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On the fourth moment of the periodic zeta-function

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Let $s=\sigma+it$ be a complex variable, and let $\mathfrak{A}=\{a_m, m\in\mathbb{Z}\}$ be a sequence of complex numbers with period $k>0$. The periodic zeta-function $\zeta(s;\mathfrak{A})$, for $\sigma>1$, is defined by

$$\zeta(s;\mathfrak{A})=\sum_{m=1}^{\infty}\frac{a_m}{m^s},$$

and by analytic continuation elsewhere. In [3], [4], [5] the asymptotics for the mean square

$$\int_0^T\left|\zeta(\sigma+it,\mathfrak{A})\right|^2\mathrm{d}t,\quad T\to\infty,$$

in the region $\frac{1}{2}\leqslant\sigma\leqslant 1$ was obtained. The aim of this note is to obtain the estimate for

$$I_2\Big(T,\frac{1}{2}\Big)=\int_0^T\left|\zeta\Big(\frac{1}{2}+it,\mathfrak{A}\Big)\right|^4\mathrm{d}t.$$

Denote by B a quantity bounded by a constant. Let $K(k)=\sum_{q=1}^{k}|a_q|^2$ and $\varphi(k)$, as usual, denote the Euler function.

THEOREM. *Let k be a prime number, and $T\to\infty$. Then*

$$I_2\Big(T,\frac{1}{2}\Big)=Bk^3K^2(k)T\log^4 T.$$

Proof. Let χ,χ_1,χ_2 denote Dirichlet characters modulo k, $L(s,\chi)$, $L(s,\chi_1)$, $L(s,\chi_2)$ be corresponding Dirichlet L-functions, and let $\zeta(s)$ be the Riemann zeta-function. Then in [6] it was obtained that

$$\zeta^2(s,\mathfrak{A})=\frac{1}{\varphi^2(k)}\sum_{q_1=1}^{k-1}\sum_{q_2=1}^{k-1}\sum_{\chi_1(\mathrm{mod}k)}\sum_{\chi_2(\mathrm{mod}k)}\overline{\chi}_1(q_1)\overline{\chi}_2(q_2)a_{q_1}a_{q_2}L(s,\chi_1)L(s,\chi_2)$$
$$+\frac{2a_k\zeta(s)}{k^s\varphi(k)}\sum_{q=1}^{k-1}\sum_{\chi(\mathrm{mod}k)}\overline{\chi}(q)a_qL(s,\chi)+\frac{a_k^2}{k^{2s}}\zeta^2(s).$$

Hence we find

$$
\begin{aligned}
&\left|\zeta(s,\mathfrak{A})\right|^4 = \zeta^2(s,\mathfrak{A})\overline{\zeta^2(s,\mathfrak{A})} \\
&= \Bigg(\frac{1}{\varphi^2(k)}\sum_{q_1=1}^{k-1}\sum_{q_2=1}^{k-1}\sum_{\chi_1(\mathrm{mod}k)}\sum_{\chi_2(\mathrm{mod}k)} \overline{\chi}_1(q_1)\overline{\chi}_2(q_2)a_{q_1}a_{q_2}L(s,\chi_1)L(s,\chi_2) \\
&\quad + \frac{2a_k\zeta(s)}{k^s\varphi(k)}\sum_{q=1}^{k-1}\sum_{\chi(\mathrm{mod}k)} \overline{\chi}(q)a_qL(s,\chi) + \frac{a_k^2}{k^{2s}}\zeta^2(s)\Bigg) \\
&\quad \times \Bigg(\frac{1}{\varphi^2(k)}\sum_{q_1=1}^{k-1}\sum_{q_2=1}^{k-1}\sum_{\chi_1(\mathrm{mod}k)}\sum_{\chi_2(\mathrm{mod}k)} \chi_1(q_1)\chi_2(q_2)\overline{a}_{q_1}\overline{a}_{q_2}\overline{L(s,\chi_1)L(s,\chi_2)} \\
&\quad + \frac{2\overline{a}_k\overline{\zeta(s)}}{k^{\overline{s}}\varphi(k)}\sum_{q=1}^{k-1}\sum_{\chi(\mathrm{mod}k)} \chi(q)\overline{a}_q\overline{L(s,\chi)} + \frac{\overline{a_k^2}\overline{\zeta^2(s)}}{k^{2\overline{s}}}\Bigg) \\
&= \frac{1}{\varphi^4(k)}\left|\sum_{q_1=1}^{k-1}\sum_{q_2=1}^{k-1}\sum_{\chi_1(\mathrm{mod}k)}\sum_{\chi_2(\mathrm{mod}k)} \overline{\chi}_1(q)\overline{\chi}_2(q)a_{q_1}a_{q_2}L(s,\chi_1)L(s,\chi_2)\right|^2 \\
&\quad + \frac{4|a_k|^2|\zeta(s)|^2}{k^{2\sigma}\varphi^2(k)}\left|\sum_{q=1}^{k-1}\sum_{\chi(\mathrm{mod}k)} \overline{\chi}(q)a_qL(s,\chi)\right|^2 + \frac{|a_k|^4|\zeta(s)|^4}{k^{4\sigma}} \\
&\quad + \frac{2\overline{a}_k\overline{\zeta(s)}}{k^{\overline{s}}\varphi^3(k)}\sum_{q_1=1}^{k-1}\sum_{q_2=1}^{k-1}\sum_{\chi_1(\mathrm{mod}k)}\sum_{\chi_2(\mathrm{mod}k)} \overline{\chi}_1(q_1)\overline{\chi}(q_2)a_{q_1}a_{q_2} \\
&\quad \times L(s,\chi_1)L(s,\chi_2)\sum_{q=1}^{k-1}\sum_{\chi(\mathrm{mod}k)} \chi(q)\overline{a}_q\overline{L(s,\chi)} \\
&\quad + \frac{\overline{a_k^2}\overline{\zeta^2(s)}}{k^{2\overline{s}}\varphi^2(k)}\sum_{q_1=1}^{k-1}\sum_{q_2=1}^{k-1}\sum_{\chi_1(\mathrm{mod}k)}\sum_{\chi_2(\mathrm{mod}k)} \overline{\chi}_1(q_1)\overline{\chi}_2(q_2)a_{q_1}a_{q_2}L(s,\chi_1)L(s,\chi_2) \\
&\quad + \frac{2a_k\zeta(s)}{k^2\varphi^3(k)}\sum_{q=1}^{k-1}\sum_{\chi(\mathrm{mod}k)} \overline{\chi}(q)a_qL(s,\chi) \\
&\quad \times \sum_{q_1=1}^{k-1}\sum_{q_2=1}^{k-1}\sum_{\chi_1(\mathrm{mod}k)}\sum_{\chi_2(\mathrm{mod}k)} \chi_1(q_1)\chi_2(q_2)\overline{a}_{q_1}\overline{a}_{q_2}\overline{L(s,\chi_1)}\ \overline{L(s,\chi_2)} \\
&\quad + \frac{2a_k\zeta(s)\overline{\zeta^2(s)}\overline{a_k^2}}{k^sk^{2\overline{s}}\varphi(k)}\sum_{q=1}^{k-1}\sum_{\chi(\mathrm{mod}k)} \overline{\chi}(q)a_qL(s,\chi) \\
&\quad + \frac{a_k^2\zeta^2(s)}{\varphi^2(k)k^{2s}}\sum_{q_1=1}^{k-1}\sum_{q_2=1}^{k-1}\sum_{\chi_1(\mathrm{mod}k)}\sum_{\chi_2(\mathrm{mod}k)} \chi_1(q_1)\chi_2(q_2)\overline{a}_{q_1}\overline{a}_{q_2}\overline{L(s,\chi_1)}\ \overline{L(s,\chi_2)}
\end{aligned}
$$

$$+\frac{2\overline{a}_k a_k^2\overline{\zeta(s)}\zeta^2(s)}{k^{\overline{s}}k^{2s}\varphi(k)}\sum_{q=1}^{k-1}\sum_{\chi(\mathrm{mod}k)}\chi(q)\overline{a}_q\overline{L(s,\chi)}.$$

Therefore

$$\int_0^T\left|\zeta\left(\frac{1}{2}+it,\mathfrak{A}\right)\right|^4\mathrm{d}t=\frac{Bk^2}{\varphi^2(k)}\sum_{q_1=1}^{k-1}\sum_{q_2=1}^{k-1}\sum_{\chi_1(\mathrm{mod}k)}\sum_{\chi_2(\mathrm{mod}k)}|a_{q_1}|^2|a_{q_2}|^2$$
$$\times\int_0^T\left|L\left(\frac{1}{2}+it,\chi_1\right)\right|^2\left|L\left(\frac{1}{2}+it,\chi_2\right)\right|^2\mathrm{d}t+\frac{B|a_k|^2}{\varphi(k)}\sum_{q=1}^{k-1}\sum_{\chi(\mathrm{mod}k)}|a_q|^2$$
$$\times\int_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\left|L\left(\frac{1}{2}+it,\chi\right)\right|^2\mathrm{d}t+\frac{B|a_k|^4}{k^2}\int_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^4\mathrm{d}t$$
$$+\frac{B|a_k|}{k^{\frac{1}{2}}\varphi^3(k)}\sum_{q=1}^{k-1}\sum_{q_2=1}^{k-1}\sum_{q=1}^{k-1}|a_{q_1}||a_{q_2}||a_q|\sum_{\chi(\mathrm{mod}k)}\sum_{\chi_1(\mathrm{mod}k)}\sum_{\chi_2(\mathrm{mod}k)}\int_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|$$
$$\times\left|L\left(\frac{1}{2}+it,\chi\right)\right|\left|L\left(\frac{1}{2}+it,\chi_1\right)\right|\left|L\left(\frac{1}{2}+it,\chi_2\right)\right|\mathrm{d}t$$
$$+\frac{B|a_k|^2}{k\varphi^2(k)}\sum_{q_1=1}^{k-1}\sum_{q_2=1}^{k-1}|a_{q_1}||a_{q_2}|\sum_{\chi_1(\mathrm{mod}k)}\sum_{\chi_2(\mathrm{mod}k)}\int_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^2$$
$$\times\left|L\left(\frac{1}{2}+it,\chi_1\right)\right|\left|L\left(\frac{1}{2}+it,\chi_2\right)\right|\mathrm{d}t$$
$$+\frac{B|a_k|}{k^{\frac{1}{2}}\varphi^3(k)}\sum_{q=1}^{k-1}\sum_{q_1=1}^{k-1}\sum_{q_2=1}^{k-1}|a_q||a_{q_1}||a_{q_2}|\sum_{\chi(\mathrm{mod}k)}\sum_{\chi_1(\mathrm{mod}k)}\sum_{\chi_2(\mathrm{mod}k)}\int_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|$$
$$\times\left|L\left(\frac{1}{2}+it,\chi\right)\right|\left|L\left(\frac{1}{2}+it,\chi_1\right)\right|\left|L\left(\frac{1}{2}+it,\chi_2\right)\right|\mathrm{d}t$$
$$+\frac{B|a_k|^3}{k^{\frac{3}{2}}\varphi(k)}\sum_{q=1}^{k-1}|a_q|\sum_{\chi(\mathrm{mod}k)}\int_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^3\left|L\left(\frac{1}{2}+it,\chi\right)\right|\mathrm{d}t$$
$$+\frac{B|a_k|^2}{k\varphi^2(k)}\sum_{q_1=1}^{k-1}\sum_{q_2=1}^{k-1}|a_{q_1}||a_{q_2}|\sum_{\chi_1(\mathrm{mod}k)}\sum_{\chi_2(\mathrm{mod}k)}\int_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^2$$
$$\times\left|L\left(\frac{1}{2}+it,\chi_1\right)\right|\left|L\left(\frac{1}{2}+it,\chi_2\right)\right|\mathrm{d}t$$
$$+\frac{B|a_q|^3}{k^{\frac{3}{2}}\varphi(k)}\sum_{q=1}^{k-1}|a_q|\sum_{\chi(\mathrm{mod}k)}\left|\zeta\left(\frac{1}{2}+it\right)\right|^3\left|L\left(\frac{1}{2}+it,\chi\right)\right|\mathrm{d}t. \tag{1}$$

By the Cauchy-Schwarz inequality we have

$$\int_0^T \left|L\left(\frac{1}{2}+it,\chi_1\right)\right|^2 \left|L\left(\frac{1}{2}+it,\chi_2\right)\right|^2 \mathrm{d}t \leqslant \left(\int_0^T \left|L\left(\frac{1}{2}+it,\chi_1\right)\right|^4 \mathrm{d}t \int_0^T \left|L\left(\frac{1}{2}+it,\chi_2\right)\right|^4 \mathrm{d}t\right)^{\frac{1}{2}},$$

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \left|L\left(\frac{1}{2}+it,\chi\right)\right|^2 \mathrm{d}t \leqslant \left(\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \mathrm{d}t \int_0^T \left|L\left(\frac{1}{2}+it,\chi\right)\right|^4 \mathrm{d}t\right)^{\frac{1}{2}},$$

$$\begin{aligned}\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right| \left|L\left(\frac{1}{2}+it,\chi\right)\right| \left|L\left(\frac{1}{2}+it,\chi_1\right)\right| \left|L\left(\frac{1}{2}+it,\chi_2\right)\right| \mathrm{d}t \\ \leqslant \left(\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \left|L\left(\frac{1}{2}+it,\chi\right)\right|^2 \mathrm{d}t \int_0^T \left|L\left(\frac{1}{2}+it,\chi_1\right)\right|^2 \left|L\left(\frac{1}{2}+it,\chi_2\right)\right| \mathrm{d}t\right)^{\frac{1}{2}} \\ \leqslant \left(\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \mathrm{d}t \int_0^T \left|L\left(\frac{1}{2}+it,\chi\right)\right|^4 \mathrm{d}t \right. \\ \left. \times \int_0^T \left|L\left(\frac{1}{2}+it,\chi_1\right)\right|^2 \mathrm{d}t \int_0^T \left|L\left(\frac{1}{2}+it,\chi_2\right)\right|^4 \mathrm{d}t\right)^{\frac{1}{4}},\end{aligned}$$

$$\begin{aligned}\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \left|L\left(\frac{1}{2}+it,\chi_1\right)\right| \left|L\left(\frac{1}{2}+it,\chi_2\right)\right| \mathrm{d}t \\ \leqslant \left(\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \mathrm{d}t \int_0^T \left|L\left(\frac{1}{2}+it,\chi_1\right)\right|^2 \left|L\left(\frac{1}{2}+it,\chi_2\right)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}} \\ \leqslant \left(\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^T \left|L\left(\frac{1}{2}+it,\chi_1\right)\right|^4 \mathrm{d}t \int_0^T \left|L\left(\frac{1}{2}+it,\chi_2\right)\right|^4 \mathrm{d}t\right)^{\frac{1}{4}},\end{aligned}$$

$$\begin{aligned}\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^3 \left|L\left(\frac{1}{2}+it,\chi\right)\right| \mathrm{d}t \\ \leqslant \left(\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \mathrm{d}t \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \left|L\left(\frac{1}{2}+it,\chi\right)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}} \\ \leqslant \left(\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \mathrm{d}t \int_0^T \left|L\left(\frac{1}{2}+it,\chi\right)\right|^4 \mathrm{d}t\right)^{\frac{1}{4}}. \qquad (2)\end{aligned}$$

Let $K(k) = \sum_{q=1}^{k} |a_q|^2$. Moreover, is well known that

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \mathrm{d}t = BT\log^4 T,$$

and

$$\int_0^T \left|L\left(\frac{1}{2}+it,\chi\right)\right|^4 \mathrm{d}t = BTk\log^4 T.$$

Hence and from (1) and (2) we find

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it,\mathfrak{A}\right)\right|^4 \mathrm{d}t = Bk^3K^2(k)T\log^4 T + B|a_k|^2k^{\frac{1}{2}}K(k)T\log^4 T$$

$$+ \frac{B|a_k|^4}{k^2}T\log^4 T + B|a_k|k^{\frac{7}{4}}K^{\frac{3}{2}}(k)T\log^4 T$$

$$+ B|a_k|^2k^{\frac{1}{2}}K(k)T\log^4 T + B|a_k|k^{\frac{7}{4}}K^{\frac{3}{2}}(k)T\log^4 T$$

$$+ \frac{B|a_k|^3}{k^{\frac{3}{4}}}K^{\frac{1}{2}}(k)T\log^4 T + B|a_k|^2k^{\frac{1}{2}}K(k)T\log^4 T$$

$$+ \frac{B|a_k|^3}{k^{\frac{3}{4}}}K^{\frac{1}{2}}(k)T\log^4 T.$$

This proves the theorem.

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REZIUMĖ

D. Šiaučiūnas, A.A. Laurutis. Apie periodinės dzeta funkcijos ketvirtąjį momentą

Straipsnyje gautas periodinės dzeta funkcijos ketvirtojo momento įvertis kritinėje tiesėje, kai periodo ilgis k yra pirminis skaičius.