Liet. mat. rink. LMD darbai, **48/49**, 2008, 412–416

The discounted local limit theorems for large deviations

Leonas SAULIS, Dovilė DELTUVIENĖ (VGTU)
e-mail: lsaulis@fm.vgtu.lt

Abstract. Theorems of large deviations, both in the Cramer zone and the Linnik power zones, for the normal approximation of the distribution density function of normalized sum $S_v = \sum_{k=0}^{\infty} v^k X_k$, $0 < v < 1$, of i.i.d. random variables (r.v.) $X_0, X_1, \ldots$ satisfying the generalized Bernstein's condition are obtained.

Keywords: distribution density function, characteristic function, cumulant, large deviations, discount factor.

1. Introduction

Let $X_0, X_1, \ldots$ be a sequence of independent r.v. with the common distribution function $F(x)$, and let v, $0 < v < 1$, be a discount factor. We define r.v. S_v by

$$S_v = \sum_{k=0}^{\infty} v^k X_k, \tag{1}$$

which may be interpreted as the present value of the sum of certain periodic and identically distribution payments X_k. We assume that the first two moments of r.v. X_k are finite

$$\mu = \int_{-\infty}^{\infty} x \,\mathrm{d}F(x) < \infty, \qquad \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \,\mathrm{d}F(x) < \infty, \tag{2}$$

and that the centered moments $\mathbf{E}(X_k - \mu)^s$, $s = 3, 4, \ldots$ satisfy the generalized S.N. Bernstein condition: there exist constants $\gamma \geqslant 0$, $K > 0$ such that

$$\bigl|\mathbf{E}(X_k - \mu)^s\bigr| \leqslant (s!)^{1+\gamma} K^{s-2} \sigma^2, \quad s = 3, 4, \ldots. \tag{B_γ}$$

Notice that the mean and variance of the r.v. S_v are, respectively,

$$\mathbf{E}S_v = \mu(1-v)^{-1} \quad \text{and} \quad \mathbf{D}S_v = \sigma^2\bigl(1-v^2\bigr)^{-1}. \tag{3}$$

It has been shown in [2], that the normalized r.v.

$$Z_v = \sigma^{-1}(1-v)^{1/2}\bigl(S_v - \mu(1-v)^{-1}\bigr), \tag{4}$$

with mean $\mathbf{E}Z_v = 0$ and variance $\mathbf{D}Z_v = (1+v)^{-1}$, is asymptotically normal if $v \to 1$. Let $F_v(x)$ and $p_v(x)$ be the distribution and density function, respectively, of the r.v.

Z_v. We denote the normal distribution with zero mean and variance $(1+v)^{-1}$ by N_v, i.e., $N_v(x) = \int_{-\infty}^{x} \varphi_v(y)\,\mathrm{d}y$ where

$$\varphi_v(x) = \Big(\frac{1+v}{2\pi}\Big)^{\frac{1}{2}} \exp\Big\{-\frac{1+v}{2}x^2\Big\}, \quad -\infty < x < \infty\,. \tag{5}$$

The distribution $F_v(x)$ of random variable Z_v has been approximated by normal law $N_v(x)$ and the exact estimate has been derived by H.U. Gerber [1]. The authors of the current paper proved in [2] the theorems of large deviations. Let us notice that the asymptotic analysis of a density function $p_v(x)$ of a random variables Z_v is more complicated than asymptotic analysis of distribution $F_v(x)$.

Note that the characteristic function $f(t) = \mathbf{E}\exp\{itX_0\}$ of the r.v. X_0 is analytic in the vicinity of the point $t=0$ if condition (B_γ) is satisfied with $\gamma = 0$. In this case, large deviation theorems are proved in the Cramer zone. In the case when the moments of r.v. X_0 satisfy condition (B_γ) with $\gamma > 0$, $f(t)$ is not analytical. In this case, theorems for large deviations in power Linnik zones are proved in [3].

2. The discounted version of the large deviations

In order to prove theorems for large deviations for the r.v. Z_v defined by (4), it is necessary to obtain upper estimates for its cumulants $\Gamma_l(Z_v)$, $l = 3, 4, \ldots$.

PROPOSITION 1. *If for r.v. X_k, $k = 0, 1, 2, \ldots$ the condition (B_γ) is satisfied, then*

$$\big|\Gamma_l(Z_v)\big| \leqslant \frac{1}{1+v+v^2} \cdot \frac{(l!)^{1+\gamma}}{\Delta_v^{l-2}}, \quad l = 3, 4, \ldots, \tag{6}$$

where

$$\Delta_v = \frac{\sigma}{2(\sigma \vee K)\sqrt{1-v}}, \tag{7}$$

$a \vee b = \max\{a, b\}$.

The proof is presented in [2].

Denote

$$\Delta_v(\gamma) = c_v(\gamma)\Delta_v^{\frac{1}{1+2\gamma}}, \quad c_v(\gamma) = (1/6)\big(\sqrt{2}/\big(6(1+v)^{1+\gamma}\big)\big)^{\frac{1}{1+2\gamma}}, \tag{8}$$
$$T_v(\gamma) = \big(3/8\big)\big(1 - x\big/\Delta_v(\gamma)\big)\Delta_v(\gamma), \quad 0 < x < \Delta_v(\gamma).$$

In what follows, let θ_i, $i = 1, 2, \ldots$ denote some quantities, not exceeding 1 in absolute value. Further, suppose that the density $p_{X_0}(x)$ of the r.v. X_0 is bounded, i.e.,

$$\sup_x p_{X_0}(x) \leqslant C < \infty. \tag{D}$$

THEOREM 1. *If for the r.v.* X_k *with* $\mu = \mathbf{E}X_k$ *and* $\sigma^2 = \mathbf{E}(X_k - \mu)^2 > 0$, $k = 0, 1, 2, \ldots$ *conditions* (B_γ) *and* (D) *are fulfilled, then for each integer* l, $l \geqslant 3$ *in the interval* $0 \leqslant x < \Delta_v(\gamma)$ *the following relation holds:*

$$\frac{p_v(x)}{\varphi_v(x)} = \exp\{L_\gamma(x)\}\biggl(1 + \sum_{\nu=0}^{l-3} M_\nu(x) + \theta_1 q(\gamma, l) \cdot \Bigl(\frac{x+1}{\Delta_v}\Bigr)^{l-2} + \theta_2 \frac{5\pi^2 x^2}{8} T_v(\gamma) \exp\Bigl\{-\frac{1}{\pi^2} T_v^2(\gamma)\Bigr\} + \theta_3 \cdot c(K, \sigma, \gamma) \times C v^{-\frac{3}{2}} \exp\Bigl\{-\frac{c_3}{4(K \vee \sigma)C^2} \cdot \frac{1}{1 - v^2}\Bigr\}\biggr), \tag{9}$$

where $q(\gamma, l)$ *is defined by* (6.7) *in* [3], $c(K, \sigma, \gamma) = 384\sqrt{2\pi} e^2 24^\gamma (K \vee \sigma)$, *and*

$$L_\gamma(x) = \sum_{3 \leqslant k < p} \lambda_k x^k, \quad p = \begin{cases} (1/\gamma) + l + 1, & \gamma > 0, \\ \infty, & \gamma = 0, \end{cases} \tag{10}$$

where $\lambda_k = -b_{k-1}/k$ *and* b_k *are determined from the equation*

$$\sum_{r=1}^{j} \frac{1}{r!} \Gamma_{r+1}(Z_v) \sum_{j_1 + \ldots + j_r = j} \prod_{i=1}^{r} b_{j_i} = \begin{cases} 1, & j = 1, \\ 0, & j = 2, 3, \ldots. \end{cases} \tag{11}$$

In particular,

$$b_1 = \Gamma_2^{-1}(Z_v) = 1 + v, \quad b_2 = -\frac{1}{2}(1+v)^3 \Gamma_3(Z_v),$$

$$b_3 = -\frac{1}{6}(1+v)^4 \bigl(\Gamma_4(Z_v) - 3(1+v)\Gamma_3^2(Z_v)\bigr), \ldots.$$

Polynomials $M_\nu(x)$ *are defined by formula*

$$M_\nu(x) = \sum_{k=0}^{\nu} K_k(x) q_{\nu-k}(x), \tag{12}$$

where

$$K_\nu(x) = \sum \prod_{m=1}^{\nu} \frac{1}{k_m!} \Bigl(-\lambda_{m+2} x^{m+2}\Bigr)^{k_m}, \quad K_0(x) \equiv 1,$$

$$q_\nu(x) = \sum H_{\nu+2}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!} \Bigl(\Gamma_{m+2}(Z_v) \big/ (m+2)!\Bigr)^{k_m},$$

$q_0(x) \equiv 1$, $H_l(x)$ *are Chebyshew–Hermite polynomials, and the summation is taken over all integer solutions of the equation* $k_1 + 2k_2 + \ldots + \nu k_\nu = \nu$. *In particular,*

$$M_0(x) \equiv 0, \quad M_1(x) = -\frac{1}{2}\Gamma_3(Z_v) x,$$

$$M_2(x) = (1/8)\Big(5\Gamma_3^2(Z_v) - 2\Gamma_4(Z_v)x^2 + (1/24)\big(3\Gamma_4(Z_v) - 5\Gamma_3^2(Z_v)\big)\Big),$$

where

$$\Gamma_l(Z_v) = \Big(\frac{\sqrt{1-v}}{\sigma}\Big)^l \frac{1}{1-v^l}\,\Gamma_l(X_0), \quad l = 2, 3, \dots . \tag{13}$$

Proof of Theorem 1. The proof of this theorem is based on the results obtained in [4] for the distribution density function $p_{Z_n}(x)$ of normed sum of $Z_n = (\sqrt{\mathbf{D}S_n})^{-1}S_n$, $S_n = \sum_{j=1}^{n}\xi_j^{(n)}$, of independent non-indentically distributed random variables in the scheme of series. First of all, we notice that condition (D) implies the following inequality for the density function $p_{Y_j}(x)$ of r.v. $Y_j = v^j X_j$: $\sup_x p_{Y_j}(x) \leqslant Cv^{-j}$. The quantity K_n in [4] corresponds $K_v := 2\sup_{j\geqslant 0} v^j \cdot \{K \vee \sigma\} = 2\{K \vee \sigma\}$. The expression of the remaining terms in the statement of the Theorem 1 is derived on the base of the estimate of $R_{n,\gamma}$ (24) in [4].

THEOREM 2. *Let for the r.v.'s X_k, $k = 0, 1, 2, \dots$ conditions (B_γ) and (D) are fulfilled. Then for*

$$x \geqslant 0, \quad x = \mathrm{o}\big((1-v)^{-\nu}\big), \quad \nu = \big(2 + 4(1 \vee \gamma)\big)^{-1} \tag{14}$$

the relation

$$\frac{p_v(x)}{\varphi_v(x)} \to 1, \quad v \to 1 \tag{15}$$

holds. In particular, if $\gamma = 0$, the relation (16) *holds for $x \geqslant 0$, $x = \mathrm{o}((1-v)^{-1/6})$.*

Proof of Theorem 2. For all $x = \mathrm{o}((1-v)^{-\frac{1}{2}\nu})$, where $\nu = \nu(\gamma) = (1+2\max\{1,\gamma\})^{-1}$, we get $x\Delta_v^{-1} = \mathrm{o}((1-v)^{(1+\gamma)/(1+2\max\{1,\gamma\})}) \to 0$ for all $\gamma \geqslant 0$ if $v \to 1$. We have to show that $L_\gamma(x) \to 0$ for all $x = \mathrm{o}((1-v)^{-\frac{1}{2}\nu})$. Recalling expression (11) of $L_\gamma(x)$ and making use of estimates (6) of the cumulants $\Gamma_l(Z_v)$, we derive

$$\begin{aligned}\big|\lambda_3 x^3\big| &= \frac{1}{6}(1+v)^3\big|\Gamma_3(Z_v)x^3\big| \leqslant \frac{(1+v)^2 6^\gamma}{\Delta_v}\mathrm{o}\big((1-v)^{-\frac{3}{2}\nu}\big)\\ &= \mathrm{o}\Big((1-v)^{\frac{1}{2}(1-3\nu)}\Big) \to 0, \quad v \to 1,\end{aligned}$$

because $1 - 3\nu = 2(\max\{1,\gamma\} - 1)(1 + 2\max\{1,\gamma\})^{-1} \geqslant 0$. Further, having in mind the expression of the polynomials $M_r(x)$ (formula (13)) included into the statement of Theorem 1, and using the estimates (7) of the cumulants $\Gamma_l(Z_v)$, $l = 3, 4, \dots$, we get $M_r(x) \to 0$, $r = 0, 1, \dots, l-3$. In view of Theorem 1, this yealds the proof.

References

1. H.U. Gerber, The discounted central theorem and its Berry–Esseen analogue, *Ann. Math. Stat.*, **42**, 389–392 (1971).
2. L. Saulis and D. Deltuviene, The Discounted Limit Theorems, *Acta Applicandae Mathematicae*, **90**, 219–226 (2006).
3. L. Saulis, V. Statulevičius, *Limit Theorems for Large Deviations*, Kluwer Academic Publisher, Dordrecht, Boston, London (1991).
4. D. Deltuviene and L. Saulis, Asymptotic expansion of the distribution density function for the sum of random variables in the series scheme in large deviation zones, *Acta Applicandae Mathematicae*, **78**, 87–97 (2003).

REZIUMĖ

L. Saulis, D. Deltuvienė. Didžiųjų nuokrypių diskontavimo lokalinės teoremos

Darbe gautos normuotos sumos $S_v = \sum_{k=0}^{\infty} v^k X_k$, $0 < v < 1$ skirstinio tankio funkcijos $p_v(x)$ aproksimacijos normaliuoju dėsniu, atsižvelgiant į asimptotinius skleidinius, didžiųjų nuokrypių teoremos Kramerio ir laipsninėse Liniko zonose, kai nepriklausomi vienodai pasiskirstę atsitiktiniai dydžiai $X_0, X_1, X_2, \ldots$ tenkina apibendrintą N.S. Bernsteino sąlygą.

Raktiniai žodžiai: skirstinio tankio funkcija, charakteristinė funkcija, kumulantas, didieji nuokrypiai, diskontavimo faktorius.