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On the uniform distribution of endomorphisms of s-dimensional torus, II

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Let $\Omega = \Omega_s$ $(s \geqslant 2)$ be s-dimensional torus, i.e., the set of points

$$\mathbf{x} = (x_1, \dots, x_s), \quad 0 \leqslant x_i < 1, \ i = 1, \dots, s.$$

An endomorphism of torus $T\colon\ \Omega \to \Omega$ is defined by

$$T\mathbf{x} = \mathbf{x}W \ (\text{mod } 1), \quad \mathbf{x} \in \Omega,$$

where W is a nonsingular matrix with integer elements.

We continue our work [1] on the investigation of the conditions on the initial point $\mathbf{x}$ and the matrix W for the sequence $\mathbf{x}W, \mathbf{x}W^2, \dots, \mathbf{x}W^k, \dots$ be uniformly distributed on torus Ω. Therefore all the notations are the same as in [1]. Here we remind some of them:

$$\boldsymbol{\xi} = \boldsymbol{\xi}(t) = \big(\varphi_1(t), \dots, \varphi_s(t)\big), \quad a \leqslant t \leqslant b, \tag{1}$$

is a parametric curve on Ω_s, functions $\varphi_i(t)$ have bounded derivatives of order $s-1$, $W(t)$ is the Wronskian of these functions, $W(t) \neq 0$ for $t \in [a,b]$, the characteristic polynomial of the matrix W satisfies certain conditions on its roots.

D. Moskvin [2] proved that the sequence

$$\boldsymbol{\xi} W, \boldsymbol{\xi} W^2, \dots, \boldsymbol{\xi} W^k, \dots \tag{2}$$

is uniformly distributed on torus Ω_s for almost all $t \in [a,b]$ in the sense of the Lebesgue measure μ.

In [1] the condition

$$\left|\varphi_i''(t)\right| \geqslant \kappa > 0, \quad i = 1, \dots, s \tag{3}$$

was used instead of $|W(t)| \neq 0$, and a restricted condition was imposed on the roots $\theta_1, \dots, \theta_s$ of the characteristic polynomial of the matrix W.

In this paper the condition (3) is replaced by another one.

The following theorem is proved.

THEOREM. *Let W be nonsingular matrix with integer elements, $\theta_1, \theta_2, \dots, \theta_s$ be its eigenvalues, $|\theta_1| > |\theta_2| > \dots > |\theta_s|$, $\mathbf{w}_i = (w_{i1}, \dots, w_{is})$ be the corresponding eigen-*

vectors, and $g(t)=\mathbf{w}_1\boldsymbol{\xi}(t)$*,*

$$\kappa^2=\bigl(\varphi_1''(t)\bigr)^2+\cdots+\bigl(\varphi_s''(t)\bigr)^2>0,\quad t\in[a,b].$$

If for each $t_0\in[a,b]$ *such that* $g(t_0)=0$*, there exists* k*,* $1<k\leqslant s$*,* $g^{(k)}(t_0)\neq 0$*, then the sequence* (2) *is uniformly distributed on* Ω_s *for almost all* $t\in[a,b]$ *in the sense of the Lebesgue measure.*

Consider the linear combination

$$g(t)=w_{11}\varphi_1(t)+w_{12}\varphi_2(t)+\cdots+w_{1s}\varphi_s(t).$$

The following auxilary result for this function is true.

LEMMA. *If* $g(t_0)=0$*,* $t_0\in[a,b]$*, and* $g^{(k)}(t_0)\neq 0$ *for some* k*,* $1<k\leqslant s$*, then there exist two sufficiently small constants* λ*,* λ^* *such that*

$$\bigl|g(t)\bigr|\geqslant\lambda|t-t_0|^{s-1}\quad\textit{for } |t-t_0|\leqslant\lambda^*.$$

Proof of Lemma. According to the Taylor formula in the neighbourhood of $t=t_0$,

$$g(t)=\sum_{k=1}^{s-1}\frac{g^{(k)}(t_0)}{k!}(t-t_0)^k+\frac{g^{(s)}(t^*)}{s!}(t-t_0)^s,\quad t^*\in[a,b]. \tag{4}$$

Let $g'(t_0)\neq 0$. Then

$$g(t)=(t-t_0)\bigl(g'(t_0)+(t-t_0)B_1(t)\bigr)$$

with a function $B_1(t)$ bounded for $t\in[a,b]$, $\max_t|B_1(t)|=B_1$. Therefore the inequality

$$\bigl|g(t)\bigr|\geqslant\frac{|g'(t_0)|}{2B_1}|t-t_0|$$

is true in the interval $|t-t_0|\leqslant\frac{|g'(t_0)|}{2B_1}$, and the statement of Lemma is true.

Now let $g'(t_0)=0$, but $g''(t_0)\neq 0$. In the same manner we obtain from (4) that

$$g(t)=(t-t_0)^2\bigl(g''(t_0)+(t-t_0)B_2(t)\bigr)$$

with a function $B_2(t)$ bounded for $t\in[a,b]$, $\max_t|B_2(t)|=B_2$, and the inequality

$$\bigl|g(t)\bigr|\geqslant\frac{|g''(t_0)|}{2B_2}|t-t_0|^2$$

is true in the interval $|t-t_0|\leqslant\frac{|g''(t_0)|}{2B_2}$.

After a final number of similar steps, since $g^{(k)}(t_0)\neq 0$ for some k, we get the proof of Lemma.

So, every zero of the function $g(t)$ is isolated and the number of zeroes is finite.

Proof of Theorem. The eigenvectors $\mathbf{w}_i$, $i = 1, \ldots, s$, corresponding to different eigenvalues form the complete basis in the space $\mathbf{R}_s$. So, any vector can be written as follows:

$$\boldsymbol{\xi} W^m = \sum_{i=1}^{s} \left(\sum_{j=1}^{s} w_{ij} \varphi_j(t) \right) \theta_i^m \mathbf{w}_i = \sum_{i=1}^{s} L_i(t) \theta_i^m \mathbf{w}_i. \tag{5}$$

The function $L_1(t)$ as well as $L_1'(t)$ have finite number of isolated zeroes. Let these zeroes be t_i, $i = 1, 2, \ldots, r$. According to Lemma

$$\left| L_1(t) \right| \geqslant \lambda_i |t - t_i|^{s-1}, \quad i = 1, \ldots, r.$$

For a sufficiently small δ, $\delta > 0$, we take nonintersecting intervals of length 2δ:

$$\Delta_i = (t_i - \delta, t_i + \delta), \quad i = 1, \ldots, r.$$

For every t, $t \notin \bigcup \Delta_i$, i.e., $t \in [a, b] \setminus (\bigcup \Delta_i) = A$, analogously to [2] the inequality

$$\left| L_1(t) \right| \geqslant d \delta^{s-1}, \quad d = \min_i \lambda_i \tag{6}$$

can be proved.

The set A is a union of intervals, and let $[a_1, b_1]$ be one of them. Consider the integral

$$I_m = \int_{a_1}^{b_1} g\big(\boldsymbol{\xi}(t) W^m\big)\, \mathrm{d}t, \quad g(\mathbf{x}) \in E_s^\alpha(c),$$

where $E_s^\alpha(c)$ is the Korobov class of functions [3]. We obtain from (5) and (6) that

$$\begin{aligned}
\boldsymbol{\xi} W^m &= L_1(t)\theta_1^m \mathbf{w}_1 + L_2(t)\theta_2^m \mathbf{w}_2 + \cdots \\
&= \big(L_1(t)\theta_1^m w_{11} + L_2(t)\theta_2^m w_{21} + \cdots, \\
&\qquad L_1(t)\theta_1^m w_{12} + L_2(t)\theta_2^m w_{22} + \cdots, \ldots\big) \\
&= \big(L_1(t) w_{11}\big(\theta_1^m + \mathrm{O}\big(\delta^{-s}\theta_2^m\big)\big), L_1(t) w_{12}\big(\theta_1^m + \mathrm{O}\big(\delta^{-s}\theta_2^m\big)\big), \ldots\big) \\
&= \big(w_{11} L_1(t), w_{12} L_2(t), \ldots\big)\big(\theta_1^m + \mathrm{O}\big(\delta^{-s}\theta_2^m\big)\big)
\end{aligned} \tag{7}$$

Let $t = t_0$ be such that $L_1'(t_0) = 0$. Then the integral I_m can be divided into three parts and each of them is evaluated separately:

$$I_m = \left(\int_{a_1}^{t_0 - \delta} + \int_{t_0 - \delta}^{t_0 + \delta} + \int_{t_0 + \delta}^{b_1} \right) g\big(\boldsymbol{\xi}(t) W^m\big)\, \mathrm{d}t = I_{m1} + I_{m2} + I_{m3}.$$

It is evident that the middle term $I_{m2} = \mathrm{O}(\delta)$. Both I_{m1} and I_{m3} have the same estimation. Let us take I_{m1}:

$$I_{m1} = \int_{a_2}^{b_2} g\big(\boldsymbol{\xi}(t) W^m\big)\, \mathrm{d}t, \quad \text{where } a_2 = a_1,\ b_2 = t_0 - \delta.$$

According to (7) we can write:

$$I_{m1} = \int_{a_2}^{b_2} g\bigl(w_{11}\theta_1^m L_1(t)\bigl(1 + \mathrm{O}\bigl(\delta^{-s}\varrho\bigr)\bigr), \ldots\bigr)\,\mathrm{d}t, \quad \text{where } \varrho = \left|\frac{\theta_2}{\theta_1}\right|^m.$$

After the change of variables $u = \theta_1^m L_1(t)$ we obtain

$$I_{m1} = \frac{1}{\theta_1^m}\int_{D_m}^{D'_m} g\bigl(w_{11}u\bigl(1 + \mathrm{O}\bigl(\varrho\delta^{-s}\bigr)\bigr), \ldots\bigr)\Phi(u)\,\mathrm{d}u,$$

with $D_m = \theta_1^m L_1(a_2)$, $D'_m = \theta_1^m L_1(b_2)$, $\Phi(u) = \frac{d}{du}L_1^{-1}(\frac{u}{\theta_1^m})$.

Suppose D_m and D'_m are integers. Otherwise the estimation of I_{m1} differs only in $\mathrm{O}(\theta_1^m\delta^{s-1})$. Thus by the Abel transformation and analogously to [4] we get

$$\begin{aligned}
I_{m1} &= \frac{1}{\theta_1^m}\sum_{k=D_m}^{D'_m-1}\int_k^{k+1} g(\ldots)\Phi(u)\,\mathrm{d}u = \frac{1}{\theta_1^m}\sum_{k=D_m}^{D'_m-1}\int_0^1 g(\ldots)\Phi(u+k)\,\mathrm{d}u \\
&= \frac{1}{\theta_1^m}\int_0^1\sum_{k=D_m}^{D'_m-1}\bigl(\Phi(u+k) - \Phi(u+k+1)\bigr)\sum_{l=D_m}^{k} g(\ldots)\,\mathrm{d}u \\
&= \frac{1}{\theta_1^m}\int_0^1\sum_{k=D_m}^{D'_m-1}\bigl(\Phi(u+k) - \Phi(u+k+1)\bigr)(k - D_m + 1) \\
&\quad\times\left\{\int_{\Omega_s} g(\mathbf{x})\,\mathrm{d}\mathbf{x} + \mathrm{O}\left(\frac{1}{k - D_m + 1} + (k - D_m + 1)\bigl(\varrho\delta^{-s}\bigr)^{\frac{\alpha-1}{1+\varepsilon}}\right)\right\}
\end{aligned}$$

with

$$\alpha = 1 + \frac{2(1+\varepsilon)\ln|\theta_1|}{\ln|\theta_1| - \ln|\theta_2|}, \quad \varepsilon > 0.$$

From (6) we get the estimate

$$\bigl|\Phi(u+k) - \Phi(u+k+1)\bigr| = \mathrm{O}\bigl(\theta_1^{-m}\delta^{-3s}\bigr)$$

and then we have the equality

$$I_{m1} = (b_2 - a_2)\int_{\Omega_s} g(\mathbf{x})\,\mathrm{d}\mathbf{x} + \mathrm{O}\left(\delta + \frac{1}{\theta_1^m\delta^{3s}} + \frac{\theta_1^m}{\delta^{3s}}\left(\left|\frac{\theta_2^m}{\theta_1^m}\right|\frac{1}{\delta}\right)^{\frac{\alpha-1}{1+\varepsilon}}\right).$$

The remaining part of the proof of Theorem is the same as in [4] (see also [1]).

References

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REZIUMĖ

B. Kryžienė, G. Misevičius. s-mačio toro endomorfizmų tolygus pasiskirstymas, II

Darbe apibendrinama D. Moskvino teorema apie s-mačio toro Ω_s endomorfizmų (mod 1) tolygų pasiskistymą. Vietoje apribojimo – funkcijų $\varphi_1(t), \ldots, \varphi_s(t)$ vronskijanas $W(t) \neq 0,\ t \in [a, b]$, naudojama kita sąlyga $(\varphi_1''(t))^2 + \ldots + (\varphi_s''(t))^2 > 0,\ t \in [a, b]$.