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On joint universality for general Dirichlet series

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Let $s = \sigma + it$ be a complex variable, and let $\mathbb{C}$ denote the complex plane. The series

$$f(s) = \sum_{m=1}^{\infty} a_m e^{-\lambda_m s}, \quad \sigma > \sigma_a,$$

is called the general Dirichlet series. Here $a_m \in \mathbb{C}$ and $\{\lambda_m\}$ is an increasing sequence of positive numbers, $\lim\limits_{m\to\infty} \lambda_m = +\infty$. Let

$$\nu_T(\ldots) = \frac{1}{T}\text{meas}\{\tau \in [0, T]: \ldots\},$$

where $T > 0$, $\text{meas}\{A\}$ denotes the Lebesgue measure of the set A, and in place of dots a condition satisfied by τ is to be written. Note that the problem of the universality for zeta-functions comes back to S.M. Voronin. In 1975 he proved [6] that any analytic function can be approximated by translations $\zeta(s + i\tau)$ of the Riemann zeta-function $\zeta(s)$. The Voronin theorem states [2] that if K is a compact subset of the strip $\{s \in \mathbb{C}: \frac{1}{2} < \sigma < 1\}$ with connected complement, and $g(s)$ is a non-vanishing continuous function on K which is analytic in the interior of K, then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T\Big(\sup_{s\in K} |\zeta(s + i\tau) - g(s)| < \varepsilon\Big) > 0.$$

For the universality of general Dirichlet series theorem we need some conditions.

We suppose that the system of exponents $\{\lambda_m\}$ is linearly independent over the field of rational numbers, the function $f(s)$ is meromorphically continuable to the half-plane $\sigma > \sigma_1$ with some $\sigma_1 < \sigma_a$ and it is analytic in the strip

$$D = \{s \in \mathbb{C}: \sigma_1 < \sigma < \sigma_a\}.$$

We also require that, for $\sigma > \sigma_1$, the estimates

$$f(\sigma + it) = B|t|^{\alpha}, \quad |t| \geqslant t_0,\ \alpha > 0,$$

and

$$\int_{-T}^{T} \big|f(\sigma + it)\big|^2 \,\mathrm{d}t = BT, \quad T \to \infty,$$

should be satisfied. Here and in the sequel B denotes a quantity bounded by a constant. Denote, for $x > 0$,

$$r(x) = \sum_{\lambda_m \leqslant x} 1,$$

and let $c_m = a_m e^{-\lambda_m \sigma_a}$. Suppose that, for some $\theta > 0$,

$$\sum_{\lambda_m \leqslant x} |c_m|^2 = \theta r(x)(1 + \mathrm{o}(1))$$

as $x \to \infty$, $|c_m| \leqslant d$ with some $d > 0$, and

$$r(x) = C_1 x^{\varkappa} + B, \tag{1}$$

where $\varkappa \geqslant 1$, $C_1 > 0$ and $|B| \leqslant C_2$. Finally, we assume that $f(s)$ cannot be represented in the region $\sigma > \sigma_a$ by an Euler product over primes. Then we have the following statement [5].

THEOREM A. *Suppose that the function $f(s)$ satisfies all the conditions stated above. Let K be a compact subset of the strip D with connected complement, and let $g(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T\Big(\sup_{s\in K} \big|f(s+i\tau) - g(s)\big| < \varepsilon\Big) > 0.$$

For simplicity, we will consider a collection of two functions, only. Let, for $\sigma > \sigma_{aj}$, the series

$$f_j(s) = \sum_{m=1}^{\infty} a_{mj} e^{-\lambda_m s}$$

converges absolutely, $j = 1, 2$. As above, suppose that $f_j(s)$ is meromorphically continuable to the half-plane $\sigma > \sigma_{1j}$ with some $\sigma_{1j} < \sigma_{aj}$, all poles being included in a compact set, it is analytic in the strip $\{s \in \mathbb{C}\colon \sigma_{1j} < \sigma < \sigma_{aj}\}$, and that $f_j(s)$ cannot be represented by an Euler product over primes in the region $\sigma > \sigma_{aj}$, $j = 1, 2$. Moreover, let, for $\sigma > \sigma_{1j}$, the estimates

$$f_j(\sigma + it) = B|t|^{\alpha_j}, \quad |t| \geqslant t_0,\ \alpha_j > 0, \tag{2}$$

and

$$\int_{-T}^{T} \big|f_j(\sigma + it)\big|^2 \,\mathrm{d}t = BT, \quad T \to \infty, \tag{3}$$

be satisfied. Let $c_{mj} = a_{mj} e^{-\lambda_m \sigma_{aj}}$, $j = 1, 2$. Then we suppose that there exist $r \geqslant 2$ sets $\mathbb{N}_k$, $\mathbb{N}_{k_1} \bigcap \mathbb{N}_{k_2} = \varnothing$, for $k_1 \neq k_2$, $\mathbb{N} = \bigcup_{k=1}^{r} \mathbb{N}_k$, such that $c_{mj} = b_{kj}$ for $m \in \mathbb{N}_k$,

$k = 1, \ldots, r$, $j = 1, 2$. Let

$$L = \begin{pmatrix} b_{11} & b_{12} \\ \ldots & \ldots \\ b_{r1} & b_{r2} \end{pmatrix},$$

and we assume that the sequence $\{\lambda_m\}$ satisfies (1), and that

$$\sum_{\lambda_m \leqslant x,\ m \in \mathbb{N}_k} 1 = \varkappa_k r(x)(1 + \mathrm{o}(1)), \quad x \to \infty, \tag{4}$$

with positive $\varkappa_k$, $k = 1, \ldots, r$. Then in [3] the following assertion was obtained.

THEOREM B. *Suppose that conditions* (1)–(4) *are satisfied, the set* $\{\log 2\} \bigcup \bigcup_{m=1}^{\infty} \{\lambda_m\}$ *is linearly independent over the field of rational numbers, and that* $\operatorname{rank}(L) = 2$. *Let* K_j *be a compact subset of the strip* D_j *with connected complement, and let* $g_j(s)$ *be a continuous function on* K_j *wchich is analytic in the interior of* K_j, $j = 1, 2$. *There, for every* $\varepsilon > 0$,

$$\liminf_{T \to \infty} \nu_T\Big(\sup_{1 \leqslant j < 2} \sup_{s \in K_j} \big| f_j(s + i\tau) - g_j(s) \big| < \varepsilon \Big) > 0.$$

The requirement that the set $\{\log 2\} \bigcup \bigcup_{m=1}^{\infty} \{\lambda_m\}$ should be linearly independent over the field of rational numbers is not natural. It turns out that the number $\log 2$ can be removed from the later set. The aim of this paper is the following statement.

THEOREM. *Suppose that conditions* (1)–(4) *are satisfied, the system* $\{\lambda_m\}$ *is linearly independent over the field of rational numbers, and that* $\operatorname{rank}(L) = 2$. *Then the assertion of Theorem B is true.*

Let G be a region on the complex plane. Denote by $H(G)$ the space of analytic on G functions equipped with the topology of uniform convergence on compacta. Let, for $N > 0$,

$$D_{j,N} = \big\{ s \in \mathbb{C}\colon\ \sigma_{1j} < \sigma < \sigma_{aj},\ |t| < N \big\}, \quad j = 1, 2,$$

and

$$H_{2,N} = H_2(D_{1,N}, D_{2,N}) = H(D_{1,N}) \times H(D_{2,N}).$$

Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S. Let

$$P_T(A) = \nu_T\big(f_1(s_1 + i\tau), f_2(s_2 + i\tau) \in A \big), \quad A \in \mathcal{B}(H_{2,N}),$$

and let γ be the unit circle on the complex plane, and

$$\Omega = \prod_{m=1}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}$. Since Ω is a compact topological Abelian group, the probability Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$ exists. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ onto the coordinate space γ_m.

On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define an $H_{2,N}$-valued random element

$$f(s_1, s_2; \omega) = \big(f_1(s_1, \omega), f_2(s_2, \omega)\big),$$

where

$$f_j(s_j, \omega) = \sum_{m=1}^{\infty} a_{mj}\omega(m)e^{-\lambda_m s}, \quad s_j \in D_{j,N}, \ j = 1, 2.$$

Let P_f stand for the distribution of the random element $f(s_1, s_2; \omega)$, i.e.,

$$P_f(A) = m_H\big(\omega \in \Omega\colon \ f(s_1, s_2; \omega) \in A\big), \quad A \in \mathcal{B}(H_{2,N}).$$

LEMMA 1. *The probability measure P_T converges weakly to the measure P_f as $T \to \infty$.*

Proof is based on a limit theorem from [1].

We consider the support S of the measure P_f in Lemma 1. The support S is the minimal closed set of $H_{2,N}$ such that $P_f(S_{P_f}) = 1$.

LEMMA 2. *The support of the random element $f(s_1, s_2; \omega)$ is the whole of $H_{2,N}$.*

Proof uses lemmas from [2] and [4]. A full proof of the lemma is sufficiently long, it will be given elsewhere.

Proof of the theorem. First we suppose that the functions $g_1(s), g_2(s)$ have analytical continuation to the regions $D_{1,N}, D_{2,N}$, respectively. Let G consist of $(y_1, y_2) \in H_{2,N}$ satisfying the inequality

$$\sup_{1\leqslant j\leqslant 2}\ \sup_{s\in K_j} \big|y_j(s) - g_j(s)\big| < \frac{\varepsilon}{4}.$$

Clearly, the set G is open. Therefore, properties of the weak convergence of probability measures, Lemmas 1 and 2 yield

$$\liminf_{T\to\infty} \nu_T\Big(\sup_{1\leqslant j\leqslant 2} \sup_{s\in K_j} \big|f_j(s+i\tau) - g_j(s)\big| < \frac{\varepsilon}{4}\Big) \geqslant P_f(G) > 0.$$

Now let the functions $g_1(s), g_2(s)$ and the sets K_1, K_2 satisfy the conditions of the theorem. Then by the Mergelyan theorem, see, for exmaple, [7], there exist polynomials $p_1(s), p_2(s)$ such that

$$\sup_{1\leqslant j\leqslant 2} \sup_{s\in K_j} \big|p_j(s) - g_j(s)\big| < \frac{\varepsilon}{2}. \tag{5}$$

By the begining of the proof

$$\liminf_{T\to\infty} \nu_T\Big(\sup_{1\leqslant j\leqslant 2} \sup_{s\in K_j} \big|f_j(s+i\tau) - p_j(s)\big| < \frac{\varepsilon}{2}\Big) > 0. \tag{6}$$

In virtue of (5)

$$\Big\{\tau\colon \sup_{1\leqslant j\leqslant 2} \sup_{s\in K_j} \big|f_j(s+i\tau) - p_j(s)\big| < \frac{\varepsilon}{2}\Big\}$$
$$\subseteq \Big\{\tau\colon \sup_{1\leqslant j\leqslant 2} \sup_{s\in K_j} \big|f_j(s+i\tau) - g_j(s)\big| < \varepsilon\Big\}.$$

This together with (6) shows that

$$\liminf_{T\to\infty} \nu_T\Big(\sup_{1\leqslant j\leqslant 2} \sup_{s\in K_j} \big|f_j(s+i\tau) - g_j(s)\big| < \varepsilon\Big) > 0.$$

The theorem is proved.

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REZIUMĖ

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Patikslinta viena bendųjų Dirichle eilučių jungtinė universalumo teorema.