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Cut elimination for knowledge logic with interaction

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Abstract. In the article the multimodal logic T_n with central agent interaction axiom is analysed. The Hilbert type calculi is presented, then Gentzen type calculi with cut is derived and the proof of cut-elimination theorem is outlined. The work shows that it is possible to construct a Gentzen type calculi without cut for this logic.

Keywords: multimodal logic, T_n, interaction axiom, cut elimination.

1. Introduction

Multimodal logics (K_n, T_n, $S4_n$) is often used to model the behaviour of agents. However they do not include knowledge of interaction between them, so they are often enriched with interaction axioms. Some examples of most popular interaction axioms can be found in [1]. In this article we analyse the single case: when there is one agent (called the central agent), who knows everything that other agents are aware of. For simplicity the base logics is T_n. The aim of the article is to find the sequent calculi without cut for logics T_n with central agent interaction axiom.

We define propositional formula in standart way, including operators $\neg$, $\vee$, $\wedge$, $\supset$. Then to get modal formula we add modal logic operator $K(l)$ in similar way, where l is either c, meaning central agent, or a number of agent starting from 1. In the article capital latin letters (A, B, ...) means any modal formula, capital greek letters (Γ, Δ, Γ_1, Γ^*, $\Gamma^\sharp$) means a (possibly empty) multiset of modal formulas (the order of the formulas in a multiset does not matter).

2. Hilbert type calculus

We use the standart Hilbert type propositional calculus (HPC), which can be found in [2].

DEFINITION 2.1. Hilbert type calculus for logic T_n (HT_n) consists of all the rules and axioms of HPC and:

- axiom k: $K(l)(A \supset B) \supset (K(l)A \supset K(l)B)$;
- axiom t: $K(l)A \supset A$;
- rule for knowledge operator: $\frac{A}{K(l)A}$;

where l is any agent.

DEFINITION 2.2. Hilbert type calculus for logic T_n with interaction axiom (HT_n^I) consists of all the rules and axioms of HT_n and axiom $K(i)A \supset K(c)A$, where i is any agent, exept the central one.

3. Gentzen type calculus with cut

We use the standart Gentzen type propositional calculus (GPC), which can be found in [3], but we do not include structural rules of exchange, because the order of the members of multisets is not important.

DEFINITION 3.1. Gentzen type calculus for logic T_n (GT_n) consists of axiom $A \to A$, all the rules of GPC and rules for modal operator:

$$\frac{A, K(l)A, \Gamma \to \Delta}{K(l)A, \Gamma \to \Delta}\,(K(l) \to), \qquad \frac{\Gamma_1 \to A}{K(l)\Gamma_1, \Gamma_2 \to K(l)A, \Delta}\,(\to K(l)),$$

where l is any agent.

DEFINITION 3.2. Gentzen type calculus with cut for logic T_n with central agent interaction axiom ($GT_n^I cut$) consists of axiom $A \to A$, all the rules of GT_n and the rule for central agent interaction axiom:

$$\frac{\Gamma \to \Delta, K(i)A}{\Gamma \to \Delta, K(c)A}\,(\to K^c),$$

where i is any agent, exept the central one;

THEOREM 3.3. *Formula is provable in $GT_n^I cut$ if and only if it is provable in HT_n^I.*

Proof. The "if" part. By induction on the proof in $GT_n^I cut$ it can be shown that $\Gamma \to \Delta$ is provable in $GT_n^I cut$ if and only if $\wedge_{F\in\Gamma} F \supset \vee_{G\in\Delta} G$ is provable in HT_n^I.

The "only if" part. It is easy to show that all the axioms of HT_n^I are provable in $GT_n^I cut$. It remains to show, that the rules of Hilbert type calculus can be replaced by the proof in Gentzen type calculus. Actually, MP rule can be replaced by cut in $GT_n^I cut$ and rule for knowledge operator can be replaced by $(\to K(l))$.

4. Gentzen type calculus without cut

DEFINITION 4.1. Gentzen type calculus without cut for logic T_n with central agent interaction axiom (GT_n^I) consists of axiom $A \to A$, all the rules of $GT_n^I cut$, except the cut rule, and the rule:

$$\frac{K(c)A, \Gamma \to \Delta}{K(i)A, \Gamma \to \Delta}\,(K^c \to),$$

where i is any agent, exept the central one;

THEOREM 4.2. *Formula is provable in GT_n^I if and only if it is provable in $GT_n^I cut$.*

Proof. The "if" part. By induction on applications of $(K^c \to)$ rule. If we have the proof in GT_n^I with application of this rule:

$$\dfrac{P_1 \left\{ \dfrac{\dots}{K(c)A, \Gamma \to \Delta} \right.}{P_2 \left\{ \dfrac{K(i)A, \Gamma \to \Delta}{\dots} \right.} (K^c \to)$$

then we can exchange this fragment of proof with a fragment without application of $(K^c \to)$ rule:

$$\dfrac{\dfrac{K(i)A \to K(i)A}{K(i)A \to K(c)A} (\to K^c) \qquad \left. \dfrac{\dots}{K(c)A, \Gamma \to \Delta} \right\} P_1}{P_2 \left\{ \dfrac{K(i)A, \Gamma \to \Delta}{\dots} \right.} (K(c)A\ cut).$$

This way we can get a proof in $GT_n^I cut$.

The "only if" part. We follow the cut-elimination theorem proof given in [3]. First we change the cut rule with the mix rule:

$$\dfrac{\Gamma \to \Delta, A \qquad A, \Pi \to \Lambda}{\Gamma, \Pi^* \to \Delta^*, \Lambda} (A\ mix),$$

where Π^* and Δ^* are obtained from Π and Δ respectively by deleting all the occurencies of formula A (which is called the mix formula). We call $\Gamma \to \Delta, A$ the upper left sequent of a mix, $A, \Pi \to \Lambda$ the upper right sequent of a mix and $\Gamma, \Pi^* \to \Delta^*, \Lambda$ the lower sequent of a mix. It can be easily shown that formula is provable in $GT_n^I cut$ if and only if it is provable in $GT_n^I cut$ with the cut rule replaced by the mix rule.

We analyse only those proofs, which have only one mix, occuring as their last rule. By induction on applications of the mix rule we can extend it to all the proofs.

Finaly we define the grade and the range of a proof.

DEFINITION 4.3. The grade of a proof, which has only one mix, occuring as its last rule, is the number of logical symbols in the mix formula.

DEFINITION 4.4. Let P be a proof, which has only one mix, occuring as its last rule:

$$\dfrac{\dfrac{\dots}{\Gamma \to \Delta, A} \qquad \dfrac{\dots}{A, \Pi \to \Lambda}}{\Gamma, \Pi^* \to \Delta^*, \Lambda} (A\ mix).$$

The rank of upper left (right) sequent of the mix is the maximum number of occurencies of mix formula (A) in all of its threads. The rank of a proof P is the sum of ranks of upper left and right sequents.

For cut-elimination theorem for GPC it is enough to have induction on the grade of a proof and on the rank of it, as shown in [3]. For GT_n^I in some cases we must also have induction on the height of a proof, which is the maximum height of any of the threads in a proof. The proof of cut-elimination theorem for GT_n^I is as follows.

1. Let the rank of the proof be 2.
   a) If the left (or right) upper sequent of a mix is an axiom, then we eliminate the mix as shown in [3].

   b) If the left (or right) upper sequent of a mix is lower sequent of structural rule, then we eliminate the mix as shown in [3].

   c) If the left and right upper sequents of a mix are not axioms and both are lower sequents of logical rules, then we must distinguish cases according to the rules applied. Since the rank of the proof is 2, the rules in the upper and lower sequents are applied to the mix formula A.

   How to lower the grade if $A \equiv B \wedge C$ is shown in [3]. Here we analyze just the case when the mix formula is of the form $K(c)B$. In that case (because the rank of the proof is 2) the only formula, which can be applied to the mix formula in the right upper sequent, is $\to K(i)$. We treat only the case, where the rule applied to the left upper sequent is $\to K^c$. Then there are four cases different in the further proof of upper left sequent. Here we present one.

   Assume the proof P is:

$$\dfrac{\dfrac{\dfrac{Q\left\{\dfrac{\cdots}{\Gamma_1 \to B}\right.}{K(i)\Gamma_1, \Gamma_2 \to \Delta, K(i)B}}{K(i)\Gamma_1, \Gamma_2 \to \Delta, K(c)B} \qquad \dfrac{R\left\{\dfrac{\cdots}{B, \Pi_1 \to C}\right.}{K(c)B, K(c)\Pi_1, \Pi_2 \to \Lambda, K(c)C}}{K(i)\Gamma_1, \Gamma_2, K(c)\Pi_1, \Pi_2 \to \Delta, \Lambda, K(c)C}.$$

   Then first we must change the mix:

$$\dfrac{Q\left\{\dfrac{\cdots}{\Gamma_1 \to B}\right. \qquad R\left\{\dfrac{\cdots}{B, \Pi_1 \to C}\right.}{\Gamma_1, \Pi_1^\sharp \to C}.$$

   $\Pi_1^\sharp$ is obtained from Π by deleting all occurencies of formula B. Because the grade of this mix is lower than of the previous one, by the induction hypothesis we will be able to get the proof P' of $\Gamma_1, \Pi_1^\sharp \to B$ without a mix. Then we will have to change the proof P like this:

$$\dfrac{\dfrac{P'\left\{\dfrac{\cdots}{\Gamma_1, \Pi_1^\sharp \to C}\right.}{K(c)\Gamma_1, \Gamma_2, K(c)\Pi_1, \Pi_2 \to \Delta, \Lambda, K(c)C}}{K(i)\Gamma_1, \Gamma_2, K(c)\Pi_1, \Pi_2 \to \Delta, \Lambda, K(c)C}.$$

From this case we can se that elimination of cut from $GT_n^I cut$ requires that $(K^c \rightarrow)$ rule be added to GT_n^I.

2. Let the rank of the right upper sequent of the mix be larger than 1. In these cases the aim is to lower the rank of the mix and leave the grade unchanged. In several cases it is needed to leave the grade ant the rank unchanged and to lower the height of the proof. The main idea is presented in [3].

3. Let the rank of the left upper sequent of the mix be larger than 1. Similar to previous.

References

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REZIUMĖ

J. Andrikonis, R. Pliuškevičius. Pjūvio eliminavimas žinių logikai su sąveika

Straipsnyje nagrinėjama multimodalinė logika T_n su centrinio agento sąveikos aksioma. Pristatomas Hilberto tipo skaičiavimas, išvedamas Gentzeno tipo skaičiavimas su pjūvio taisykle ir pateikiami pjūvio pašalinimo teoremos įrodymo kontūrai. Darbas demonstruoja, kad šiai logikai įmanoma sukonstruoti Gentzeno tipo skaičiavimą be pjūvio.