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Multiplicative sets and left strongly prime ideals in rings

Algirdas KAUČIKAS (VPU)
e-mail: algiskau@yahoo.com

All considered rings are associative with identity element. $A \subset B$ means that A is a proper subset of B.

Let R be a nonzero ring. We recall that a left ideal $\mathfrak{p} \subset R$ is called *(left) strongly prime* if for each $u \notin \mathfrak{p}$ there exist the finite set $\{\alpha_1, \dots, \alpha_n\} \subseteq R, \quad n = n(u)$, such that $r\alpha_1 u, \dots, r\alpha_n u \in \mathfrak{p}$ implies $r \in \mathfrak{p}$. Each subset $\{\alpha_1 u, \dots, \alpha_n u\}$ is called an insulator of u for $\mathfrak{p}$. Of course, when R is commutative, strongly prime ideals coincide with prime ideals of R. Basic properties of the left strongly prime ideals were investigated in [2, 3, 4]. All left maximal ideals evidently are strongly prime.

Recall that a subset $S \subset R$ is called multiplicative if it is multiplicatively closed, contains unity element and $0 \notin S$. Standard very important fact of the commutative algebra is that each ideal $\mathfrak{p} \subset R$, maximal with respect to $\mathfrak{p} \cap S = \oslash$ is prime. See [1], Chapter 2, §2, Cor. 1. We generalize this fact for any associative ring.

Let S be a commutative multiplicative set of the ring R. We call a left ideal $L \subset R$ an S- ideal if $LS \subseteq L$, i.e., if $ls \in L$ for all $l \in L$ and $s \in S$.

Consider the family $\{L_i, i \in I\}$ of proper left S-ideals of the ring R disjoint with S. This family is not empty because the zero ideal belongs to it. Evidently, this family is inductive and, by Zorn's lemma contains the maximal element.

THEOREM 1. *Each left S-ideal $\mathfrak{p}$, maximal with respect to being disjoint with S is strongly prime.*

Proof. Let $u \notin \mathfrak{p}$. Then S-ideal $\mathfrak{p} + RuS$ properly contains $\mathfrak{p}$, so intersects with S. Thus we have

$$p + \alpha_1 u a_1 + \dots + \alpha_n u a_n = a$$

with some $\alpha_1, \dots, \alpha_n \in R, \ a, a_1, \dots, a_n \in S$. We show that $\{\alpha_1 u, \dots, \alpha_n u\}$ is an insulator of u for $\mathfrak{p}$. Indeed, let $r\alpha_1 u, \dots, r\alpha_n u \in \mathfrak{p}$. Then, because $\mathfrak{p}$ is an S-ideal, $r\alpha_1 u a_1, \dots, r\alpha_n u a_n$ also belong to $\mathfrak{p}$, thus $ra \in \mathfrak{p}$. If $r \notin \mathfrak{p}$, we would have

$$q + \beta_1 r b_1 + \dots + \beta_m r b_m = b$$

with some $\beta_1, \dots, \beta_m \in R, \quad b, b_1, \dots, b_m \in S$. Multiplying this equality by a from the right and using commutativity of S, we obtain that $ab \in \mathfrak{p}$. But $\mathfrak{p}$ is disjoint with S – so the contradiction. So $r \in \mathfrak{p}$ and $\mathfrak{p}$ is strongly prime. Moreover we have showed that elements from S are insulators for $\mathfrak{p}$, i.e., $ra \in \mathfrak{p}$ with $a \in S$ implies that $r \in \mathfrak{p}$.

Particulary, let $a \in R$ be non-nilpotent element. Then $S = \{1, a, ..., a^n, ...\}$ is commutative multiplicative set and S-ideals are left ideals L having property $La \subseteq L$. By Theorem 1, left S-ideal $\mathfrak{p}$, a maximal one with respect to $\mathfrak{p} \cap S = \oslash$ is strongly prime.

Theorem 1 can be generalized. Let $\mathcal{A}$ be a set, which elements are finite subsets of the ring R. Let $\mathcal{A}$ be multiplicatively closed, $\{1\} \in \mathcal{A}$ and $\{0\} \notin \mathcal{A}$. We call such set $\mathcal{A}$ multiplicative F-set.

Left ideal $L \subset R$ is called an $\mathcal{A}$-ideal if $Ls \subseteq L$ for all $s \in \mathcal{A}$. We say that L is disjoint with $\mathcal{A}$ if $s \nsubseteq L$ for all $s \in \mathcal{A}$. Analogously, we prove

THEOREM 2. *Let $\mathcal{A}$ be a commutative multiplicative F-set. Each left $\mathcal{A}$-ideal $\mathfrak{p}$ maximal with respect to being disjoint with $\mathcal{A}$ is strongly prime.*

Of course, multiplicative sets also are multiplicative F-sets.

Let $s = \{a_1, ..., a_n\} \subseteq R$ be a non-nilpotent subset of the ring. Then $\mathcal{A} = \{1, s, ..., s^n, ...\}$ is commutative multiplicative F-set.

We recall, that the intersection of all left strongly prime ideals is called left strongly prime radical of the ring R, which is denoted by $rad_l R$. By A.L. Rosenberg's theorem (see [4]), $rad_l R$ coincides with Levitzky radical $L(R)$ of the ring R. Unfortunately A.L. Rozenberg's proof is very long and highly complicated.

It is very easy to get this result from the Theorem 2.

References

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REZIUMĖ

A. Kaučikas. Multiplikatyvios aibės ir kairieji stipriai pirminiai idealai žieduose

Įrodyta, kad maksimalus kairysis žiedo S-idealas, nesikertantis su komutatyvia multiplikatyvia aibe S, yra stipriai pirminis.