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Discrete limit theorem for general Dirichlet series in the space of meromorphic function

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1. Introduction

Denote by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all positive integers, of all real numbers, and all complex numbers, respectively. Let $s = \sigma + it$ be a complex variable, $\{\lambda_m\colon m \in \mathbb{N}\}$ be an increasing sequence of positive numbers such that $\lim_{m\to\infty} \lambda_m = +\infty$, and let $\{a_m\colon m \in \mathbb{N}\}$ be a sequence of complex numbers. The series

$$\sum_{m=1}^{\infty} a_m \mathrm{e}^{-\lambda_m s} \tag{1}$$

is called a general Dirichlet series with coefficients a_m and exponents λ_m. If $\lambda_m = \log m$, then we have the ordinary Dirichlet series.

Suppose that series (1) absolutely converges for $\sigma > \sigma_a$ and has a sum $f(s)$. In the majority of classical cases the function $f(s)$ is analytically or meromorphically continuable to the whole complex plane. In [4] we began the investigation of the discrete value-distribution of series (1) by probabilistic methods and we proved for it limit theorems in the sense of the weak convergence of probability measures on the complex plane. Discrete limit theorems for $f(s)$ in the space of analytic functions were obtained in [6].

Let, for positive integer N,

$$\mu_N(\ldots) = \frac{1}{N+1}\#\{0 \leqslant m \leqslant N\colon \ldots\},$$

where in place of dots a condition satisfied by m is to be written. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S.

The aim of this paper is to obtain a discrete limit theorem for the function $f(s)$ on the left of the line $\sigma = \sigma_a$. Suppose that the function $f(s)$ is meromorphically continuable to the region $\sigma > \sigma_1$ with some $\sigma_1 < \sigma_a$, all poles in this region are included in a compact set. Denote by B a number (not always the same) bounded by a constant. Moreover, we assume that, for $\sigma > \sigma_1$, the estimates

$$f(s) = B|t|^{\alpha}, \quad |t| \geqslant t_0,\ \alpha > 0, \tag{2}$$

where t_0 is a fixed positive number, and

$$\int_0^T \left|f(\sigma+it)\right|^2 \mathrm{d}t = BT, \quad T \to \infty, \tag{3}$$

are satisfied.

Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, and let $d(s_1, s_2)$ be a metric given by the formulae

$$d(s_1,s_2) = \frac{2|s_1-s_2|}{\sqrt{1+|s_1|^2}\sqrt{1+|s_2|^2}}, \quad d(s,\infty) = \frac{2}{\sqrt{1+|s|^2}}, \quad d(\infty,\infty) = 0,$$

where $s, s_1, s_2 \in \mathbb{C}$. This metric is compatible with the topology of $\mathbb{C}_\infty$. Let $D = \{s \in \mathbb{C}\colon \sigma > \sigma_1\}$. Denote by $M(D)$ the space of meromorphic on D functions $g\colon D \to (\mathbb{C}_\infty, d)$ equipped with the topology of uniform convergence on compacta. In this topology, a sequence $\{g_n\colon g_n \in M(D)\}$ converges to the function $g \in M(D)$ if

$$d\big(g_n(s), g(s)\big) \to 0, \quad n \to \infty,$$

uniformly on compact subsets of $\mathbb{C}$. Consider the weak convergence of the probability measure

$$P_N(A) = \mu_N\big(f(s+imh) \in A\big), \quad A \in \mathcal{B}(M(D)).$$

Then we have the following statement.

THEOREM. *Suppose that the function $f(s)$ satisfies conditions* (2) *and* (3). *Then there exists a probability measure P on $(M(D), \mathcal{B}(M(D)))$ such that the measure P_N converges weakly to P as $N \to \infty$.*

2. Auxiliary results

Denote by $s_1, \dots, s_r$ all poles of the function $f(s)$ lying in the half-plane $\sigma > \sigma_1$. Since all poles are included in compact set, we have that the number r is finite. Let

$$f_1(s) = \prod_{j=1}^{r} \big(1 - \mathrm{e}^{\lambda_1(s_j - s)}\big).$$

Then, clearly, $f_1(s)$ is a Dirichlet polynomial, and $f_1(s_j) = 0$ for $j = 1, \dots, r$. We can write $f_1(s) = \sum_{m=1}^{r} b_m \mathrm{e}^{-\lambda_1 m s}$. Therefore, by Theorem 2 of [5] there exists a probability measure P' on $(H(D), \mathcal{B}(H(D)))$ such that the measure

$$P_{N,f_1}(A) = \mu_N\big(f_1(s+imh) \in A\big), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to P' as $N \to \infty$.

Moreover, let $f_2(s) = f_1(s) f(s)$. Then we have, that the function $f_2(s)$ is regular on D, and for $\sigma > \sigma_a$, we have

$$f_2(s) = \prod_{j=1}^{r} \big(1 - \mathrm{e}^{\lambda_1(s_j - s)}\big) \sum_{m=1}^{\infty} a_m \mathrm{e}^{-\lambda_m s} = \sum_{j=0}^{r} \sum_{m=1}^{\infty} a_{mj} \mathrm{e}^{-(\lambda_m + j\lambda_1)s}$$

with some coefficients a_{mj} satisfying $a_{mj} = B|a_m|$ for $m \in \mathbb{N}$ and $j = 0, 1, \dots, r$. On account of (2) and (3), we have, for $\sigma > \sigma_1$,

$$f_2(s) = B|t|^{\alpha}, \quad |t| \geqslant t_0,\ \alpha > 0, \tag{4}$$

and

$$\int_0^T \left|f_2(\sigma + it)\right|^2 \mathrm{d}t = BT, \quad T \to \infty. \tag{5}$$

Thus, now we have a similar but more general situation as in [4] and [6], and applying a similar method, we can prove the following lemma.

LEMMA 1. *There exists a probability measure P'' on $(H(D), \mathcal{B}(H(D)))$ such that the probability measure*

$$P_{N,f_2}(A) = \mu_N\big(f_2(s + imh) \in A\big), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to P'' as $N \to \infty$.

We begin with a limit theorem for the Dirichlet polynomial

$$p_n(s) = \sum_{j=0}^{r} \sum_{m=1}^{n} a_{mj} \mathrm{e}^{-(\lambda_m + j\lambda_1)s}. \tag{6}$$

LEMMA 2. *There exists a probability measure P_{p_n} on $(H(D), \mathcal{B}(H(D)))$ such that the probability measure*

$$P_{N,p_n}(A) = \mu_N\big(p_n(s + imh) \in A\big), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to P_{p_n} as $N \to \infty$.

Proof. The lemma is obtained in the same way as Theorem 2 in [4].

Now we will approximate the function $f_2(s)$ by absolutely convergent Dirichlet series in the mean. Let $\sigma_2 = \sigma_a - \sigma_1$. For $\sigma \in [-\sigma_2, \sigma_2]$ define

$$l_n(s) = \frac{s}{\sigma_2}\Gamma\left(\frac{s}{\sigma_2}\right)\mathrm{e}^{(\lambda_n + j\lambda_1)s}.$$

Clearly, $\sigma_2 > 0$. For $\sigma > \sigma_1$ we consider the function

$$g_n(s) = \frac{1}{2\pi i}\int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} f_2(s + z) l_n(z) \frac{\mathrm{d}z}{z}.$$

LEMMA 3. *The function $g_n(s)$ has the expansion*

$$g_n(s) = \sum_{j=0}^{r}\sum_{m=1}^{\infty} a_{mj} \exp\left\{ -\mathrm{e}^{-(\lambda_m - \lambda_n)\sigma_2}\right\} \mathrm{e}^{-(\lambda_m + j\lambda_1)s},$$

the series being absolutely convergent for $\sigma > \sigma_1$.

Proof. The lemma is obtained in the same way as Lemma 8 in [3].

LEMMA 4. *Let K be a compact subset of D. Then*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{m=0}^{N}\sup_{s\in K}\left|f_2(s+imh)-g_n(s+imh)\right|=0.$$

Proof of this lemma is similar to that of Lemma 6 in [4].

Lemma 3 shows that $g_n(s)$ is an analytic function on D. Let

$$P_{N,n}(A)=\mu_N\bigl(g_n(s+imh)\in A\bigr),\quad A\in\mathcal{B}(H(D)).$$

To investigate the weak convergence of these measures we need a metric on $H(D)$ which induces its topology. It is known , see, for example, Lemma 1.7.1 of [2], that there exists a sequence $\{K_n\}$ of compact subsets of D such that $D=\cup_{n=1}^{\infty}K_n$, $K_n\subset K_{n+1}$, and if K is a compact subset of D, then $K\subseteq K_n$ for some n. Then

$$\varrho(f,g)=\sum_{n=1}^{\infty}\frac{\varrho_n(f,g)}{1+\varrho_n(f,g)},\quad f,g\in H(D),$$

where

$$\varrho_n(f,g)=\sup_{s\in K_n}\left|f(s)-g(s)\right|$$

is a metric on $H(D)$ which induces its topology.

LEMMA 5. *There exists a probability measure P_n on $(H(D),\mathcal{B}(H(D)))$ such that the measure $P_{N,n}$ converges weakly to P_n as $N\to\infty$.*

Proof. The lemma is obtained in the same way as Lemma 10 in [3].

Proof of Lemma 1. We will use the same reasoning as in the proof of Lemma 5. Taking into account Lemma 4, by the Chebyshev inequality we find that for every $\varepsilon>0$

$$\lim_{n\to\infty}\limsup_{N\to\infty}\mu_N\bigl(\varrho(f_2(s+imh),g_n(s+imh))\geqslant\varepsilon\bigr)$$
$$\leqslant\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{(N+1)\varepsilon}\sum_{m=0}^{N}\varrho\bigl(f_2(s+imh),g_n(s+imh)\bigr)=0. \tag{7}$$

Let θ_N be a random variable defined on a certain probability space $(\Omega_0,\mathcal{B}(\Omega_0),\mathbb{P})$ with values mh and distribution $\mathbb{P}(\theta_N=mh)=\frac{1}{N+1}$, $m=0,1,\dots,N$. Define

$$X_{N,n}(s)=g_n(s+i\theta_N)\quad\text{and}\quad Y_N(s)=f_2(s+i\theta_N).$$

Then we can write relation (7) in the form

$$\lim_{n\to\infty}\limsup_{N\to\infty}\mathbb{P}\bigl(\varrho(Y_N(s),X_{N,n}(s))\geqslant\varepsilon\bigr)=0. \tag{8}$$

Denote by $\xrightarrow{\mathcal{D}}$ the convergence in distribution. From Lemma 5 we have

$$X_{N,n}\underset{N\to\infty}{\xrightarrow{\mathcal{D}}}X_n, \tag{9}$$

where X_n is an $H(D)$-valued random element with distribution P_n. Using relation (9), similarly to the proof of Lemma 5 we obtain that the family of probability measures $\{P_n\}$ is tight, hence it is relatively compact. From this the existence of a subsequence $\{P_{n_1}\} \subset \{P_n\}$ which weakly converges to some measure P'' as $n_1 \to \infty$ follows. Since P_n is the distribution of the random element X_n, we have that

$$X_{n_1} \xrightarrow[n_1\to\infty]{\mathcal{D}} P''. \tag{10}$$

Now Theorem 4.2 of [1] together with (8), (9) and (10) imply

$$Y_N \xrightarrow[N\to\infty]{\mathcal{D}} P'',$$

i.e., P_{N,f_2} converges weakly to P'' as $N \to \infty$.

Define $H^2(D) = H(D) \times H(D)$, and let

$$P_{N,f_1,f_2}(A) = \mu_N\big((f_1(s+imh), f_2(s+imh)) \in A\big), \quad A \in \mathcal{B}(H^2(D)).$$

We have limit theorems for the functions $f_1(s)$ and $f_2(s)$ in the space $H(D)$. Now we will prove a joint discrete limit lemma for these functions.

LEMMA 6. *There exists a probability measure $\widetilde{P}$ on $(H^2(D), \mathcal{B}(H^2(D)))$ such that the measure P_{N,f_1,f_2} converges weakly to $\widetilde{P}$ as $N \to \infty$.*

For the proof of this lemma we need the following results. Let $p_n(s)$ is defined in (6).

LEMMA 7. *There exists a probability measure $\widetilde{P}_{f_1,p_n}$ on $(H^2(D), \mathcal{B}(H^2(D)))$ such that the probability measure*

$$P_{N,f_1,p_n}(A) = \mu_N\big((f_1(s+imh), p_n(s+imh)) \in A\big), \quad A \in \mathcal{B}(H^2(D)).$$

converges weakly to $\widetilde{P}_{f_1,p_n}$ as $N \to \infty$.

Proof. Let γ denote the unit circle on $\mathbb{C}$, i.e., $\gamma = \{s \in \mathbb{C}\colon\ |s| = 1\}$, and let

$$\Omega = \prod_{m=1}^{n} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}$. Define the function $v\colon \Omega_n \to H^2(D)$ by the formula

$$v(x_1,\dots,x_n) = \left(\sum_{m=0}^{r} b_m e^{-\lambda_1 ms} x_1^{-m}, \sum_{j=0}^{r}\sum_{m=1}^{n} a_{mj} e^{-(\lambda_m+j\lambda_1)s} x_1^{-j} x_m^{-1}\right),$$

$(x_1,\dots,x_n) \in \Omega_n$.

Now we can prove that the probability measure

$$\mu_N\big((e^{i\lambda_1 mh},\dots,e^{i\lambda_n mh}) \in A\big), \quad A \in \mathcal{B}(\Omega_n)$$

converges weakly to the some measure $\widehat{P}$. Proof of this proposition is similar to that of Lemma 1 in [5], only the limit measure $\widehat{P}$ is not necessarily the Haar measure.

By the definition of the function $v(x_1, \dots, x_n)$, we have

$$v(e^{i\lambda_1 mh}, \dots, e^{i\lambda_n mh}) = \bigl(f_1(s+imh), p_n(s+imh)\bigr).$$

The function v is continuous. Therefore, from this and in view of Theorem 5.1 from [1], we obtain that the measure P_{N,f_1,p_n} converges weakly to the measure $\widetilde{P}_{f_1,p_n} = \widehat{P}v^{-1}$. The lemma is proved.

LEMMA 8. *On $(H^2(D), \mathcal{B}(H^2(D)))$ there exists a probability measure $\widetilde{P}_n$ such that the measure*

$$P_{N,f_1,g_n}(A) = \mu_N\bigl((f_1(s+imh), g_n(s+imh)) \in A\bigr), \quad A \in \mathcal{B}(H^2(D)),$$

weakly converges to $\widetilde{P}_n$ as $N \to \infty$.

Proof of the lemma uses Lemma 7 and is similar to that of Lemma 5.

Proof of Lemma 6 follows in the same way from Lemma 8 as Lemma 1 follows from Lemma 5.

3. Proof of the Theorem

Define the function u: $H^2(D) \to M(D)$ by the formula

$$u(g_1, g_2) = \frac{g_2}{g_1}, \quad g_1, g_2 \in H(D).$$

The metric d satisfies the equality

$$d(g_1, g_2) = d\Bigl(\frac{1}{g_1}, \frac{1}{g_2}\Bigr).$$

Therefore, the function u is continuous. Hence, by Theorem 5.1 from [1] and Lemma 6, we obtain that the measure $P_{N,f_1,f_2}u^{-1}$ converges weakly to $\widetilde{P}u^{-1}$ as $N \to \infty$, i.e., the measure

$$P_N(A) = \mu_n\bigl(f(s+imh) \in A\bigr) = \mu_N\Bigl(\frac{f_2(s+imh)}{f_1(s+imh)} \in A\Bigr), \quad A \in \mathcal{B}(M(D)),$$

converges weakly to $P = \widetilde{P}u^{-1}$ as $N \to \infty$. The theorem is proved.

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REZIUMĖ

R. Macaitienė. Diskreti ribinė teorema bendrosioms Dirichlet eilutėms meromorfinių funkcijų erdvėje

Įrodyta diskreti ribinė teorema bendrosioms Dirichlet eilutėms tikimybinių matų silpnojo konvergavimo prasme meromorfinių funkcijų erdvėje.