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A weighted discrete universality theorem for L-functions of elliptic curves

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Let E be an elliptic curve over rational given by the equation

$$E:\ y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}.$$

Suppose that the discriminant of E $\Delta = -16(a^3 + 27b^2) \neq 0$. Then the curve E is non-singular.

For each prime p, denote by $\nu(p)$ the number of solutions of the congruence

$$y^2 \equiv x^3 + ax + b \ (\mathrm{mod}\, p),$$

and let $\lambda(p) = p - \nu(p)$. Then the L-function of the curve E is the Euler product

$$L_E(s) = \prod_{p|\Delta} \Big(1 - \frac{\lambda(p)}{p^s}\Big)^{-1} \prod_{p\nmid\Delta} \Big(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\Big)^{-1},$$

where $s = \sigma + it$ is a complex variable. In view of the Hasse estimate

$$|\lambda(p)| \leqslant 2\sqrt{p}$$

the product defining $L_E(s)$ converges absolutely and uniformly on compact subsets of the half-plane $D_a = \{s \in \mathbb{C}\colon\ \sigma > \frac{3}{2}\}$, and defines there an analytic function. Moreover, the function $L_E(s)$ is analytically continuable to an entire function.

In [3] the universality of $L_E(s)$ has been obtained. Denote by $\mathrm{meas}\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

THEOREM A. *Suppose that E is a non-singular elliptic curve over the field of rational numbers. Let K be a compact subset of the strip $D = \{s \in \mathbb{C}\colon\ 1 < \sigma < \frac{3}{2}\}$ with connected complement, and let $f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\Big\{\tau \in [0, T]\colon\ \sup_{s\in K} \big|L_E(s + i\tau) - f(s)\big| < \varepsilon\Big\} > 0.$$

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In [1] a weighted universality theorem for the function $L_E(s)$ has been proved. Let T_0 be a fixed positive number and let $w(\tau)$ be a positive function of bounded variation on $[T_0, \infty)$. Define

$$V = V(T, w) = \int_{T_0}^{T} w(\tau)\,\mathrm{d}\tau,$$

and suppose that $\lim_{T\to\infty} V(T, w) = +\infty$. Moreover, let $X(\tau, \omega)$, $\tau \in \mathbb{R}$, be an ergodic process, $E|X(\tau, \omega)| < \infty$, with sample paths almost surely integrable in the Riemann sense over every finite interval. Suppose that the function $w(\tau)$ satisfies

$$\frac{1}{V}\int_{T_0}^{T} w(\tau)X(t+\tau, \omega)\,\mathrm{d}\tau = EX(0, \omega) + \mathrm{o}(1+|t|)^{\delta} \tag{1}$$

almost surely for all $t \in \mathbb{R}$ with some $\delta > 0$ as $T \to \infty$.

Let I_A be the indicator function of the set A.

THEOREM B. *Suppose that condition* (1) *is satisfied. Let K and $f(s)$ be the same as in Theorem A. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{V}\int_{T_0}^{T} w(\tau) I_{\{\tau:\ \sup_{s\in K}|L_E(s+i\tau)-f(s)|<\varepsilon\}}\,\mathrm{d}\tau > 0.$$

The paper [2] is devoted to a discrete universality theorem for the function $L_E(s)$.

THEOREM C. *Suppose that* $\exp\{\frac{2\pi k}{h}\}$ *is an irrational number for all $k \in \mathbb{Z}\setminus\{0\}$. Let K and $f(s)$ be the same as in Theorem A. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1}\sharp\left\{0 \leqslant m \leqslant N:\ \sup_{s\in K}\left|L_E(s+imh) - f(s)\right| < \varepsilon\right\} > 0.$$

The aim of this note is to obtain a weighted discrete universality theorem for the function $L_E(s)$. Let $w(x)$ be a non-negative function on $[0, \infty)$. Suppose that

$$U = U(N, w) = \sum_{m=0}^{N} w(m) \to \infty$$

as $N \to \infty$. We will prove the following statement.

THEOREM 1. *Suppose that $w(x)$ is a continuous non-vanishing function and that* $\exp\{\frac{2\pi k}{h}\}$ *is an irrational number for all $k \in \mathbb{Z}\setminus\{0\}$. Let K and $f(s)$ be the same as in Theorem A. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{U} \sum_{\substack{m=0\\ \sup_{s\in K}|L_E(s+imh)-f(s)|<\varepsilon}}^{N} w(m) > 0.$$

The proof of Theorem 1 like that of Theorems A, B and C is based on a discrete limit theorem with weight in the sense of weak convergence of probability measures in the space of analytic functions for the function $L_E(s)$. Let $D_V = \{s \in \mathbb{C}\colon\ 1 < \sigma < \frac{3}{2}, |t| < V\}$, where V is an arbitrary positive number. Denote by $H(D_V)$ the space of analytic on D_V functions equipped with the topology of uniform convergence on compacta. Then a discrete limit theorem for the Matsumoto zeta-function obtained in [4] implies the following assertion. Let $\gamma = \{s \in \mathbb{C}\colon\ |s| = 1\}$, and

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. With the product topology and pointwise multiplication the torus Ω is a compact topological group. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S. Then on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H exists, and we obtain a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(p)$ be the projection of $\omega \in \Omega$ to the coordinate space γ_p. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define the $H(D_V)$-valued random element $L_E(s, \omega)$ by the formula

$$L_E(s,\omega) = \prod_{p|\Delta}\left(1 - \frac{\lambda(p)\omega(p)}{p^s}\right)^{-1}\prod_{p\nmid\Delta}\left(1 - \frac{\lambda(p)\omega(p)}{p^s} + \frac{\omega^2(p)}{p^{2s-1}}\right)^{-1}.$$

LEMMA 1. *Suppose that* $\exp\{\frac{2\pi k}{h}\}$ *is an irrational number for all* $k \in \mathbb{Z} \setminus \{0\}$. *Then the probability measure*

$$\frac{1}{N+1}\sharp\left\{0 \leqslant m \leqslant N\colon\ L_E(s + imh) \in A\right\}, \quad A \in \mathcal{B}(H(D_V)),$$

converges weakly to the distribution of the random element $L_E(s, \omega)$.

The support of the random element $L_E(s, \omega)$ has been considered in [2] and [3]. Let

$$S_V = \left\{g \in H(D_V)\colon\ g(s) \neq 0 \quad \text{or} \quad g(s) \equiv 0\right\}.$$

LEMMA 2 [2]. *The support of the random element* $L_E(s, \omega)$ *is the set* S_V.

A weighted limit theorem in the space $H(D_V)$ for the function $L_E(s)$ follows from the following general result. Let S be a metric space, and let $g(t)$ be a S-valued function defined for $t \geqslant 0$.

LEMMA 3 [5]. *Suppose that* $w(x)$ *is a continuous non-increasing function on* $[0, \infty)$, *and that the probability measure*

$$\frac{1}{N+1}\sharp\left(0 \leqslant m \leqslant N\colon\ g(mh) \in A\right), \quad A \in \mathcal{B}(S),$$

converges weakly to some probability measure Q *on* $(S, \mathcal{B}(S))$ *as* $N \to \infty$. *Then also the probability measure*

$$\frac{1}{U} \sum_{\substack{m=0 \\ g(mh) \in A}}^{N} w(m), \quad A \in \mathcal{B}(S),$$

converges weakly to the measure Q *as* $N \to \infty$.

LEMMA 4. *Suppose that* $\exp\{\frac{2\pi k}{h}\}$ *is an irrational number for all* $k \in \mathbb{Z} \setminus \{0\}$, *and that* $w(x)$ *is a continuous non-increasing function on* $[0, \infty)$. *Then the probability measure*

$$\frac{1}{U} \sum_{\substack{m=0 \\ L_E(s+imh) \in A}}^{N} w(m), \quad A \in \mathcal{B}(H(D_V)),$$

converges weakly to the distribution of the random element $L_E(s, \omega)$ *as* $N \to \infty$.

Proof. The lemma is an immediate consequence of Lemmas 1 and 4.

Proof of Theorem 1. Obviously, we can find $V > 0$ such that $K \subset D_V$. First we suppose that the function $f(s)$ has a non-vanishing analytic continuation to the region D_V. Let

$$G = \left\{ g(s) \in H(D_V) \colon \sup_{s \in K} \left| g(s) - f(s) \right| < \varepsilon \right\}.$$

Then G is an open set in the space $H(D_V)$, and $G \subset S_V$. Denote by P_{L_E} the distribution of the random element $L_E(s, \omega)$. Then from the properties of the weak convergence of probability measures and of the support, in view of Lemma 4 we have that

$$\liminf_{N \to \infty} \frac{1}{U} \sum_{\substack{m=0 \\ L_E(s+imh) \in G}}^{N} w(m) \geqslant P_{L_E}(G) > 0. \tag{2}$$

Now let $f(s)$ be as in the statement of Theorem 1. Then by the Mergelyan theorem there exists a polynomial $p(s)$, $p(s) \neq 0$ on K, such that

$$\sup_{s \in K} \left| f(s) - p(s) \right| < \frac{\varepsilon}{4}. \tag{3}$$

The polynomial $p(s)$ has only finitely many zeros. Therefore, there exists a region G_1 with connected complement such that $K \subset G_1$, and $p(s) \neq 0$ on G_1. Then we can choose a continuous branch of $\log p(s)$ on G_1 which is analytic in the interior of G_1. By the Mergelyan theorem again there exits a polynomial $q(s)$ such that

$$\sup_{s \in K} \left| p(s) - \mathrm{e}^{q(s)} \right| < \frac{\varepsilon}{4}.$$

This and (3) yield

$$\sup_{s\in K}\left|f(s)-e^{q(s)}\right|<\frac{\varepsilon}{2}. \tag{4}$$

However, $e^{q(s)}\neq 0$. Therefore, (2) and the definition of the set G show that

$$\liminf_{N\to\infty}\frac{1}{U}\sum_{\substack{m=0\\ \sup_{s\in K}|L_E(s+imh)-e^{q(s)}|<\varepsilon}}^{N} w(m)>0.$$

Now from this and (4) Theorem 1 follows.

References

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REZIUMĖ

V. Garbaliauskienė. Diskreti universalumo teorema su svoriu elipsinių kreivių L-funkcijoms

Įrodyta diskreti universalumo Voronino prasme teorema su svoriu elipsinių kreivių virš racionaliųjų skaičių kūno L-funkcijoms.