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Multiquantaloids

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1. Introduction

The "formulas-as-types" idea (in the type-theoretical approach) is the idea that a formula may be identified with the set of its proofs. Let $\mathbb{F}$ be a set of such "formulas", a set of abstract sets whose elements are viewed as their "proofs". Let $Seq = \mathcal{P}(\mathbb{F}) \times \mathbb{F}$ (where $\mathcal{P}(\mathbb{F})$ denotes the power set of $\mathbb{F}$) be the set of "sequents", pairs $\Gamma \vdash A$ consisting of a family $\Gamma \subseteq \mathbb{F}$ of formulas and of a formula $A \in \mathbb{F}$ and representing the metalogical claim that A is a consequence of Γ. Finally, let Q be a quantale which is introduced in order to grade sequents. In this paper we show that graded sequents form a multicategory in the sense of J. Lambek [2]. Moreover, it appears that this multicategory has additional properties analogous to those of a quantaloid [5]. We call it "multiquantaloid".

2. Quantales and quantal formulas

We begin with a discussion of quantales. This term was first suggested in 1986 by C.J. Mulvey [3] to model "the logic of quantum mechanics", a logic involving an associative (in general noncommutative) operation "AND THEN".

DEFINITION 2.1 ([4], [5]). *A quantale is a complete lattice Q together with an associative binary operation $\circ$ satisfying:*

$$q \circ \bigvee_i q_i = \bigvee_i q \circ q_i \quad \textit{and} \quad \Big(\bigvee_i q_i\Big) \circ q = \bigvee_i (q_i \circ q)$$

for all $q \in Q$ and all families $\{q_i\} \subseteq Q$. A quantale Q is called unital if it has an element 1 *such that $1 \circ q = q = q \circ 1$ for all $q \in Q$. It is called commutative if $p \circ q = q \circ p$ for all $p, q \in Q$.*

Examples of quantales include real or complex numbers with usual multiplication or addition, the interval [0, 1] of real numbers with a "triangular norm", a lower semicontinuous semigroup operation, complete Boolean algebras, complete Heyting algebras and complete MV-algebras.

DEFINITION 2.2. *Let Q be a quantale and A and B be two formulas (sets of "proofs"). A Q-matrix X from A to B is a mapping assigning to each pair a, b of* *elements of* $A \times B$ *an element* x_{ab} *of* Q. Q*-matrices compose by "matrix multiplication": for* $X\colon A \to B$ *and* $Y\colon B \to C$, *the composite* $X \circ Y = Z\colon A \to C$ *has its general element given by*

$$x_{ac} = \bigvee_{b \in B} x_{ab} \circ y_{bc}.$$

It is clear that this composition is associative. If Q is unital, then the Q-matrix $1_A\colon A \to A$ defined by

$$(1_A)_{aa'} = \begin{cases} 1, & \text{if } a = a', \\ \bot, & \text{if } a \neq a' \end{cases}$$

(where $\bot$ denotes the bottom element of Q) is the unit Q-matrix on A (neutral with respect to $\circ$). Now we turn to a "multidimensional" version of this definition.

DEFINITION 2.3. *Let* Q *be a quantale,* $\Gamma = \{A_i\}$ *a family of formulas and* A *a formula. A* Q*-multimatrix* $f\colon \Gamma \to A$ *from* Γ *to* A *is a mapping assigning to each family* $\gamma = \{a_i \in A_i\}$ *of proofs of respective formulas and to each proof* $a \in A$ *an element* $f_{\gamma a}$ *of* Q. *If* $f\colon \Gamma \to A$ *and* $g\colon \Delta\langle A\rangle \to B$ *(where* $\Delta\langle A\rangle$ *denotes a family* $\Delta = \{B_j\}$ *of formulas with a fixed formula* A, *i.e.,* $B_{j_0} = A$ *for some* j_0*) are* Q*-matrices, then they can be "composed" to produce a* Q*-matrix* $f\hat{A}g\colon \Delta\langle\Gamma\rangle \to B$ *(where the notation* $\Delta\langle\Gamma\rangle$ *indicates that the family* Γ *is "substituted" into the family* Δ *instead of the formula* A*) having its general element given by*

$$(f\hat{A}g)_{\delta\langle\gamma\rangle b} = \bigvee_{a \in A} f_{\gamma a} \circ g_{\delta\langle a\rangle b},$$

where $\delta\langle a\rangle = \{b_j \in B_j \mid b_{j_0} = a\}$, $\delta\langle\gamma\rangle = \{b_j \in B_j \mid b_{j_0} = \gamma\}$, $b \in B$. *(The notation* $f\hat{A}g$ *indicates the "orientation" of composition where* f *is substituted into* g.*)*

It is clear that in the case when $\Gamma = \{A_1\}$ and $\Delta\langle A\rangle = A$ Q-multimatrices and their composites reduce to Q-matrices and their composites.

DEFINITION 2.4. *A* Q*-formula is a formula* A *together with a* Q*-matrix* $X\colon A \to A$ *which is idempotent, i.e.,* $X \circ X = X$. *This* Q*-matrix is called the graduation of a* Q*-formula. (To economize on brackets, we shall write* A *for the (graduated)* Q*-formula* (A, X).*)*

It is clear that a formula A together with the unit Q-matrix $1_A\colon A \to A$ (when Q is unital) forms a Q-formula.

DEFINITION 2.5. *Let* $\Gamma (= \{A_i\}) \vdash A$ *be a sequent. A (graded)* Q*-sequent* $f\colon \Gamma \to A$ *is a* Q*-multimatrix from* Γ *to* A *(forgetting graduations) such that* $X_i \circ f = f = f \circ X$ *for every* i, *where* X_i *and* X *are graduations of formulas* A_i *and* A, *respectively. Composites of underlying* Q*-multimatrices of* Q*-sequents will be called cuts.*

It is clear that any sequent $\Gamma(=\{A_i\}) \vdash A$ is a Q-sequent (putting $X = 1_A, X_i = 1_{A_i}$ for all i). Note that the concept of graded sequents (graded consequence relations) was proposed by M.K. Chakraborty [1] as a certain fuzzy subset of Seq with values in some lattice. In our setting the role of this lattice could play Q-multimatrices.

3. Multicategories and multiquantaloids

We will see that Q-sequents together with cuts form a multicategory in the sense of J. Lambek.

DEFINITION 3.1. *([2]) Let $\mathcal{M}$ be a class of "objects" and let $\bar{\mathcal{M}}$ be the free monoid generated by $\mathcal{M}$ (its elements are strings $\Gamma = A_1 \dots A_n$ of objects, where n may be zero, in which case Γ is the empty string, which will be denoted by a blank) together with two mappings as follows:*

(i) *A mapping assigning to each string $\Gamma \in \bar{\mathcal{M}}$ and each $A \in \mathcal{M}$ a set $\mathcal{M}(\Gamma, A)$ in this set is called a morphism (in the original text multiarrow or arrow) $f\colon \Gamma \to A$ of $\mathcal{M}$, with domain Γ and codomain A. Each such morphism has a unique domain and a unique codomain.*

(ii) *A mapping assigning to each triple (Γ, Δ, Θ) of strings of objects of $\mathcal{M}$ a map $\mathcal{M}(\Gamma, A) \times \mathcal{M}(\Delta A \Theta, B) \to \mathcal{M}(\Delta \Gamma \Theta, B)$. For morphisms $f\colon \Gamma \to A$ and $g\colon \Delta A \Theta \to B$, this mapping is written as $(f, g) \mapsto f\hat{A}g$ and the morphism $f\hat{A}g\colon \Delta\Gamma\Theta \to B$ will here be called the cut (of orientation A) of f with g.*

The class $\mathcal{M}$ with these two mappings is called a multicategory when the following axioms hold:

Associativity: *If $f\colon \Gamma \to A$, $g\colon \Delta A \Theta \to B$ and $h\colon \Phi B \Psi \to C$ are morphisms of $\mathcal{M}$ (with indicated domains, codomains and orientations of cuts), then*

$$(f\hat{A}g)\hat{B}h = f\hat{A}(g\hat{B}h).$$

Commutativity: *If $f\colon \Gamma \to A$, $g\colon \Delta \to B$, $h\colon \Phi A \Theta B \Psi \to C$ are morphisms of $\mathcal{M}$, then*

$$f\hat{A}(g\hat{B}h) = g\hat{B}(f\hat{A}h).$$

Identity: *For each object A of $\mathcal{M}$ there exists a morphism $1_A\colon A \to A$ (called the identity morphism of $\mathcal{M}$) such that*

$$f\colon \Gamma \to A \Rightarrow f\hat{A}1_A = f; \quad g\colon \Gamma A \Delta \to B \Rightarrow 1_A\hat{A}g = g.$$

Observe that the axioms for a multicategory are much like the axioms for a category, except that morphisms $f\colon C \to A$ have been replaced by morphisms $f\colon \Gamma \to A$ (where Γ may be empty) and that for the cut of morphisms f and g in a multicategory we have non-single posibility but maybe a lot of choices for orientations in the domain of g. In the following we will take a freedom to modify the terminology proposed by J. Lambek – instead of strings of objects we will take arbitrary families of objects.

PROPOSITION 3.2. *Let Q be a commutative quantale. Then Q-sequents form a multicategory Q–Seq: its objects are Q-formulas, its morphisms are Q-sequents, and its identity morphisms are graduations of Q-formulas. (In the case when Q is non-commutative the axiom of commutativity of Definition 3.1 is not valid. Then Q–Seq satisfies only two axioms: Associativity and Identity.)*

Proof. Verifying each of the axioms in turn, we argue as follows. The axiom of associativity: for any Q-sequents $f\colon \Gamma \to A$, $g\colon \Delta\langle A\rangle \to B$ and $h\colon \Phi\langle B\rangle \to C$, $(f\hat{A}g)\hat{B}h = f\hat{A}(g\hat{B}h)$ is satisfied, because for each families γ, $\delta\langle A\rangle$, $\phi\langle B\rangle$ of proofs of families Γ, $\Delta\langle A\rangle$, $\Phi\langle B\rangle$ of Q-formulas, respectively, and all $c \in C$, obviously, the equality

$$\bigvee_{b\in B}\Big(\bigvee_{a\in A} f_{\gamma a}\circ g_{\delta\langle a\rangle b}\Big)\circ h_{\phi\langle b\rangle c} = \bigvee_{a\in A} f_{\gamma a}\circ\Big(\bigvee_{b\in B} g_{\delta\langle a\rangle b}\circ h_{\phi\langle b\rangle c}\Big)$$

holds. The axiom of commutativity: for any $f\colon \Gamma \to A$, $g\colon \Delta \to B$, $h\colon \Phi\langle A\rangle\langle B\rangle \to C$, $f\hat{A}(g\hat{B}h) = g\hat{B}(f\hat{A}h)$ is satisfied because, by the commutativity of Q, for each families γ, δ, $\phi\langle A\rangle\langle B\rangle$ of proofs of families Γ, Δ, $\Phi\langle A\rangle\langle B\rangle$ of formulas, respectively, and each $c \in C$, the equality

$$\bigvee_{a\in A} f_{\gamma a}\circ\Big(\bigvee_{b\in B} g_{\delta b}\circ h_{\phi\langle a\rangle\langle b\rangle c}\Big) = \bigvee_{b\in B} g_{\delta b}\circ\Big(\bigvee_{a\in A} f_{\gamma a}\circ h_{\phi\langle a\rangle\langle b\rangle c}\Big)$$

holds. Finally, let $X\colon A \to A$ be the graduation of a Q-formula A. Then it is easy to check that it is the identity morphism of A. The first part of the axiom of identity: if $f\colon \Gamma \to A$, then $f\hat{A}X = f$, because, by Definition 2.5, for all $\gamma \in \Gamma$, $a \in A$, $\bigvee_{a'\in A} f_{\gamma a'}\circ x_{a'a} = f_{\gamma a}$. Finally, the second part of the axiom of identity: if $g\colon \Gamma\langle A\rangle \to B$, then $X\hat{A}g = g$ holds, since, for all $a \in A$, $\gamma\langle a\rangle \in \Gamma\langle A\rangle$, $b \in B$, $\bigvee_{a'\in A} x_{aa'}\circ g_{\gamma\langle a'\rangle b} = g_{\gamma\langle a\rangle b}$.

Let us now confine attention to important additional properties of the multicategory Q–Seq.

PROPOSITION 3.3. *For every family Γ of objects of Q–Seq and for every object A of Q–Seq, the "hom-set" Q–Seq(Γ, A) of all morphisms from Γ to A is a complete lattice. Moreover, the cut of Q–Seq preserves arbitrary joins in both sides: for all objects A, B, for all families Γ, $\Delta\langle A\rangle$ of objects of Q–Seq, for all morphisms $f\colon \Gamma \to A$, $g\colon \Delta\langle A\rangle \to B$ and for all families $\{f_i\colon \Gamma \to A\}$ and $\{g_i\colon \Delta\langle A\rangle \to B\}$ of morphisms of Q–Seq,*

$$f\hat{A}\bigvee_i g_i = \bigvee_i (f\hat{A}g_i) \quad \text{and} \quad \Big(\bigvee_i f_i\Big)\hat{A}g = \bigvee_i (f_i\hat{A}g). \tag{1}$$

Proof. Since Q is a complete lattice, it follows that every Q–Seq(Γ, A) is also a complete lattice for the pointwise partial order: if $\{f_i\colon \Gamma \to A\}$ is a family of Q-

sequents then the relations

$$\Big(\bigvee_i f_i\Big)_{\gamma a} = \bigvee_i (f_i)_{\gamma a} \quad \text{and} \quad \Big(\bigwedge_i f_i\Big)_{\gamma a} = \bigwedge_i (f_i)_{\gamma a}$$

define Q-sequents $\bigvee_i f_i$ and $\bigwedge_i f_i$, respectively. Since cut of Q-sequents is defined in terms of *joins* and $\circ$, it will preserve *joins* of Q-sequents in each variable: for every formula A, for all $\gamma \in \Gamma$, $\delta\langle A\rangle \subseteq \Delta\langle A\rangle$, $b \in B$,

$$\bigvee_{a\in A} f_{\gamma a} \circ \Big(\bigvee_i g_i\Big)_{\delta\langle a\rangle b} = \bigvee_i \Big(\bigvee_{a\in A} f_{\gamma a} \circ (g_i)_{\delta\langle a\rangle b}\Big)$$

and

$$\bigvee_{a\in A} \Big(\bigvee_i f_i\Big)_{\gamma a} \circ g_{\delta\langle a\rangle b} = \bigvee_i \Big(\bigvee_{a\in A} (f_i)_{\gamma a} \circ g_{\delta\langle a\rangle b}\Big),$$

i.e., (1) holds. This proves the assertion.

We generalize these properties of Q–Seq to arbitrary multicategories.

DEFINITION 3.4. *A multiquantaloid is a multicategory such that its hom-sets are complete lattices and its cut preserves arbitrary joins in both variables, i.e.,* (1) *holds. In the absence of commutativity in the sence of Definition 3.1 (e.g., Q–Seq in the case when Q is noncommutative), I prefer to speak of "noncommutative" multiquantaloids.)*

Note that multiquantaloids generalize the notion of quantaloid, a category whose hom-sets are complete lattices and composition preserves arbitrary joins in both variables [5].

References

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REZIUMĖ

R.P. Gylys. Multikvantaloidai

Apibrėžiama nauja multikvantaloido sąvoka. Nusakome multikvantaloidą kaip multikategoriją, kurios morfizmai sudaro pilnas gardeles, o pjūviai išsaugo bet kokius supremumus. Iš vienos pusės multikvantaloidai apibendrina kvantaloido sąvoką, o iš kitos jų morfizmai bei jų pjūviai turi loginius atitikmenis.