{5}$$

that could be partitioned into three functions. Here $c_0 = -a(1 - c_1)$, $0 < c_1 < a$, $d_0 = a(1 - d_1)$, $0 < d_1 < a$.

The process noise $v(k)$ and the measurement noise $e(k)$ are added to an intermediate signal $x(k)$ and the output $y(k)$, respectively. Noises are mutually noncorrelated sequences of independent Gaussian variables with zero means and variances σ_v^2, σ_e^2, respectively.

The aim of the given paper is to estimate parameters (2) of the linear part (1), as well as parameters $\eta = (c_0, c_1, d_0, d_1)^T$ and the threshold a of nonlinearity (5) by processing N pairs of observations $u(k)$ and $y(k)$ of the Hammerstein system (Fig. 1).

3. The data reordering

Let us rearrange the data $u(k)\ \forall\, k \in \overline{1,N}$ in an ascending order of their values. Thus, the observations of the rearranged input $\tilde{u}(k)$ of the Hammerstein system should be partitioned into three data sets: left-hand side data set (N_1 samples) with values lower than or equal to negative a, middle data set (N_2 samples) with values higher than negative a but lower or equal to a, and right-hand side data set (N_3 samples) with values higher than a. Here $N = N_1 + N_2 + N_3$. In spite of the unknown a, from the engineering point of view it is assumed that no less than 50% observations are concentrated on the middle-set and approximately by 25% or less on any side set. Hence, the observations of the rearranged input $\tilde{u}(k)$ with the highest and positive values will be concentrated on the right-hand side set, while the observations with the lowest and negative values on the left-hand side one.

Let us suppose now that the process noise $v(k)$ is absent. Then the observations of the middle data set $\tilde{u}(k)$ are coincident with the respective observations of the intermediate signal $x(k)$ equivalent to those input observations $u(k)$ that passed the piecewise nonlinearity (5) without any processing. In such a case, one could get these observations simply by choosing the upper interval bound lower than the 75 percentage and the lower interval bound higher than the 25 percentage of the sampled reordered observations of $\tilde{u}(k)$. Thus, the middle data set $\tilde{u}(k)\ \forall\, k \in \overline{N_1 + l_1, N_2 - l_2}$ is, really, reordered in an ascending order of their values $u(k)\ \forall\, k \in \overline{1, N}$ with small portions of missing observations within it that belong to the left-hand and right-hand side sets of the data. Here arbitrary integers $l_1, l_2 > 0$. If N_1 and N_3 are unknown beforehand then an approach used in robust estimation could be applied here, too [2].

Let us now partially reconstruct an unmeasurable intermediate signal $x(k)$, choosing in the initial order only those values of $u(k)\ \forall\, k \in \overline{1, N}$ that are present in the middle data set of $\tilde{u}(k)\ \forall\, k \in \overline{N_1 + l_1, N_2 - l_2}$. In such a case one could get $x(k) \equiv u(k)$ for $k = 1 + l(k)$, such that $l(k) \leqslant l(k+1)$, where $l(k)$ is a positive time-varying integer. Really, assuming that the process noise $v(k)$ is absent, the available sequence $x(k)$ is equivalent to the input sequence $u(k)$ but with some portions of missing observations in it that belong to the left-hand or right-hand side sets of the rearranged data. It could be used to calculate the estimates of parameters (2) of the transfer function $G(q, \boldsymbol{\Theta})$ according to

$$\widehat{\boldsymbol{\Theta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}, \tag{6}$$

if a number of equations of the initial system of linear equations

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\Theta}, \tag{7}$$

with respective observations of the input signal in the matrix $\mathbf{X}$ are deleted. Here

$$\widehat{\boldsymbol{\Theta}}^T = (\hat{\mathbf{b}}, \hat{\mathbf{a}})^T, \quad \hat{\mathbf{b}}^T = (\hat{b}_0, \hat{b}_1, \ldots, \hat{b}_m), \quad \hat{\mathbf{a}}^T = (\hat{a}_1, \ldots, \hat{a}_m) \tag{8}$$

are $(2m + 1) \times 1$, $(m + 1) \times 1$, $m \times 1$ vectors of the estimates of parameters (2), respectively,

$$\mathbf{X} = \begin{bmatrix} \tilde{u}(m+1) \ldots & \tilde{u}(1) & \tilde{y}(m) & \ldots & \tilde{y}(1) \\ \tilde{u}(m+2) \ldots & \tilde{u}(2) & \tilde{y}(m+1) \ldots & & \tilde{y}(2) \\ \vdots \quad \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{u}(N) & \ldots \tilde{u}(N-m) & \tilde{y}(N-1) \ldots & & \tilde{y}(N-m) \end{bmatrix} \tag{9}$$

is the $(N-m) \times (2m+1)$ matrix, consisting of observations of the reordered input $\tilde{u}(k)$ and the associated output signal $\tilde{y}(k)$, and $\mathbf{Y}$ is the $(N-m) \times 1$ vector, consisting of $\tilde{y}(k)$.

Afterwards, one could completely reconstruct the unknown intermediate signal $x(k)\ \forall\, k \in \overline{1, N}$ according to the formula

$$\hat{x}(k) = \frac{1}{\hat{b}_0} y(k) + \frac{\hat{a}_1}{\hat{b}_0} y(k-1) +, \ldots, + \frac{\hat{a}_m}{\hat{b}_0} y(k-m) - \frac{\hat{b}_1}{\hat{b}_0} \hat{x}(k-1), \ldots, - \frac{\hat{b}_m}{\hat{b}_0} \hat{x}(k-m), \tag{10}$$

if in (1) their estimates are substituted instead of respective parameter sets (2). Here $\hat{x}(k)$ is an estimate of $x(k)$, $\hat{b}_0 \neq 0$.

Estimates of the parameters c_0, d_0 and c_1, d_1 are calculated by the ordinary least squares, too. In such a case, the sums of the form

$$I(c_0, c_1) = \sum_{i=1}^{N_1} \left[\tilde{\hat{x}}(i) - c_0 - c_1 \tilde{u}(i)\right]^2 = \min!, \tag{11}$$

$$I(d_0, d_1) = \sum_{j=N_1+L+1}^{N} \left[\tilde{\hat{x}}(j) - d_0 - d_1 \tilde{u}(j)\right]^2 = \min!, \tag{12}$$

are to be minimized in respect of parameters c_0, c_1 and d_0, d_1, respectively, using side-set data particles of $\hat{x}(k)$ and respective observations of the rearranged input signal $\tilde{u}(k)$. Here $\tilde{\hat{x}}(k)$ are observations of the signal $\hat{x}(k)$ that were rearranged in accordance with $\tilde{u}(k)$.

The estimates of the threshold a for the right-hand side and left-hand side sets are found according to

$$\hat{a} = \hat{d}_0/(1 - \hat{d}_1), \quad \hat{a} = \hat{c}_0/(1 - \hat{c}_1), \tag{13}$$

respectively.

In order to determine how the same process noise realization and different realizations of measurement noise affect the accuracy of estimation of unknown parameters, we have used the Monte Carlo simulation with 10 data samples, each containing 100 input-output observation pairs. 10 experiments with the same realization of the process noise $v(k)$ and different realizations of the measurement noise $e(k)$ of different levels of its intensity were carried out. The intensity of noises was assured by choosing respective signal-to-noise ratios (SNR) (the square root of the ratio of signal and noise variances). For the process noise $\mathrm{SNR}^{\mathrm{v}}$ was equal to 100 and for the measurement noise $\mathrm{SNR}^{\mathrm{e}}$: 1, 10, 100. As inputs for all given nonlinearities, the periodical signal and white Gaussian noise with variance 1 were chosen. In each ith experiment the estimates of parameters were calculated. During the Monte Carlo simulation, averaged values of the estimates of parameters as well as of the threshold

Table 1. Averaged estimates of the parameters $b_1, a_1, c_0, c_1, d_0, d_1$, and thresholds a, $-a$ with their confidence intervals (the first line for each estimate corresponds to the periodical signal, while the second line to the Gaussian white noise as inputs)

Estimates	$\mathrm{SNR}^e = 1$	$\mathrm{SNR}^e = 10$	$\mathrm{SNR}^e = 100$
$\hat{b}_1$	0.39 ± 0.10 0.34 ± 0.05	0.32 ± 0.03 0.32 ± 0.01	0.3 ± 0.01 0.3 ± 0.00
$\hat{a}_1$	-0.09 ± 0.09 -0.17 ± 0.11	-0.39 ± 0.04 -0.42 ± 0.04	-0.48 ± 0.01 -0.48 ± 0.02
$\hat{c}_0$	-1.20 ± 0.39 -1.08 ± 0.51	-1.04 ± 0.09 -0.95 ± 0.16	-0.99 ± 0.03 -0.93 ± 0.05
$\hat{c}_1$	0.02 ± 0.24 0.02 ± 0.28	0.04 ± 0.07 0.07 ± 0.08	0.06 ± 0.02 0.07 ± 0.03
$\hat{d}_0$	0.19 ± 0.83 0.94 ± 0.34	0.55 ± 0.30 0.91 ± 0.13	0.78 ± 0.1 0.89 ± 0.04
$\hat{d}_1$	0.57 ± 0.61 0.07 ± 0.18	0.31 ± 0.21 0.08 ± 0.07	0.15 ± 0.07 0.09 ± 0.02
$\hat{a}$	0.62 ± 0.83 0.97 ± 0.22	0.57 ± 0.55 0.98 ± 0.08	0.91 ± 0.06 0.98 ± 0.03
$-\hat{a}$	-1.23 ± 0.36 -1.05 ± 0.25	-1.09 ± 0.08 -1 ± 0.09	-1.05 ± 0.02 -1 ± 0.03

and their confidence intervals were calculated. In Table 1 for each input the averaged estimates of parameters and threshold a of the simulated Hammerstein system (Fig. 1) with the linear part (1) ($b_1 = 0.3$; $a_1 = -0.5$) and piecewise nonlinearity (5) ($c_0 = -0.9, c_1 = 0.1, d_0 = 0.9, d_1 = 0.1, a = 1$) and with their confidence intervals (the significance level $\alpha = 0.05$) are presented. It should be noted that, in each experiment here, the value of SNR^v was fixed and the same while, the values of SNR^e were varying due to different realizations of $e(k)$. The Monte Carlo simulation (Table 1) implies that the accuracy of parametric identification of the Hammerstein system depends on the intensity of process and measurement noises as well as on the type of the input signal.

The problem of identification of Hammerstein systems could be essentially reduced by a simple input data rearrangement in an ascending order of their values. Thus, the available data are partitioned into three data sets that correspond to distinct threshold regression models. Later on the estimates of unknown parameters of linear regression models could be calculated by processing the respective sets of rearranged input and associated output observations.

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REZIUMĖ

R. Pupeikis. Apie Hameršteino sistemų identifikavimą

Straipsnyje nagrinėjamas Hameršteino sistemų laipsniškas dalimis tiesiško netiesiškumo su nežinomais nuožulnumais bei nežinomų slenksčių ir tiesinės dalies, aprašomos skirtumine lygtimi su nežinomais koeficientais, junginys. Parodyta, kad pertvarkius įėjimo signalo stebėjimus pagal didėjančias ju reikšmes, galima išskirti vidurinę stebėjimų dalį, atitinkančią nestebimo tarpinio signalo stebėjimus. Pasiūlytas pilno tarpinio signalo atstatymo būdas pagal įėjimo signalo vidurinės dalies ir atitinkamus išėjimo signalo stebėjimus. Nežinomų tiesinės Hameršteino sistemos dalies koeficientų ir dalimis tiesiško netiesiškumo parametrų bei slenksčių įverčiai gaunami mažiausiųjų kvadratų metodo algoritmais, apdorojant stebimų pertvarkytų įėjimo, išėjimo bei atkurto tarpinio signalų duomenis. Pateikti modeliavimo rezultatai.